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Existence and uniqueness of solutions for the nonlinear retarded and advanced implicit Hadamard fractional differential equations with nonlocal conditions

Moussa Haoues[1] , Abdelouaheb Ardjouni [2*], Ahcene Djoudi[3]

[1] Laboratory of Informatics and Mathematics,
Department of Mathematics and Informatics, University of Souk-Ahras,
P.O. Box 1553, Souk-Ahras, 41000, Algeria
E-mail: moussakoussa84@yahoo.fr

[2] Department of Mathematics and Informatics,
University of Souk Ahras, P.O. Box 1553, Souk Ahras, 41000, Algeria
Applied Mathematics Lab, Faculty of Sciences,
Department of Mathematics, University of Annaba,
P.O. Box 12, Annaba 23000, Algeria

[3] Applied Mathematics Lab, Faculty of Sciences,
Department of Mathematics, University of Annaba,
P.O. Box 12, Annaba 23000, Algeria
E-mail: adjoudi@yahoo.com

[*] *Corresponding Author*. E-mail: abd_ardjouni@yahoo.fr

Abstract. In this paper, we use the Banach contraction mapping principle and the Krasnoselskii fixed point theorem to obtain the existence and uniqueness of solutions for nonlinear retarded and advanced implicit Hadamard fractional differential equations with nonlocal conditions. The results obtained here extend the work of Benchohra, Bouriah and Henderson [5]. Two examples are also given to illustrate the results.

1 Introduction

Fractional differential equations with and without delay arise from a variety of applications including in various fields of science and engineering such as applied sciences, practical problems concerning

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mechanics, the engineering technique fields, economy, control systems, physics, chemistry, biology, medicine, atomic energy, information theory, harmonic oscillator, nonlinear oscillations, conservative systems, stability and instability of geodesic on Riemannian manifolds, dynamics in Hamiltonian systems, etc. In particular, problems concerning qualitative analysis of linear and nonlinear fractional differential equations with and without delay have received the attention of many authors, see [1]–[14], [16], [17] and the references therein.

In interesting contributions, Benchohra, Bouriah and Henderson [5] discussed the existence and uniqueness of solutions for the nonlinear implicit Hadamard fractional differential equation with retarded and advanced arguments

$$\begin{cases} D^{\alpha} y(t) = f(t, y_t, D^{\alpha} y(t)), \ t \in [1, e], \\ y(t) = \varkappa(t), \ t \in [1-r, 1], \ r > 0, \\ y(t) = \psi(t), \ t \in [e, e+h], \ h > 0, \end{cases}$$

where D^{α} is the Hadamard fractional derivative of order $1 < \alpha \leq 2$. By employing the Schauder fixed point theorem and the Banach contraction mapping principle, the authors obtained existence and uniqueness results.

Anh and Ke [4] investigated the existence and asymptotic stability of solutions for the following retarded fractional differential equation with nonlocal conditions

$$\begin{cases} {}^{C}D^{\alpha} y(t) = Ay(t) + f(t, y(t), y_t), \ t > 0, \\ y(t) + (Hy)(t) = \varkappa(t), \ t \in [-r, 0], \ r > 0, \end{cases}$$

where ${}^{C}D^{\alpha}$ is the standard Caputo fractional derivative of order $0 < \alpha \leq 1$. By using the fixed point theory for condensing maps,the authors obtained existence and stability results.

In this paper, we are interested in the analysis of qualitative theory of the problems of the existence and uniqueness of solutions to nonlinear retarded and advanced Hadamard fractional differential equations. Inspired and motivated by the works mentioned above and the references in this paper, we concentrate on the existence and uniqueness of solutions for the nonlinear retarded and advanced implicit Hadamard fractional differential equation with nonlocal conditions

$$\begin{cases} D^{\alpha} y(t) = f(t, y_t, D^{\alpha} y(t)), \text{ for each } t \in J := [1, e], \\ y(t) + (H_1 y)(t) = \varkappa(t), \ t \in [1-r, 1], \ r > 0, \\ y(t) + (H_2 y)(t) = \psi(t), \ t \in [e, e+h], \ h > 0, \end{cases} \tag{1.1}$$

where D^{α} is the Hadamard fractional derivative of order $1 < \alpha \leq 2$, $f : J \times C([-r, h], \mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, $H_1 : C([1-r, e+h], \mathbb{R}) \to C([1-r, 1], \mathbb{R})$ and $H_2 : C([1-r, e+h], \mathbb{R}) \to C([e, e+h], \mathbb{R})$ are given continuous mappings, $\varkappa \in C([1-r, 1], \mathbb{R})$ and $\psi \in C([e, e+h], \mathbb{R})$. For each function y defined on $[1-r, e+h]$ and for any $t \in J$, we denote by y_t the element of $C([-r, h], \mathbb{R})$ defined by

$$y_t(\theta) = y(t+\theta), \ \theta \in [-r, h].$$

To show the existence and uniqueness of solutions, we transform (1.1) into an integral equation and then use the Banach contraction mapping principle and the Krasnoselskii fixed point theorem. Finally, we provide two examples to illustrate our obtained results. The results obtained here extend the work of Benchohra, Bouriah and Henderson [5].

2 Preliminaries

In this section, we introduce notations, definitions and preliminary facts that are used throughout this article. By $C([a,b],\mathbb{R})$ we denote the Banach space of all continuous functions equipped with the norm

$$\|y\|_{[a,b]} = \sup\{|y(t)| : a \le t \le b\}.$$

Let $L^1(J,\mathbb{R})$ be the space of Lebesgue integrable functions $w : J \longrightarrow \mathbb{R}$ with the norm

$$\|w\|_1 = \int_1^e |w(s)|\,ds.$$

Definition 2.1 ([10]). The Hadamard fractional order integral of the function $y \in L^1(J,\mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$, is given by

$$I^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \left(\log\frac{t}{s}\right)^{\alpha-1} y(s)\frac{ds}{s},$$

where $\Gamma(\alpha) = \int_0^\infty \exp(-t)\,t^{\alpha-1}dt$ is the Gamma function.

Definition 2.2 ([10]). The Hadamard fractional order derivative of order $\alpha \in \mathbb{R}_+$ of the function $y : [1,\infty) \to \mathbb{R}$ is defined by

$$D^\alpha y(t) = \frac{1}{\Gamma(n-\alpha)} \left(t\frac{d}{dt}\right)^n \int_1^t \left(\log\frac{t}{s}\right)^{n-\alpha-1} y(s)\frac{ds}{s},$$

where $n = [\alpha]+1$ and $[\alpha]$ denotes the integer part of α.

Definition 2.3 ([10]). Let $\alpha > 0$ and $n = [\alpha]+1$. The equality $D^\alpha y(t) = 0$ is valid if and only if

$$y(t) = \sum_{j=1}^{n} c_j (\log t)^{\alpha-j} \text{ for each } t \in J,$$

where $c_j \in \mathbb{R}$ $(j = 1,\dots,n)$ are arbitrary constants.

Lastly in this section, we state the fixed point theorems which enable us to prove the existence and uniqueness of a solution of (1.1).

Definition 2.4. Let $(X,\|.\|)$ be a Banach space and $N : X \to X$. The operator N is a contraction operator if there is an $\lambda \in (0,1)$ such that $x,y \in X$ imply

$$\|Nx - Ny\| \le \lambda \|x-y\|.$$

Theorem 2.1 (Banach contraction mapping principle [15]). Let X be a Banach space X and $N : X \to X$ be a contraction operator. Then there is a unique $x \in X$ with $Nx = x$.

Theorem 2.2 (Krasnoselskii fixed point theorem [15]).

If $\mathcal{K}$ is a nonempty bounded, closed and convex subset of a Banach space X, A and B two operators defined on $\mathcal{K}$ with values in X such that

i) $Ax+By \in \mathcal{K}$, for all $x,y \in \mathcal{K}$,

ii) A is continuous and compact,

iii) B is a contraction.

Then there exists $z \in \mathcal{K}$ such that $z = Az+Bz$.

3 Existence and uniqueness results

Let us list some assumptions to prove our existence and uniqueness results.

(H1) The function $f : J \times C([-r,h],\mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ is continuous.

(H2) There exist constants $K_1 \in \mathbb{R}_+$ and $K_2 \in (0,1)$ such that

$$|f(t,u,v) - f(t,\tilde{u},\tilde{v})| \leq K_1 \|u-\tilde{u}\|_{[-r,h]} + K_2 |v-\tilde{v}|,$$

for any $t \in J,\ u,\tilde{u} \in C([-r,h],\mathbb{R})$ and $v,\tilde{v} \in \mathbb{R}$.

(H3) There exist constants $K_3, K_4 \in (0,1)$ such that

$$\|H_1 y_1 - H_1 y_2\|_{[1-r,1]} \leq K_3 \|y_1 - y_2\|_{[1-r,e+h]},$$

and

$$\|H_2 y_1 - H_2 y_2\|_{[e,e+h]} \leq K_4 \|y_1 - y_2\|_{[1-r,e+h]},$$

for any $y_1, y_2 \in C([1-r,e+h],\mathbb{R})$.

(H4) There exist constants $M_{H_1} > 0$ and $M_{H_2} > 0$ such that

$$\|H_1 y\|_{[1-r,1]} \leq M_{H_1} \text{ and } \|H_2 y\|_{[e,e+h]} \leq M_{H_2},$$

for any $y \in C([1-r,e+h],\mathbb{R})$.

(H5) There exist $p,q,r \in C(J,\mathbb{R}_+)$ with $r^* = \sup_{t\in J} r(t) < 1$ such that

$$|f(t,u,w)| \leq p(t) + q(t)\|u\|_{[-r,h]} + r(t)|w|,$$

for $t \in J$, $u \in C([-r,h],\mathbb{R})$ and $w \in \mathbb{R}$.

Definition 3.1. A function $y \in C^2([1-r,e+h],\mathbb{R})$ is said to be a solution of (1.1) if y satisfies the implicit fractional differential equations $D^\alpha y(t) = f(t,y_t,D^\alpha y(t))$ on J, and the conditions $y(t) + (H_1 y)(t) = \varkappa(t)$ on $[1-r,1]$ and $y(t) + (H_2 y)(t) = \psi(t)$ on $[e,e+h]$.

The proof of the following lemma is close to the proof of Lemma 1 given in [5].

Lemma 3.1. Let σ be a continuous function. Then the linear problem

$$\begin{cases} D^\alpha y(t) = \sigma(t),\ t \in J, \\ y(t) + (H_1 y)(t) = \varkappa(t),\ t \in [1-r,1], \\ y(t) + (H_2 y)(t) = \psi(t),\ t \in [e,e+h], \end{cases}$$

has a unique solution which is given by

$$y(t) = \begin{cases} \varkappa(t) - (H_1 y)(t),\ \text{if } t \in [1-r,1], \\ \left(1 - (\log t)^{\alpha-1}\right)(\varkappa(1) - (H_1 y)(1)) + (\log t)^{\alpha-1}(\psi(e) - (H_2 y)(e)) \\ -\int_1^e G(t,s)\sigma(s)\dfrac{ds}{s},\ \text{if } t \in J, \\ \psi(t) - (H_2 y)(t),\ \text{if } t \in [e,e+h], \end{cases}$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} (\log t)^{\alpha-1}(1-\log s)^{\alpha-1} - (\log t - \log s)^{\alpha-1},\ 1 \leq s \leq t \leq e, \\ (\log t)^{\alpha-1}(1-\log s)^{\alpha-1},\ 1 \leq t \leq s \leq e. \end{cases} \tag{3.1}$$

Our first result is based on the Banach contraction mapping principle.

Theorem 3.1. Assume that (H1)–(H3) hold. If

$$\delta+\frac{2K_1}{(1-K_2)\Gamma(\alpha+1)}<1 \text{ where } \delta=\max\{K_3,K_4\}, \tag{3.2}$$

then there exists a unique solution of (1.1).

Proof. Consider the operator $N : C([1-r,e+h],\mathbb{R}) \to C([1-r,e+h],\mathbb{R})$ defined by

$$(Ny)(t)=\begin{cases}\varkappa(t)-(H_1y)(t), \text{ if } t\in[1-r,1],\\ \left(1-(\log t)^{\alpha-1}\right)(\varkappa(1)-(H_1y)(1))+(\log t)^{\alpha-1}(\psi(e)-(H_2y)(e))\\ -\int_1^e G(t,s)\,g(s)\,\frac{ds}{s}, \text{ if } t\in J,\\ \psi(t)-(H_2y)(t), \text{ if } t\in[e,e+h],\end{cases}$$

where $g\in C(J,\mathbb{R})$ is such that

$$g(t)=f(t,y_t,g(t)).$$

Clearly, the fixed points of operator N are solution of problem (1.1). Let $u,w\in C([1-r,e+h],\mathbb{R})$. If $t\in[1-r,1]$, then

$$\begin{aligned}|(Nu)(t)-(Nw)(t)| &= |(H_1u)(t)-(H_1w)(t)|\\ &\le \|H_1u-H_1w\|_{[1-r,1]}\\ &\le K_3\|u-w\|_{[1-r,e+h]}.\end{aligned}$$

And if $t\in[e,e+h]$, then

$$\begin{aligned}|(Nu)(t)-(Nw)(t)| &= |(H_2u)(t)-(H_2w)(t)|\\ &\le \|H_2u-H_2w\|_{[e,e+h]}\\ &\le K_4\|u-w\|_{[1-r,e+h]}.\end{aligned}$$

Also, for $t\in J$, we have

$$\begin{aligned}&|(Nu)(t)-(Nw)(t)|\\ &\le \left|1-(\log t)^{\alpha-1}\right||(H_1u)(1)-(H_1w)(1)|+(\log t)^{\alpha-1}|(H_2u)(e)-(H_2w)(e)|\\ &+\int_1^e|G(t,s)||g(s)-z(s)|\frac{ds}{s},\\ &\le \left|1-(\log t)^{\alpha-1}\right|\|H_1u-H_1w\|_{[1-r,1]}+(\log t)^{\alpha-1}\|H_2u-H_2w\|_{[e,e+h]}\\ &+\int_1^e|G(t,s)||g(s)-z(s)|\frac{ds}{s},\end{aligned} \tag{3.3}$$

where $g,z\in C(J,\mathbb{R})$ is such that

$$g(t)=f(t,u_t,g(t)),$$

and

$$z(t)=f(t,w_t,z(t)).$$

By (H2) we have

$$\begin{aligned} |g(t)-z(t)| &= |f(t,u_t,g(t))-f(t,w_t,z(t))| \\ &\leq K_1 \|u_t-w_t\|_{[-r,h]} + K_2 |g(t)-z(t)|. \end{aligned}$$

Then

$$|g(t)-z(t)| \leq \frac{K_1}{1-K_2} \|u_t-w_t\|_{[-r,h]}. \tag{3.4}$$

By (H3) we have

$$\|H_1 u - H_1 w\|_{[1-r,1]} \leq K_3 \|u-w\|_{[1-r,e+h]} \tag{3.5}$$

and

$$\|H_2 u - H_2 w\|_{[e,e+h]} \leq K_4 \|u-w\|_{[1-r,e+h]} \tag{3.6}$$

By considering (3.4), (3.5) and (3.6) in (3.3), we have

$$\begin{aligned} |(Nu)(t)-(Nw)(t)| &\leq \left(K_3\left(1-(\log t)^{\alpha-1}\right)+K_4(\log t)^{\alpha-1}\right)\|u-w\|_{[1-r,e+h]} \\ &\quad + \frac{K_1}{1-K_2}\|u-w\|_{[1-r,e+h]}\int_1^e |G(t,s)|\frac{ds}{s}. \end{aligned} \tag{3.7}$$

On the other hand, we have for each $t \in J$

$$\begin{aligned} \int_1^e |G(t,s)|\frac{ds}{s} &\leq \frac{1}{\Gamma(\alpha)}\left[\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{ds}{s} + (\log t)^{\alpha-1}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1}\frac{ds}{s}\right] \\ &\leq \frac{2}{\Gamma(\alpha)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1}\frac{ds}{s} = \frac{2}{\Gamma(\alpha+1)}. \end{aligned} \tag{3.8}$$

By considering (3.8) in (3.7), we have

$$\begin{aligned} &|(Nu)(t)-(Nw)(t)| \\ &\leq \left(\left((K_4-K_3)(\log t)^{\alpha-1}+K_3\right)+\frac{2K_1}{(1-K_2)\Gamma(\alpha+1)}\right)\|u-w\|_{[1-r,e+h]} \\ &\leq \left(\delta+\frac{2K_1}{(1-K_2)\Gamma(\alpha+1)}\right)\|u-w\|_{[1-r,e+h]}, \end{aligned}$$

where

$$\delta = \max_{t\in J}\left((K_4-K_3)(\log t)^{\alpha-1}+K_3\right) = \max\{K_3,K_4\}.$$

Therefore,

$$\|Nu-Nw\|_{[1-r,e+h]} \leq \left(\delta+\frac{2K_1}{(1-K_2)\Gamma(\alpha+1)}\right)\|u-w\|_{[1-r,e+h]}.$$

By (3.2), the operator N is a contraction. Hence, by the Banach contraction mapping principle, N has a unique fixed point which is the unique solution of (1.1).

Our second result is based on the Krasnoselskii fixed point theorem.

Theorem 3.2. Assume (H1)–(H5) hold. If

$$\frac{2q^*}{(1-r^*)\Gamma(\alpha+1)} < 1,$$

where $q^* = \sup_{t\in J} q(t)$, then (1.1) has at least one solution.

Proof. Choose

$$R \geq \|\varkappa\|_{[1-r,1]} + M_{H_1} + \|\psi\|_{[e,e+h]} + M_{H_2} + \frac{2\left(p^* + q^* R\right)}{\left(1 - r^*\right)\Gamma\left(\alpha + 1\right)},$$

where $p^* = \sup_{t \in J} p\left(t\right)$, and define the set

$$D_R = \left\{ y \in C\left(\left[1 - r, e + h\right], \mathbb{R}\right) : \|y\|_{[1-r,e+h]} \leq R \right\}.$$

It is clear D_R that is a bounded, closed and convex subset of $C\left(\left[1 - r, e + h\right], \mathbb{R}\right)$. Let A and B the two operators defined on D_R by

$$\left(Ay\right)\left(t\right) = \begin{cases} 0, \text{ if } t \in \left[1 - r, 1\right], \\ -\int_1^e G\left(t, s\right) g\left(s\right) \dfrac{ds}{s}, \text{ if } t \in J, \\ 0, \text{ if } t \in \left[e, e + h\right], \end{cases}$$

and

$$\left(By\right)\left(t\right) = \begin{cases} \varkappa\left(t\right) - \left(H_1 y\right)\left(t\right), \text{ if } t \in \left[1 - r, 1\right], \\ \left(1 - \left(\log t\right)^{\alpha - 1}\right)\left(\varkappa\left(1\right) - \left(H_1 y\right)\left(1\right)\right) \\ + \left(\log t\right)^{\alpha - 1}\left(\psi\left(e\right) - \left(H_2 y\right)\left(e\right)\right), \text{ if } t \in J, \\ \psi\left(t\right) - \left(H_2 y\right)\left(t\right), \text{ if } t \in \left[e, e + h\right]. \end{cases}$$

Therefore, the existence of a solution of (1.1) is equivalent to that the operator $A + B$ has a fixed point in D_R. The proof is divided into three steps.

Step 1. We prove that $Ax + By \in D_R$ for all $x, y \in D_R$. If $t \in \left[1 - r, 1\right]$, then

$$\begin{aligned} \left|\left(Ax\right)\left(t\right) + \left(By\right)\left(t\right)\right| &= \left|\varkappa\left(t\right) - H_1\left(y\right)\left(t\right)\right| \\ &\leq \|\varkappa\|_{[1-r,1]} + \|H_1 y\|_{[1-r,1]} \\ &\leq \|\varkappa\|_{[1-r,1]} + M_{H_1} \leq R, \end{aligned}$$

and if $t \in \left[e, e + h\right]$, then

$$\begin{aligned} \left|\left(Ax\right)\left(t\right) + \left(By\right)\left(t\right)\right| &= \left|\psi\left(t\right) - H_2\left(y\right)\left(t\right)\right| \\ &\leq \|\psi\|_{[e,e+h]} + \|H_2 y\|_{[e,e+h]} \\ &\leq \|\psi\|_{[e,e+h]} + M_{H_2} \leq R. \end{aligned}$$

If also $t \in J$,

$$\begin{aligned} &\left|\left(Ax\right)\left(t\right) + \left(By\right)\left(t\right)\right| \\ &= \Bigg|\left(1 - \left(\log t\right)^{\alpha - 1}\right)\left(\varkappa\left(1\right) - \left(H_1 y\right)\left(1\right)\right) \\ &+ \left(\log t\right)^{\alpha - 1}\left(\psi\left(e\right) - \left(H_2 y\right)\left(e\right)\right) - \int_1^e G\left(t, s\right) \frac{g\left(s\right)}{s} ds\Bigg| \\ &\leq \left(1 - \left(\log t\right)^{\alpha - 1}\right)\left|\varkappa\left(1\right) - \left(H_1 y\right)\left(1\right)\right| \\ &+ \left(\log t\right)^{\alpha - 1}\left|\psi\left(e\right) - \left(H_2 y\right)\left(e\right)\right| + \int_1^e \left|G\left(t, s\right)\right|\left|g\left(s\right)\right| \frac{ds}{s} \\ &\leq \left(1 - \left(\log t\right)^{\alpha - 1}\right)\left(\left|\varkappa\left(1\right)\right| + \left|\left(H_1 y\right)\left(1\right)\right|\right) \\ &+ \left(\log t\right)^{\alpha - 1}\left(\left|\psi\left(e\right)\right| + \left|\left(H_2 y\right)\left(e\right)\right|\right) + \int_1^e \left|G\left(t, s\right)\right|\left|g\left(s\right)\right| \frac{ds}{s}, \end{aligned}$$

where $g\in C(J,\mathbb{R})$ is such that

$$g(t)=f(t,y_t,g(t)).$$

By (H5), we have for each $t\in J$,

$$\begin{aligned}|g(t)|&=|f(t,y_t,g(t))|\\&\le p(t)+q(t)\|y_t\|_{[-r,h]}+r(t)|g(t)|\\&\le p(t)+q(t)\|y\|_{[1-r,e+h]}+r(t)|g(t)|\\&\le\frac{p^*+q^*R}{1-r^*}.\end{aligned}$$

Therefore,

$$\begin{aligned}&|(Ax)(t)+(By)(t)|\\&\le\left(1-(\log t)^{\alpha-1}\right)\left(|\varkappa(1)|+\|H_1y\|_{[1-r,1]}\right)\\&+(\log t)^{\alpha-1}\left(|\psi(e)|+\|H_2y\|_{[e,e+h]}\right)+\frac{2(p^*+q^*R)}{(1-r^*)\Gamma(\alpha+1)}\\&\le\|\varkappa\|_{[1-r,1]}+M_{H_1}+\|\psi\|_{[e,e+h]}+M_{H_2}+\frac{2(p^*+q^*R)}{(1-r^*)\Gamma(\alpha+1)}\\&\le R.\end{aligned}$$

Consequently,

$$Ax+By\in D_R.$$

Step 2. We show B is a contraction mapping. By (H3), if $t\in[1-r,1]$, we have

$$\begin{aligned}|(By_1)(t)-(By_2)(t)|&=|\varkappa(t)-(H_1y_1)(t)-\varkappa(t)+(H_1y_2)(t)|\\&\le\|H_1y_1-H_1y_2\|_{[1-r,1]}\\&\le K_3\|y_1-y_2\|_{[1-r,e+h]},\end{aligned}$$

and if $t\in[e,e+h]$, then

$$\begin{aligned}|(By_1)(t)-(By_2)(t)|&=|\psi(t)-(H_2y_1)(t)-\psi(t)+(H_2y_2)(t)|\\&\le\|H_2y_1-H_2y_2\|_{[e,e+h]}\\&\le K_4\|y_1-y_2\|_{[1-r,e+h]}.\end{aligned}$$

If also $t\in J$,

$$\begin{aligned}&|(By_1)(t)-(By_2)(t)|\\&=\Big|\left(1-(\log t)^{\alpha-1}\right)(\varkappa(1)-(H_1y_1)(1))+(\log t)^{\alpha-1}(\psi(e)-(H_2y_1)(e))\\&-\left(1-(\log t)^{\alpha-1}\right)(\varkappa(1)-(H_1y_2)(1))-(\log t)^{\alpha-1}(\psi(e)-(H_2y_2)(e))\Big|\\&\le\left(1-(\log t)^{\alpha-1}\right)|(H_1y_1)(1)-(H_1y_2)(1)|+(\log t)^{\alpha-1}|(H_2y_1)(e)-(H_2y_2)(e)|\\&\le\left(1-(\log t)^{\alpha-1}\right)\|H_1y_1-H_1y_2\|_{[1-r,1]}+(\log t)^{\alpha-1}\|H_2y_1-H_2y_2\|_{[e,e+h]}\\&\le\left(1-(\log t)^{\alpha-1}\right)K_3\|y_1-y_2\|_{[1-r,e+h]}+(\log t)^{\alpha-1}K_4\|y_1-y_2\|_{[1-r,e+h]}\\&\le\left((K_4-K_3)(\log t)^{\alpha-1}+K_3\right)\|y_1-y_2\|_{[1-r,e+h]}.\end{aligned}$$

Thus,

$$\|By_1 - By_2\|_{[1-r,e+h]} \leq \delta \|y_1 - y_2\|_{[1-r,e+h]}.$$

Consequently, B is a contraction mapping.

Step 3. We prove that A is continuous and compact.

Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $C([1-r,e+h],\mathbb{R})$. If $t \in [1-r,1]$ or $t \in [e,e+h]$, then

$$|(Au_n)(t) - (Au)(t)| = 0.$$

For $t \in J$, we have

$$|(Au_n)(t) - (Au)(t)| \leq \int_1^e |G(t,s)| |g_n(s) - g(s)| \frac{ds}{s}, \tag{3.9}$$

where $g_n, g \in C(J,\mathbb{R})$ are such that

$$g_n(t) = f(t, u_{nt}, g_n(t)),$$

and

$$g(t) = f(t, u_t, g(t)).$$

By (H2), we have

$$\begin{aligned} |g_n(t) - g(t)| &= |f(t, u_{nt}, g_n(t)) - f(t, u_t, g(t))| \\ &\leq K_1 \|u_{nt} - u_t\|_{[-r,h]} + K_2 |g_n(t) - g(t)|, \end{aligned}$$

Then

$$|g_n(t) - g(t)| \leq \frac{K_1}{1-K_2} \|u_{nt} - u_t\|_{[-r,h]}.$$

Since $u_n \to u$, we get $g_n(t) \to g(t)$ as $n \to \infty$ for each $t \in J$. Now, let $\varepsilon > 0$ be such that, for each $t \in J$, we have $|g_n(t)| \leq \varepsilon$ and $|g(t)| \leq \varepsilon$. Then, we have

$$\begin{aligned} |G(t,s)| |g_n(s) - g(s)| &\leq |G(t,s)| [|g_n(s)| + |g(s)|] \\ &\leq 2\varepsilon |G(t,s)|. \end{aligned}$$

For each $t \in J$, the function $s \to 2\varepsilon |G(t,s)|$ is integrable on J. Then the Lebesgue dominated convergence theorem and (3.9) imply that

$$|(Au_n)(t) - (Au)(t)| \to 0 \text{ as } n \to \infty.$$

Hence

$$\|Au_n - Au\|_{[1-r,e+h]} \to 0 \text{ as } n \to \infty.$$

Consequently, A is continuous.

Now, we need to prove that $A(D_R)$ is uniformly bounded. Let $y \in D_R$. We observe that

$$\begin{aligned} |(Ay)(t)| &\leq \int_1^e |G(t,s)| |g(s)| \frac{ds}{s} \\ &\leq \frac{2(p^* + q^* R)}{(1-r^*)\Gamma(\alpha+1)} \leq R. \end{aligned}$$

Thus,

$$\|Ay\|_{[1-r,e+h]} \leq R.$$

This shows that $A(D_R)$ is uniformly bounded.

Next, we show that $A(D_R)$ is relatively compact. Let $t_1, t_2 \in [1,e]$, $t_1 < t_2$, and let $y \in D_R$. Then

$$\begin{aligned} |(Ay)(t_2) - (Ay)(t_1)| &\leq \int_1^e |G(t_2,s) - G(t_1,s)| \, |g(s)| \frac{ds}{s} \\ &\leq \frac{p^* + q^* R}{1 - r^*} \int_1^e |G(t_2,s) - G(t_1,s)| \frac{ds}{s}. \end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero. The equicontinuity for the other cases is obvious. By the Arezela-Ascoli theorem, $A(D_R)$ is relatively compact. Thus, A is compact.

Therefore, all the hypothesis of the Krasnoselskii fixed point theorem are satisfied and consequently $A + B$ has a fixed point in D_R. Then, the problem (1.1) has at least one solution on J.

4 Examples

Example 4.1. Consider the following implicit fractional differential equation

$$\begin{cases} D^{\frac{5}{3}} y(t) = \frac{\log(2+t)e^{-1}}{1+|y_t|} - \frac{\left|D^{\frac{5}{3}} y(t)\right|}{e^{t^2+1}\left(1+\left|D^{\frac{5}{3}} y(t)\right|\right)}, \text{ for each } t \in [1,e], \\ y(t) + \frac{|y(t)|}{e^{3-t}(1+|y(t)|)} = \varkappa(t), \; t \in [0,1], \\ y(t) + \frac{\sin(y(t))+1}{100t} = \psi(t), \; t \in [e, e+2], \end{cases} \tag{4.1}$$

where $\varkappa \in C([0,1], \mathbb{R})$ and $\psi \in C([e,e+2], \mathbb{R})$. Set

$$\begin{aligned} f(t,u,v) &= \frac{\log(2+t)e^{-1}}{1+|u|} - \frac{|v|}{e^{t^2+2}(1+|v|)}, \; t \in [1,e], \; u \in C([-1,2], \mathbb{R}), \; v \in \mathbb{R}, \\ (H_1 y)(t) &= \frac{|y(t)|}{e^{3-t}(1+|y(t)|)}, \; t \in [0,1], \; y \in C([0,e+2], \mathbb{R}), \\ (H_2 y)(t) &= \frac{\sin(y(t))+1}{100t}, \; t \in [e,e+2], \; y \in C([0,e+2], \mathbb{R}). \end{aligned}$$

Let $u, \tilde{u} \in C([-1,2], \mathbb{R})$, $v, \tilde{v} \in \mathbb{R}$, $t \in [1,e]$, then we have

$$\begin{aligned} &|f(t,u,v) - f(t,\tilde{u},\tilde{v})| \\ &\leq \log(2+e)e^{-1} \|u - \tilde{u}\|_{[-1,2]} + \frac{1}{e^3} |v - \tilde{v}|. \end{aligned}$$

and let $y_1, y_2 \in C([0,e+2], \mathbb{R})$, we have, if $t \in [0,1]$,

$$|(H_1 y_1)(t) - (H_1 y_2)(t)| \leq \frac{1}{e^{3-t}} \|y_1 - y_2\|_{[0,e+2]},$$

then

$$\|H_1 y_1 - H_1 y_2\|_{[0,1]} \leq \frac{1}{e^2} \|y_1 - y_2\|_{[0,e+2]},$$

and if $t \in [e, e+2]$,

$$|(H_2y_1)(t)-(H_2y_2)(t)|\leq\frac{1}{100t}\|y_1-y_2\|_{[0,e+2]},$$

then

$$\|H_2y_1-H_2y_2\|_{[e,e+2]}\leq\frac{1}{100e}\|y_1-y_2\|_{[0,e+2]}.$$

Denote $\alpha=\frac{5}{3}$, $K_1=\log(2+e)e^{-1}>0$, $0<K_2=\frac{1}{e^3}<1$, $0<K_3=\frac{1}{e^2}<1$, $0<K_4=\frac{1}{100e}<1$, $\delta=\max\left\{\frac{1}{e^2},\frac{1}{100e}\right\}=\frac{1}{e^2}$. Thus

$$\delta+\frac{2K_1}{(1-K_2)\Gamma(\alpha+1)}=\frac{1}{e^2}+\frac{2\log(2+e)e^{-1}}{\left(1-\frac{1}{e^3}\right)\Gamma\left(\frac{5}{3}+1\right)}=0.9337<1.$$

Now, all assumptions in Theorem 3.1 are satisfied, then the problem (4.1) has a unique solution.

Example 4.2. Consider the following implicit fractional differential equation

$$\begin{cases} D^{\frac{5}{3}}y(t)=\frac{1}{4e^t}+\frac{1}{5e^t}\left(|y_t|+\left|D^{\frac{5}{3}}y(t)\right|\right), \text{ for each } t\in[1,e], \\ y(t)+\frac{\cos(y(t))e^t}{e^{-t}+8}=\varkappa(t),\ t\in[0,1], \\ y(t)+\frac{6}{(t+9)^3}\frac{|y(t)|}{1+|y(t)|}=\psi(t),\ t\in[e,e+2], \end{cases} \tag{4.2}$$

where $\varkappa\in C([0,1],\mathbb{R})$ and $\psi\in C([e,e+2],\mathbb{R})$. Set

$$f(t,u,v)=\frac{1}{4e^t}+\frac{1}{5e^t}(|u|+|v|),\ t\in[1,e],\ u\in C([-1,2],\mathbb{R}),\ v\in\mathbb{R}.$$

Clearly, the function f is continuous and set

$$(H_1y)(t)=\frac{\cos(y(t))e^t}{e^{-t}+8},\ t\in[0,1],$$
$$(H_2y)(t)=\frac{6}{(t+9)^3}\frac{|y(t)|}{1+|y(t)|},\ t\in[e,e+2],$$

for any $y\in C([0,e+2],\mathbb{R})$. Let $u,\tilde{u}\in C([-1,2],\mathbb{R})$, $v,\tilde{v}\in\mathbb{R}$, $t\in[1,e]$, then we have

$$|f(t,u,v)-f(t,\tilde{u},\tilde{v})|\leq\frac{1}{5}\left(\|u-\tilde{u}\|_{[-1,2]}+|v-\tilde{v}|\right),$$

and let $y_1,y_2\in C([0,e+2],\mathbb{R})$, we have if $t\in[0,1]$,

$$|(H_1y_1)(t)-(H_1y_2)(t)|\leq\frac{e^t}{e^{-t}+8}|y_1(t)-y_2(t)|,$$

then

$$\|H_1y_1-H_1y_2\|_{[0,1]}\leq\frac{e^1}{e^{-1}+8}\|y_1-y_2\|_{[0,e+2]},$$

and if $t\in[e,e+2]$,

$$|(H_2y_1)(t)-(H_2y_2)(t)|\leq\frac{6}{(t+9)^3}\|y_1-y_2\|_{[0,e+2]},$$

then

$$\|H_2y_1 - H_2y_2\|_{[e,e+2]} \le \frac{6}{(e+9)^3}\|y_1 - y_2\|_{[0,e+2]}.$$

Denote $\alpha = \frac{5}{3}$, $K_1 = K_2 = \frac{1}{5}$, $K_3 = \frac{e^1}{e^{-1}+8} < 1$, $K_4 = \frac{6}{(e+9)^3} < 1$.

We have, for each $t \in [1,e]$

$$|f(t,u,v)| \le \frac{1}{4e^t} + \frac{1}{5e^t}\left(\|u\|_{[-1,2]} + |v|\right).$$

Thus condition (H4) is satisfied with

$$p(t) = \frac{1}{4e^t},\ q(t) = \frac{1}{5e^t},\ r(t) = \frac{1}{5e^t}.$$

Then we have

$$q^* = \frac{1}{5e},\ r^* = \frac{1}{5e} < 1.$$

Thus condition

$$\frac{2q^*}{(1-r^*)\Gamma(\alpha+1)} = \frac{2}{5e\left(1-\frac{1}{5e}\right)\Gamma\left(\frac{3}{5}+1\right)} = 0.17777 < 1,$$

is satisfied. Hence by Theorem 3.2, the problem (4.2) has at least one solution.

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