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Using a new hybrid conjugate gradient method with descent property

Amina Hallal *
Mohammed Belloufi †
Badreddine Sellami §
Department of Computer Science and Mathematics
Laboratory Informatics and Mathematics (LiM)
Mohamed Cherif Messaadia University
Souk Ahras 41000
Algeria

Abstract

The conjugate gradient method was an efficient technique for solving optimization problems. In this paper, we propose a new efficient (CG) coefficient β_k, is computed as a convex combination of Salleh and Alhawarat algorithm and CG-Descent algorithm. We prove the sufficient descent condition and the global convergence of the proposed method. It is established that the α_k satisfies the strong Wolfe line search conditions. The numerical results indicate that our method is robust and competitive.

Subject Classification: *49M37, 65K05, 90C06.*

Keywords: *Unconstrained optimization, Hybrid conjugate gradient method, Strong Wolfe conditions.*

1. Introduction

The Unconstrained Optimization Problem can be formulated as:

$$\min\{f(x), x \in \mathbb{R}^n\}. \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable function and its gradient is available.

The nonlinear conjugate gradient method is the most famous methods for solving (1) and especially for large problems.

* *E-mail:* **a.hallal@univ-soukahras.dz (Correponding Author)**
† *E-mail:* **m.belloufi@univ-soukahras.dz**
§ *E-mail:* **bsellami@univ-soukahras.dz**

The (CG) method generates a sequence of points $\{x_k\} \subset \mathbb{R}^n$ from an initial point x_0 as follows:

$$x_{k+1} = x_k + \alpha_k d_k, \quad k\varepsilon\mathbb{N}. \tag{2}$$

α_k is the step length gotten by exact or inexact line search, and the d_k of the (CG) is calculated by the following:

$$d_k = \begin{cases} -g_0, & \text{for } k = 0, \\ -g_k + \beta_{k-1} d_{k-1}, & \text{for } k \geq 1. \end{cases} \tag{3}$$

$g_k = \nabla f(x_k)$ is the gradient of *f*, $\beta_k \varepsilon \mathbb{R}$ the conjugate gradient parameter.

The different value for the β_k parameter correspond to different (CG) methods.

Some famoos classical conjugate gradient methods:

Hestenes-Stiefel method [23], Fletcher-Reeves method [20], Polak-Ribiére-Polyak method [31, 32], CG-Descent method [21], Liu-Storey method [28], Dai-Yuan method [10], Salleh-Alhawarat method [33], there formulas are given by

$$\beta_k^{HS} = \frac{g_{k+1}^T y_k}{y_k^T d_k}. \tag{4}$$

$$\beta_k^{FR} = \frac{\|g_{k+1}\|^2}{\|\cdot g_k\|^2}. \tag{5}$$

$$\beta_k^{PRP} = \frac{g_{k+1}^T y_k}{\|g_k\|^2}. \tag{6}$$

$$\beta_k^{CD} = -\frac{\|g_{k+1}\|^2}{g_k^T d_k}. \tag{7}$$

$$\beta_k^{DY} = \frac{\|g_{k+1}\|^2}{y_k^T d_k}. \tag{8}$$

$$\beta_k^{LS} = -\frac{g_{k+1}^T y_k}{g_k^T d_k}. \tag{9}$$

$$\beta_k^{ZA} = \begin{cases} \frac{g_{k+1}^T y_k}{y_k^T d_k} : \left| g_{k+1}^T g_k \right| < \left\| g_{k+1} \right\|^2, \\ 0 \qquad \text{otherwise.} \end{cases} \tag{10}$$

Where $\|.\|$ the Euclidean norm and $y_k = g_{k+1} - g_k$.

It is well known that the hybrid conjugate gradient method plays a main role in solving large-scale minimization problems.

Some hybrid conjugate methods are summarized:

See in [22]

$$\beta_k^{TAS} = \begin{cases} \beta_k^{PRP}, \ 0 \le \beta_k^{PRP} \le \beta_k^{FR}, \\ \beta_k^{FR}, \qquad \text{otherwise.} \end{cases} \tag{11}$$

See in [24]

$$\beta_k^{HS-DY} = \max\{0, \min\{\beta_k^{HS}, \beta_k^{DY}\}\}. \tag{12}$$

See in [3]

$$\beta_k^{c} = (1-\theta_k)\beta_k^{HS} + \theta_k \beta_k^{DY}. \tag{13}$$

See in [7]

$$\beta_k^{N} = (1-\theta_k)\beta_k^{PRP} + \theta_k \beta_k^{DY}. \tag{14}$$

See in [9]

$$\beta_k^{DDF} = \frac{\|g_{k+1}\|^2}{\max\{-d_k^T g_k, d_k^T y_k\}}. \tag{15}$$

See in [12]

$$\beta_k^{HS-DY} = \max\{0, \min\{\beta_k^{HS}, \beta_k^{DY}\}\}. \tag{16}$$

In this paper, we suggest a new hybrid method as a convex combination of the (CG) parameters: β_k^{ZA} and β_k^{CD}.

2. Algorithm of new hybrid gradient conjugate method

We defined the parameter β_k in the proposed method by

$$\beta_k^{hZACD} = (1-\theta_k)\beta_k^{ZA} + \theta_k \beta_k^{CD}. \tag{17}$$

i.e.

$$\beta_k^{hZACD} = (1-\theta_k)\frac{g_{k+1}^T y_k}{y_k^T d_k} + \theta_k \frac{\|g_{k+1}\|^2}{-g_k^T d_k}. \tag{18}$$

Where $\theta_k \varepsilon [0,1]$.

The new fomula of the search direction is defined by

$$d_0 = -g_0, d_{k+1} = -g_{k+1} + \beta_k^{hZACD} d_k. \tag{19}$$

The step $\alpha_k > 0$ scalar is determined by the strong Wolfe inexaxt line search as following

$$f(x_k + \alpha_k d_k) - f(x_k) \leq \delta \alpha_k g_k^T d_k, \tag{20}$$

$$\left| g_{k+1}^T d_k \right| \leq -\sigma g_k^T d_k. \tag{21}$$

For all $k \geq 0$.

Where the parameters $\delta \varepsilon \left]0, \sigma\right[$ and $\sigma \varepsilon \left]\rho, 1\right[$.

It is obvious that if $\theta_k = 0$ then $\beta_k^{hZACD} = \beta_k^{ZA}$, and if $\theta_k = 1$ then $\beta_k^{hZACD} = \beta_k^{CD}$, on the other hand if $0 < \theta_k < 1$ then

$$\beta_k^{hZACD} = (1 - \theta_k)\beta_k^{ZA} + \theta_k \beta_k^{CD}.$$

The parameter θ_k is determined in such a way that the search direction satisfies the condition Newton direction.this idea has been introduced in [3].

So, assuming that $\nabla^2 f(x_k)^{-1}$ exists for $k \geq 0$ such that

$$-\nabla^2 f(x_{k+1})^{-1} g_{k+1} = -g_{k+1} + \beta_k^{hZACD} d_k.$$

Where $s_k = x_{k+1} - x_k$.

From (18) and (19) we get

$$-\nabla^2 f(x_{k+1})^{-1} g_{k+1} = -g_{k+1} + (1-\theta_k) \frac{g_{k+1}^T y_k}{d_k^T y_k} d_k + \theta_k \frac{\|g_{k+1}\|^2}{-d_k^T g_k} d_k. \tag{22}$$

Multiplying (22) by $s_k^T \nabla^2 f(x_{k+1})$, we get

$$-s_k^T g_{k+1} = -s_k^T \nabla^2 f(x_{k+1}) g_{k+1} + (1-\theta_k)\beta_k^{ZA} s_k^T \nabla^2 f(x_{k+1}) d_k + \theta_k \beta_k^{CD} s_k^T \nabla^2 f(x_{k+1}) d_k.$$

We assume that (s_k, y_k) satisfies the secant equation and $s_k^T \nabla^2 f(x_{k+1}) = y_k$ then we results that

$$-s_k^T g_{k+1} = -y_k^T g_{k+1} + (1-\theta_k) \frac{g_{k+1}^T y_k}{d_k^T y_k} y_k^T d_k + \theta_k \frac{\|g_{k+1}\|^2}{-d_k^T g_k} y_k^T d_k.$$

We obtain in the end

$$\theta_k = \frac{(-d_k^T g_k)(-s_k^T g_{k+1})}{\left\|g_{k+1}\right\|^2 (y_k^T d_k) - (-d_k^T g_k)(g_{k+1}^T y_k)}. \tag{23}$$

Algorithm of hZACD method

Step 1: Initialization

Choose an initial point $x_0 \in \mathbb{R}^n$, $\varepsilon > 0$.
Compute $f(x_0) = f_0$ and $\nabla f(x_0) = g(x_0)$.
Set $d_0 = -g_0$, $\alpha_0 = 1$.

Step 2: Stopping criteria

If

$$\left\|g_k\right\| < \varepsilon, \tag{24}$$

then Stop.

Step 3: Compute α_k by the strong Wolfe line search (20) and (21).

Step 4: Generate the next iterate by $x_{k+1} = x_k + \alpha_k d_k$.
Compute $g_{k+1} = \nabla f(x_{k+1})$, $y_k = g_{k+1} - g_k$ and $s_k = x_{k+1} - x_k$.

Step 5: Compute θ_k

If $\left\|g_{k+1}\right\|^2 (y_k^T d_k) - (-d_k^T g_k)(g_{k+1}^T y_k) = 0$ then $\theta_k = 0$, else compute θ_k as in (23).

Step 6: Compute β_k^{hZACD}.

If $\theta_k \geq 1$, compute $\beta_k^{hZACD} = \beta_k^{CD}$,
and if $\theta_k \leq 0$, compute $\beta_k^{hZACD} = \beta_k^{ZA}$,
else $0 < \theta_k < 1$, then compute β_k by (18).

Step 7: Compute d_k by equation (19).

Step 8: Put $k = k+1$ and go to step 2.

3. Suffcient descent condition

Lemma 3.12 [14] : *Consider the method (2) and (3) satisfing, with* $\beta_k = \beta_k^{hZACD}$ hold.

Then

$$d_{k+1} = (1-\theta_k)d_{k+1}^{ZA} + \theta_k d_{k+1}^{CD}, \text{for all } k \geq 0. \tag{25}$$

Proof: *From (19) we have*

$$d_{k+1} = -g_{k+1} + (1-\theta_k)\beta_k^{ZA} d_k + \theta_k \beta_k^{CD} d_k.$$

We can write that

$$d_{k+1} = -((1-\theta_k)g_{k+1} + \theta_k g_{k+1}) + ((1-\theta_k)\beta_k^{ZA} + \theta_k \beta_k^{CD})d_k.$$

In the next

$$d_{k+1} = (1-\theta_k)(-g_{k+1} + \beta_k^{ZA} d_k) + \theta_k(-g_{k+1} + \beta_k^{CD} d_k).$$

Finally, we have

$$d_{k+1} = (1-\theta_k)d_{k+1}^{ZA} + \theta_k d_{k+1}^{CD},$$

which implies the result holds for $k = k+1$, and this implies that the proof is finished.

Theorem 3.3 : *Consider the method (2) and (3) with the search direction in (19) and* α_k *is computed by the strong Wolfe line search (20) and (21) then*

$$g_k^T d_k \leq -c\|g_k\|^2, \text{for each } k. \tag{26}$$

Proof 3.4: We demonstrate by induction.

For $k = 0$ we have $g_0^T d_0 = -\|g_0\|^2 < 0$, then (26) hold.

Let d_k descent search direction. Now for $k = k+1$.

From Lemma 3.1, we have

$$d_{k+1} = (1-\theta_k)d_{k+1}^{ZA} + \theta_k d_{k+1}^{CD}. \tag{27}$$

Multpliying (27) by g_{k+1}^T from the left, we get

$$g_{k+1}^T d_{k+1} = (1-\theta_k)g_{k+1}^T d_{k+1}^{ZA} + \theta_k g_{k+1}^T d_{k+1}^{CD}.$$

1) if $\theta_k = 0$, then $g_{k+1}^T d_{k+1}^{hZACD} = g_{k+1}^T d_{k+1}^{CD}$.

In this case, we obtained

$$g_{k+1}^T d_{k+1}^{ZA} = -\|g_{k+1}\|^2 + \frac{(g_{k+1}^T y_k)(g_{k+1}^T d_k)}{y_k^T d_k}$$
$$\leq -\|g_{k+1}\|^2 + \frac{|g_{k+1}^T d_k| |g_{k+1}^T d_k|}{|y_k^T d_k|}.$$

We can write $|g_{k+1}^T y_k| = |g_{k+1}^T (g_{k+1} - g_k)| \leq 2\|g_{k+1}\|^2$, by substituting

$$g_{k+1}^T d_{k+1}^{ZA} \leq -\|g_{k+1}\|^2 + \frac{2\|g_{k+1}\|^2 |g_{k+1}^T d_k|}{|y_k^T d_k|}.$$

Now from (21) we conclude the two next inequality

$$|g_{k+1}^T d_k| \leq -\sigma g_k^T d_k. \tag{28}$$

Hence

$$y_k^T d_k = (g_{k+1} - g_k)^T d_k \geq \sigma g_k^T d_k - g_k^T d_k \geq -(1-\sigma) g_k^T d_k. \tag{29}$$

Implies

$$\frac{1}{|y_k^T d_k|} \leq \frac{1}{-(1-\sigma) g_k^T d_k}. \tag{30}$$

So, with appliying (28) and (30), we have the following

$$g_{k+1}^T d_{k+1}^{ZA} \leq -\|g_{k+1}\|^2 + \frac{2\sigma}{(1-\sigma)} \|g_{k+1}\|^2$$
$$\leq -\left(\frac{1-3\sigma}{1-\sigma}\right) \|g_{k+1}\|^2,$$

$c_1 = \frac{1-3\sigma}{1-\sigma} > 0$ where $0 < \delta < \sigma < \frac{1}{3}$.

Implies

$$g_{k+1}^T d_{k+1}^{ZA} \leq -c_1 \|g_{k+1}\|^2. \tag{31}$$

2) if $\theta_k = 1$, then $g_{k+1}^T d_{k+1}^{hZACD} = g_{k+1}^T d_{k+1}^{CD}$

$$g_{k+1}^T d_{k+1}^{CD} = -\|g_{k+1}\|^2 + \frac{\|g_{k+1}\|^2}{-g_k^T d_k} (g_{k+1}^T d_k),$$
$$\leq -\|g_{k+1}\|^2 + \frac{\|g_{k+1}\|^2}{|g_k^T d_k|} |g_{k+1}^T d_k|.$$

According (28) it holds that

$$g_{k+1}^T d_{k+1}^{CD} \le -\|g_{k+1}\|^2 + \frac{\|g_{k+1}\|^2}{(-g_k^T d_k)}(-\sigma g_k^T d_k)$$
$$\le -(1-\sigma)\|g_{k+1}\|^2 .$$

In the end we obtain

$$g_{k+1}^T d_{k+1}^{CD} \le -c_2 \|g_{k+1}\|^2 . \qquad (32)$$

$c_2 = 1-\sigma > 0$ where $\sigma < \frac{1}{3}$.

[16] Finally for $0 < \theta_k < 1$ there exists b_1 and b_2 in wich that $0 < b_1 < \theta_k < b_2 < 1$ and from (31), (32)

$$g_{k+1}^T d_{k+1}^{hZACD} \le -(1-b_2)c_1 \|g_{k+1}\|^2 - b_1 c_2 \|g_{k+1}\|^2 .$$

Implies

$$g_{k+1}^T d_{k+1}^{hZACD} \le -((1-b_2)c_1 + b_1 c_2)\|g_{k+1}\|^2 ,$$
$$g_{k+1}^T d_{k+1}^{hZACD} \le -c\|g_{k+1}\|^2 . \qquad (33)$$

In which $c = ((1-b_2)c_1 + b_1 c_2)$.

The Proof is complete.

4. Convergence of analysis

In this section we will apply the following assumptions:

Assumption 1: The level set $\mathcal{S} = \{x \in \mathbb{R}^n \mid f(x) \le f(x_0)\}$ is bounded where x_0 is the starting point.

Assumption 2: In a neighborhood $\mathcal{N}$ of $\mathcal{S}$ the function f is continuously differentiable and its gradient $g(x)$ is Lipschitz continuous.

From assumption 1 and assumption 2, we conclude the next one

$$\| g(x) \| \le l, \text{ for all } x \in \mathcal{S}. \qquad (34)$$

[3].

Lemma 4.1 [27] : *Suppose the search direction d_k is descent, and assumption 2 hold, α_k is computed by the strong Wolfe line search, then*

$$\alpha_k \geq c\,\frac{(1-\sigma)\|g_k\|^2}{L\|d_k\|^2}. \tag{35}$$

Proof 4.2: From the second strong Wolfe inequality (29) and with using assumption 2, we obtain

$$\begin{aligned}(\sigma-1)g_k^T d_k &\leq (g_{k+1}-g_k)^T d_k \\ &\leq L\alpha_k \| d_k \|^2 .\end{aligned}$$

In which d_k is a descent direction $\sigma < 1,$ it follows that

$$\alpha_k \geq c\,\frac{(1-\sigma)\|g_k\|^2}{L\|d_k\|^2}.$$

This lemma indicate that $\exists\ \gamma > 0$ where

$$\alpha_k \geq \gamma. \tag{36}$$

Lemma 4.3 : [12] *Let Assumption 1 et 2 holds. Consider the iterative method (2) and (3), with* d_k *satisfies* $g_k^T d_k < 0$ *and the step size* α_k *is received from the strong Wolfe line search*

If

$$\sum_{k=1}^{\infty} \frac{1}{\|d_k\|^2} = \infty. \tag{37}$$

Then

$$\lim_{k\to\infty} \inf \|g_k\| = 0. \tag{38}$$

Theorem 4.4 : *Suppose that the Assumption 1 and 2 holds. Consider the algorithm hZACD, with the search direction* d_k *is descent.* α_k *is obtained by the strong Wolfe line search then*

$$\lim_{k\to\infty} \inf \|g_k\| = 0. \tag{39}$$

Proof 4.5: We proved the theorem by contradiction.

It meant we suppose that (39) doesn't hold.

We know if $g_k \neq 0\ \ \forall\ \ k$ there exists a constant $\bar{l} > 0,$ in which

$$\|g_k\| \geq \bar{l} \text{ for all } k \geq 0. \tag{40}$$

Since D =max $\{\|x-y\|, x, y \varepsilon \mathcal{S}\}$ is the diameter of $\mathcal{S}$.
From assumption 2 $\exists L > 0$

$$\|y_k\| = \|g_{k+1} - g_k\| \leq L\|x_{k+1} - x_k\| = L\|s_k\| \leq LD. \tag{41}$$

From (19) we obtain

$$\|d_{k+1}\| \leq \| g_{k+1} \| + |\beta_k^{hZACD}| \| d_k \|.$$

In which

$$|\beta_k^{hZACD}| = |(1-\theta_k)\beta_k^{ZA} + \theta_k \beta_k^{CD}| \leq |\beta_k^{ZA}| + |\beta_k^{CD}|.$$

Using Cauchy Schwartz inequality, (26), (30), (34), (40) and (41), we have that

$$\begin{aligned} |\beta_k^{ZA}| &= \left|\frac{(g_{k+1}^T y_k)}{y_k^T g_k}\right| \\ &\leq \frac{\|g_{k+1}\|\|y_k\|}{-(1-\sigma) g_k^T d_k} \\ &\leq \frac{lLD}{c(1-\sigma)\|g_k\|^2} \leq \frac{LlD}{m\bar{l}^2}. \end{aligned}$$

On the other hand,

$$|\beta_k^{CD}| = \frac{\|g_{k+1}\|^2}{-g_k^T d_k} \leq \frac{l^2}{c\bar{l}^2}.$$

Now we find that

$$|\beta_k^{hZACD}| \leq \frac{LlD}{m\bar{l}^2} + \frac{l^2}{c\bar{l}^2}.$$

Here we can write

$$\|d_{k+1}\| \leq \| g_{k+1} \| + |\beta_k^{hZACD}| \| d_k \| \tag{42}$$

$$\leq l + \frac{A\|s_k\|}{\alpha_k}. \tag{43}$$

According to (34), (36) and (41) we have

$$\|d_{k+1}\| \leq l + A\frac{LD}{\gamma}.$$

Hence

$$\frac{1}{\|d_{k+1}\|^2} \geq \frac{1}{\left(l + A\frac{LD}{\gamma}\right)^2}.$$

Therfore

$$\sum_{k=0}^{\infty} \frac{1}{\|d_{k+1}\|^2} = \infty. \tag{44}$$

By applyinge lemma 4.3 is a contradiction.

So we have prove (39), and we get finaly the convergence of our method.

5. Numerical Result

In this section, we choose some of the test functions from [4], [25]. We analyzed the performance of the new algorithm with the ZA and CD algorithms with different initial points and dimensions range from small scale to large scale. For the numerical tests, all codes were written or a PC computer with a CPU 1.60 GHz and 2.00GB of RAM, and the parameters in the strong Wolfe line searches are chosen to be $\alpha = 1,\ \sigma = 10^{-3},\ \delta = 10^{-4}$. We stop if $\|g(x_k)\| \leq 10^{-6}$ is satisfied.

In particular, the following result is established in [8], [19].

This is done based on the number of iterations and CPU time, which were evaluated using the profiles of Dolan and Moré [18]. Benchmark results are generated by running a solver on a set P of problems. Let S consists of n_s problems, P consists of n_p problems. For each problem $p \in P$ and solver $s \in S$, denote $t_{p,s}$ be the executing time (or the number of iterations) required to solve problem $p \in P$ by solver $s \in S$. The is formed as follows:

$$r_{p,s} = \frac{t_{p,s}}{\min\{t_{p,s} : s \in S\}}.$$

Assuming that a scalar $r_M \geq r_{p,s}$. for all p, s is chosen, if and only if solvers s does not solver problem p, we have:

$$\rho_s(\tau) = \frac{1}{n_p} size\{p \in P : \log_{p,s} r \leq T\}.$$

Then $\rho_s(\tau)$ is the probability for solver $s \in S$ that a performance ratio $r_{p,s}$ is within a factor $T \in \mathbb{R}^n$. The ρ_s is the cumulative distribution function for the performance ratio. The value of $\rho_s(1)$ is the probability that the solver will win over the rest of the solvers.

Figures 1, 2 exhibit the performance of the hZACD method versus ZA and CD methods, which show that our method better than the other.

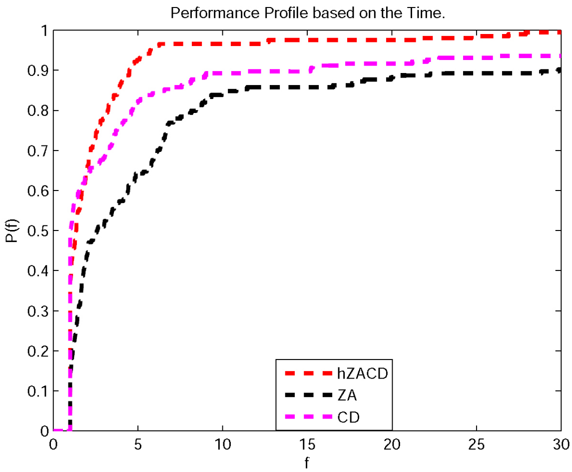

Figure 1

Performance profile based on the CPU time (inexact).

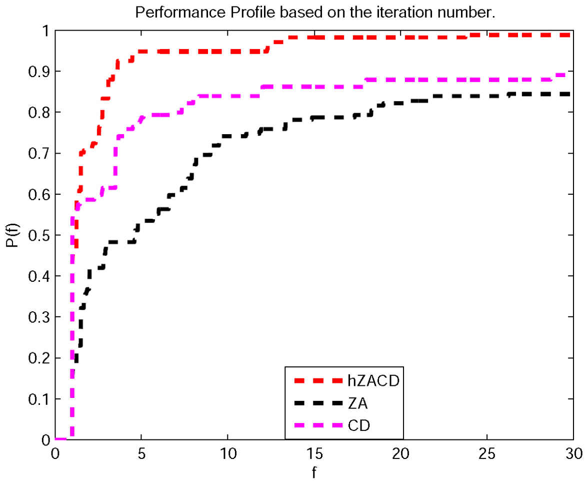

Figure 2

Performance profile based on the number of iterations (inexact).

6. Conclusion

In this work, a new form of conjugate gradient parametre has been suggested to solve an unconstrained problem. We have also demonstrated that this new method converges globally with strong Wolfe inexact line search. The presented numerical results show the robustness of our proposed method.

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References

[1] I. Abdullahi and R. Ahmad, Global convergence analysis of a new hybrid conjugate gradient method for unconstrained optimization problems, *Malaysian Journal of Fundamental and Applied Sciences*, Vol. 13, No. 2 (2017) 40-48. http://dx.doi.org/10.11113/mjfas.v13n2.540.

[2] N. Andrei, 40 Conjugate Gradient Algorithms for unconstrained. optimization, A survey on their definition, ICI Technical Report, (2008) 13-08.

[3] N. Andrei, Another hybrid conjugate gradient algorithm for unconstrained optimization, Numerical Algorithms (2008) 143-156.

[4] N. Andrei, An unconstrained optimization test functions collection, *Adv. Model. Optim*, 10 (2008) 147-161.

[5] N. Andrei, for Unconstrained Optimization, *Studies in Informatics and Control*, 17(4) (2008) 373-392.

[6] N. Andrei, Hybrid Conjugate Gradient Algorithm. *J Optim Theory Appl,* 141 (2009) 249-264.

[7] N. Andrei, New hybrid conjugate gradient algorithms for unconstrained optimization. *Encyclopedia of Optimization* (2009) 2560-2571. http://dx.doi.org/10.1007/978-0-387-74759-0-441.

[8] S. Ben hanachi, B. Sellami and M. Belloufi, New iterative conjugate gradient method for nonlinear unconstrained optimization, *RAIRO operations research*, https://doi.org/10.1051/ro/2022109.

[9] Y. H. Dai, A nonmonotone conjugate gradient algorithm for unconstrained optimization, *Journal of Systems Science and Complexity*, 15(2) (2002) 139-145.

[10] Y. H. Dai and Y. Yuan, A nonlinear conjugate gradient method with a strong global convergence property, *SIAM J Optim*, 10 (1999) 177-182. http://dx.doi.org/10.1137/s1052623497318992.

[11] Y.H. Dai, J.Y.Han, G.H. Liu, D.F.Sun, X. Yin and Y. Yuan, Convergence properties of nonlinear conjugate gradient methods, *SIAM Journal on Optimization*, 10 (1999), 348-358. http://dx.doi.org/10.1137/S1052623494268443.

[12] Y. H. Dai and Y. Yuan, A class of globally convergent conjugate gradient methods, Science China Mathematics Series A, 46(2) (2003) 251-261. http://dx.doi.org/10.1360/03ys9027.

[13] Z. Dai and F. Wen, Another improved Wei-Yao-Liu nonlinear conjugate gradient method with sufficient descent property, Applied Mathematics and Computation, 218 (2012) 7421-7430. http://dx.doi.org/10.1016/j.amc.2011.12.091.

[14] S. S. Djordjevic, New Hybrid Conjugate GradientMethod as a Convex Combination of FR and PRP Methods, Filomat, 30(11) (2016) 3083-3100. http://dx.doi.org/10.2298/FIL1611083D.

[15] S. S. Djordjevic, New Hybrid Conjugate Gradient Method as a Convex Combination of LS and CD methods, Filomat, 2(9) (2017) 1813-1825. http://dx.doi.org/10.2298/FIL1706813D.

[16] S. S. Djordjevic, New hybrid conjugate gradient method as a convex combination of HS and FR conjugate gradient methods, *Journal of Applied Mathematics and Computation* (JAMC), 2(9) (2018) 366-378. http://dx.doi.org/10.26855/jamc.2018.09.002.

[17] S. S. Djordjevic, New hybrid conjugate gradient method as a convex combination of LS and FR conjugate gradient methods, Acta Mathematica Scientia, 39B(1) (2019) 214-228. http://dx.doi.org/s10473-019-0117-6.

[18] E. D. Dolan and J. J. Moré, Benchmarking optimization software with performance profiles, Mathematical Programming 91 (2002) 201-213. http://dx.doi.org/10.1007/s101070100263.

[19] S. Delladji, M. Belloufi and B. Sellami, Behavior of the combination of PRP and HZ methods for unconstrained optimization, Numerical Algebra, *Control & Optimization* 11(3) (2021) 377.

[20] R. Fletcher and C. Reeves, Function minimization by conjugate gradients, Comput J, 7 (1964) 149-154. 149-154. http://dx.doi.org/10.1093/comjnl/7.2.149.

[21] R. Fletcher, Practical methods of optimization, 2nd ed A Wiley-Interscience Publication, John Wiley & Sons, Inc (1987) NY, USA. http://dx.doi.org/10.1002/9781118723203.

[22] J. C. Gilbert and J. Nocedal, Global convergence properties of conjugate gradient methods for optimization, *SIAM Journal on Optimization*, 2 (1992) 21-42. http://dx.doi.org/10.1137/0802003.

[23] M. R. Hestenes and E. L. Stiefel, Methods of conjugate gradients for solving linear systems, *J Research Nat Bur Standards*, 49 (1952) 409-436. http://dx.doi.org/10.6028/jres.049.044.

[24] Y. F. Hu and C. Storey, Global convergence result for conjugate gradient methods, *J Optim Theory Appl*, 71 (1991) 399-405. http://dx.doi.org/10.1007/BF00939927.

[25] M. Jamil and X. S. Yang, A Literature Survey of Benchmark Functions For Global Optimization Problems, *Int, Journal of Mathematical Modelling and Numerical Optimisation* Vol. 4(2) (2013) 150-194. http://dx.doi.org/10.1504/IJMMNO.2013.055204.

[26] F. N. Jardow and G. M. Al-Naemi, A new hybrid conjugate gradient algorithm for unconstrained optimization with inexact line search, *Indonesian Journal of Electrical Engineering and Computer Science,* 20(2) (2020) 939-947. http://dx.doi.org/10.11591/ijeecs.v20.i2.pp939-947.

[27] J. Liu and S. Li. New hybrid conjugate gradient method for unconstrained optimization, Applied Mathematics and Computation, 245 (2014) 36-43. http://dx.doi.org/0.1016/j.amc.2014.07.096.

[28] Y. Liu and C. Storey, Efficient generalized conjugate gradient algorithms, part 1: theory. JOTA, 69 (1991) 129-137. http://dx.doi.org/10.1007/bf00940464.

[29] E. Polak and G. Ribiere, Note sur la convergence de mé thodes de directions conjugués, Revue Francaise d'Informatique et de Recherche Opérationnelle, 16 (1969) 35-43. http://dx.doi.org/10.1051/m2an/196903R100351.

[30] B. T. Polyak, The conjugate gradient method in extreme problems, USSR Comp Math. Math. Phys, 9 (1969) 94-112. http://dx.doi.org/10.1016/0041-5553(69)90035-4.

[31] M. Rivaie, M. Mamat, W. J. Leong and I. Mohd, A new class of nonlinear conjugate gradient coefficients with global convergence properties, *Applied Mathematics and Computation*, 218 (2012) 11323-11332. http://dx.doi.org/10.1016/j.amc.2012.05.030.

[32] Z. Salleh and A, Alhawarat, An efficient modification of the Hestenes-Stiefel nonlinear conjugate gradient method with restart property, *Journal of Inequalities and Applications,* 110 (2016). http://dx.doi.org/10.1186/s13660-016-1049-5.

[33] A. D. Touati and C. Storey, Efficient hybrid conjugate gradient techniques, *J Optim Theory Appl*, 64 (1990) 379-397. http://dx.doi.org/10.1007/bf00939455.

[34] P. Wolfe, Convergence conditions for ascent methods, II : Some corrections, *SIAM Review*, 13 (1971) 185-188. http://dx.doi.org/10.1137/1011036.

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