SOME FIXED POINT RESULTS FOR NONEXPANSIVE MAPPINGS IN BANACH SPACES

NAJEH REDJEL AND ABDELKADER DEHICI

Abstract. In this paper, we establish some fixed point results for nonexpansive mappings defined on a finite intersection of closed, bounded and convex subsets in Banach spaces. In particular, these results can be applied to obtain the same contribution for C_λ-mappings.

1. Introduction

In Banach spaces, the investigation of the existence of fixed points for nonexpansive mappings defined on weakly compact, convex subsets is an important area in nonlinear functional analysis which was developped since the contributions of F. E. Browder, D. Göhde and W. A. Kirk (see [4, 6, 11]) where the role played by the geometry of Banach spaces in this direction was discovered. The notion of normal structure is the first tool introduced to study the existence of fixed points for nonexpansive mappings. Recall that compact and convex subsets of an arbitrary Banach spaces have normal structure (see [8], p. 39). However, it is not the case of weakly compact subsets in Banach spaces setting. The first example of a convex and weakly compact subset which does not have the normal structure is established by D. Alspach (1981) in the Lebesgue space $L^1([0,1])$. This example is given by the set

$$\widetilde{K} = \left\{ f \in L^1([0,1]), \int_0^1 f(t)dt = 1,\ 0 \leq f \leq 2 \ \ a.e. \right\}$$

The author constructed a selfmapping nonexpansive T (more precisely, an isometry) on $\widetilde{K}$ without fixed points. Alspach's solved a difficult problem remained open for a long time (for more details, see [1]).

We say that a Banach space X has the weak fixed point property (in abbreviation, wfpp) if for each weakly compact, convex subset K of X, every nonexpansive mapping $T : K \longrightarrow K$, has at least a fixed point. Thus Alspach's example enable us to assert that not all Banach spaces have wfpp. The uniformly convex Banach spaces have wfpp, indeed, it was proved that, bounded and convex subsets of such reflexive spaces have normal structure which gives as an immediate consequence that Hilbert spaces and $L^p([0,1]), 1 < p < \infty$ spaces have wfpp. But the problem wether every reflexive Banach space has or not wfpp is still open. However, reflexive closed subspaces of $L^1([0,1])$ have wfpp (see [22]), which enable us to ask if this is true for every reflexive lattice Banach spaces. Furthermore, If in a Banach space

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X, the weak compactness is equivalent to the compactness (such spaces are called having Schur property) then it is easy to assert that these spaces have wfpp since in this case the existence of fixed points is an immediate consequence of Schauder's fixed point theorem.

One of the most important problems associated to wfpp is related to Banach spaces with unconditional basis, which was solved partially by P. K. Lin (see [20]) who proved by using the ultraproduct's techniques that every Banach space X with an unconditional basis has wfpp provided that its unconditional basic constant is less than $\dfrac{\sqrt{33}-3}{2}$. But we dont't know the answer when the unconditional basic constant is greater than or equal $\dfrac{\sqrt{33}-3}{2}$. On this subject, we quote for example (see [2, 20, 21]) and the references therein.

In this paper, inspired by some ideas given in [20], we show the existence of fixed points of nonexpansive mappings defined on finite intersection of a closed, bounded and convex subsets of an arbitrary Banach spaces whenever the sum of their diameters or their maximum satisfy convenable estimations, provided that there exist bounded linear operators for which the norm and some associated norms are less than or equal to 1 satisfying that their null spaces have a nonvoid intersection with a retraction of the correspond subsets by the union of the others, we prove in particular that these results hold for the case of C_λ-mappings.

2. Preliminaries

In this section, we give some notions and notations which are used in the sequel.

Definition 2.1. Let K be a nonempty subset of a Banach space X and let $T : K \longrightarrow K$ be a selfmapping. A sequence (x_n) in K is called an approximate fixed point sequence for T if

$$\lim_{n \longrightarrow +\infty} \|x_n - Tx_n\| = 0.$$

Definition 2.2. Let K be a nonempty subset of a Banach space X. A mapping $T : K \longrightarrow K$ is said to be nonexpansive if for all $x, y \in K$, we have

$$\|Tx - Ty\| \leq \|x - y\|.$$

Remark 2.3. The family of nonexpansive selfmappings on a convex subset can be seen as a semigroup. This idea allows many authors to study common fixed points for semitopological actions on convex subsets of Banach spaces extending a well know results in this direction. For a good reading on this setting, we refer the contributions of Professor A. T. Lau and his collaborators [12, 13, 14, 15, 16, 17, 18, 19].

Let us give now the following preparatory lemma.

Lemma 2.4. (see [10, 20]) Let K be a nonempty closed convex bounded subset of a Banach space X. Let $T : K \longrightarrow K$ be nonexpansive. Then T posses an approximate fixed point sequence in K.

The following lemma has been established independently at the same time by L. A. Karlovitz [9] and K. Goebel ([7, 8]).

Lemma 2.5. Let K be a subset of a Banach space X which is minimal with respect to being nonempty, weakly compact, convex, and T-invariant for some nonexpansive mapping T, then for every approximate fixed point sequence $(x_n) \subseteq K$, we have

$$\text{for each } x \in K, \lim_{n \longrightarrow +\infty} \|x - x_n\| = diam(K).$$

The following corollary is an immediate consequence of Lemma 2.5

Corollary 2.6. Let K be a minimal weakly compact convex subset for a nonexpansive selfmapping T on K. Assume that $0 \in K$ and $diam(K) = 1$, then

$$\text{for all } \epsilon > 0, \text{ there exists } \delta > 0 \ \text{ such that } \ \|y\| > 1 - \epsilon \ \text{ whenever } \ \|Ty - y\| < \delta.$$

3. Main Results

Our first main result in this section is given by the following theorem.

Theorem 3.1. Let X be a Banach space and let $A_1, A_2, ..., A_{n+1}$ be a closed, bounded and convex subsets of X such that

$$\bigcap_{i=1}^{n+1} A_i \neq \emptyset \text{ with} \sum_{i=1}^{n+1} diam(A_i) < n.$$

Assume that there exists $j_0 \in \{1, 2, ..., n+1\}$ such that A_{j_0} is weakly compact and there exist bounded linear operators $S_1, S_2, ..., S_n$ on X satisfying the following assumptions

($\imath$): $\|S_i\| \leq 1 \quad \forall i = 1, ..., n$ and $\left\| nI - \sum_{i=1}^{n} S_i \right\| \leq 1$;

($\imath\imath$): $Ker(S_i) \bigcap \left(A_i \backslash \bigcup_{k \neq i}^{n+1} A_k \right) \neq \emptyset \quad \forall i = 1, ..., n$;

($\imath\imath\imath$): $Ker\left(nI - \sum_{i=1}^{n} S_i \right) \bigcap \left(A_{n+1} \backslash \bigcup_{i=1}^{n} A_i \right) \neq \emptyset$.

If each $A_i, i = 1, ..., n+1$ is invariant under a nonexpansive T, then T has a fixed point in $\bigcap_{i=1}^{n+1} A_i$

Proof. Assume that for all $i \in \{1, ..., n+1\}, A_i$ is invariant under T. Also, since A_{j_0} is weakly compact, then it is easy to observe that $\bigcap_{i=1}^{n+1} A_i$ is a nonempty (by assumption) weakly compact and convex subset of X which is invariant under T.

Assume that T has no fixed points in $\bigcap_{i=1}^{n+1} A_i$, then $\bigcap_{i=1}^{n+1} A_i$ must contain an approximate fixed point sequence. By assumptions ($\imath\imath$) and ($\imath\imath\imath$), there exist

$$z_i \in Ker(S_i) \bigcap \left(A_i \backslash \bigcup_{k \neq i}^{n+1} A_k \right) \quad (i = 1, ..., n)$$

and

$$z_{n+1} \in Ker \left(nI - \sum_{i=1}^{n} S_i \right) \bigcap \left(A_{n+1} \backslash \bigcup_{i=1}^{n} A_i \right).$$

It follows that for every $w_0 \in \bigcap_{i=1}^{n+1} A_i$. we have

$$\begin{aligned}
\|w_0\| =& \frac{1}{n} \left[\left\| \left(nI - \sum_{i=1}^{n} S_i \right)(w_0) + \sum_{i=1}^{n} S_i(w_0) \right\| \right] \\
\leq& \frac{1}{n} \left[\left\| \left(nI - \sum_{i=1}^{n} S_i \right)(w_0) \right\| + \sum_{i=1}^{n} \|S_i(w_0)\| \right] \\
\leq& \frac{1}{n} \left[\left\| \left(nI - \sum_{i=1}^{n} S_i \right)(w_0 - z_{n+1}) \right\| + \sum_{i=1}^{n} \|S_i(w_0 - z_i)\| \right] \\
\leq& \frac{1}{n} \left[\left\| nI - \sum_{i=1}^{n} S_i \right\| \|w_0 - z_{n+1}\| + \sum_{i=1}^{n} \|S_i\| \|w_0 - z_i\| \right] \\
\leq& \frac{1}{n} \left[diam(A_{n+1}) + \sum_{i=1}^{n} diam(A_i) \right] \\
\leq& \frac{1}{n} \sum_{i=1}^{n+1} diam(A_i) < 1,
\end{aligned}$$

which is a contradiction by Corollary 2.6.

Now, we are in position to prove the following theorem.

Theorem 3.2. Let X be a Banach space and let $(A_i)_{i=1}^{n+1} (n \geq 2)$ be a closed, bounded and convex subsets of X which are invariant under a nonexpansive mapping T. Assume that

$$\bigcap_{i=1}^{n+1} A_i \neq \emptyset,$$

and there exists $k_0 \in \{1, 2, ..., n+1\}$ such that A_{k_0} is weakly compact. If there exist bounded linear operators $(S_i)_{i=1}^{n}$ on X satisfying that

($\imath$): $\max_{1 \leq i \leq n} \|I - nS_i\| \leq \alpha_1 \;\; (\alpha_1 > 1)$ and $\left\| I - n \sum_{i=1}^{n} S_i \right\| \leq \alpha_2 \;\; (\alpha_2 > 0)$

($\imath\imath$): $Ker(S_i - I) \bigcap \left(A_i \backslash \bigcup_{k \neq i}^{n} A_k \right) \neq \emptyset \;\; \forall i = 1, ..., n$

($\imath\imath\imath$): $Ker \left(\sum_{i=1}^{n} S_i \right) \bigcap \left(A_{n+1} \backslash \bigcup_{i=1}^{n} A_i \right) \neq \emptyset.$

If

$$\max_{1\leq i\leq n} diam(A_i) < \frac{2}{\alpha_1+1},$$

and

$$diam(A_{n+1}) < \frac{n\left[2-(\alpha_1+1)\max\limits_{1\leq i\leq n} diam(A_i)\right]}{\alpha_2+1}.$$

Then T has a fixed point in $\bigcap\limits_{i=1}^{n+1} A_i$.

Proof. Assume that T has no fixed points, then $\bigcap\limits_{i=1}^{n+1} A_i$ has an invariant minimal weakly compact subset. Let $w_0 \in \bigcap\limits_{i=1}^{n+1} A_i$ and $x_i \in A_i (i = 1, 2, ..., n+1)$, without loss of generality, we assume that $\|w_0\| = 1$. By Hahn-Banach theorem, let $f_0 \in X^\star$ such that $f_0(w_0) = 1 = \|f_0\|$. Hence

$$1 - f_0(x_i) = f_0(w_0 - x_i) \leq \|f_0\|\|w_0 - x_i\| \leq diam(A_i). \tag{1}$$

So

$$1 - diam(A_i) \leq f_0(x_i), \qquad i = 1, 2, ..., n+1.$$

Putting

$$\alpha_0 = f_0\left[\left(I - \sum_{i=1}^{n} S_i\right)(w_0)\right].$$

Then

$$\begin{aligned} 1-\alpha_0 =& f_0(w_0) - f_0\left[\left(I - \sum_{i=1}^{n} S_i\right)(w_0)\right] \\ =& f_0\left(\left(\sum_{i=1}^{n} S_i\right)(w_0)\right) \\ =& f_0(S_1(w_0)) + f_0(S_2(w_0)) + ... + f_0(S_n(w_0)). \end{aligned} \tag{2}$$

Hence, there exists necessarily $l_0 \in \{1, 2, ..., n\}$ such that

$$f_0(S_{l_0}(w_0)) \leq \frac{1-\alpha_0}{n}.$$

By assumption $(\imath\imath\imath)$, there exists

$$x \in Ker\left(\sum_{i=1}^{n} S_i\right) \bigcap \left(A_{n+1} \setminus \bigcup_{i=1}^{n} A_i\right).$$

Combining (1) and (2), we get

$$n(1-\alpha_0) - diam(A_{n+1}) \leq n[f_0(S_1(w_0)) + ... + f_0(S_n(w_0))] - f_0(w_0 - x).$$

The linearity of f_0 and assumption $(\imath)$ imply that

$$\begin{aligned} n(1-\alpha_0) - diam(A_{n+1}) \leq & f_0[(nS_1 + ... + nS_n)(w_0)] - f_0(w_0 - x) \\ = & f_0\left[\left(n\sum_{i=1}^{n} S_i - I\right)(w_0 - x)\right] \\ \leq & \|f_0\| \left\| I - n\sum_{i=1}^{n} S_i \right\| \|w_0 - x\| \\ \leq & \alpha_2 \;\; diam(A_{n+1}). \end{aligned}$$

Hence

$$n(1-\alpha_0) \leq (\alpha_2 + 1) diam(A_{n+1}). \tag{3}$$

Furthermore, we have

$$\alpha_0 + (1 - diam(A_{l_0})) = (1 - diam(A_{l_0})) + 1 - (1 - \alpha_0)$$

On the other hand, for $z \in \left(A_{l_0} \setminus \bigcup_{k \neq l_0}^{n} A_k\right) \bigcap Ker(S_{l_0} - I)$, we infer

$$\begin{aligned} \alpha_0 + (1 - diam(A_{l_0})) = & 1 - diam(A_{l_0}) + 1 - (1 - \alpha_0) \\ \leq & f_0(w_0) + f_0(z) - n f_0(S_{l_0}(w_0)) \\ \leq & f_0(w_0 - z) + n f_0(z) - n f_0(S_{l_0}(w_0)) \\ = & f_0(w_0 - z) + n f_0(S_{l_0}(z)) - n f_0(S_{l_0}(w_0)) \\ = & f_0(w_0 - z) + n f_0\left[S_{l_0}(z - w_0)\right] \\ = & f_0[(I - nS_{l_0})(w_0 - z)] \\ \leq & \|f_0\| \|I - nS_{l_0}\| \|w_0 - z\| \\ \leq & \alpha_1 diam(A_{l_0}), \end{aligned}$$

it follows that

$$(\alpha_0 + 1) \leq (\alpha_1 + 1) diam(A_{l_0}). \tag{4}$$

Multiplying this inequality by n, we obtain

$$n(\alpha_0 + 1) \leq n(\alpha_1 + 1) diam(A_{l_0}). \tag{5}$$

Thus from inequalities (3) and (5), we get

$$n\left[1 - (\alpha_1 + 1) diam(A_{l_0}) + 1\right] \leq (\alpha_2 + 1) diam(A_{n+1}).$$

Thus

$$n\left[2 - (\alpha_1 + 1) \max_{1 \leq i \leq n} diam(A_i)\right] \leq (\alpha_2 + 1) diam(A_{n+1}).$$

Consequently

$$\frac{n\left[2 - (\alpha_1 + 1) \max_{1 \leq i \leq n} diam(A_i)\right]}{\alpha_2 + 1} \leq diam(A_{n+1}),$$

which is a contradiction.

Corollary 3.3. Let X be a reflexive Banach space and let $x_1, x_2, ..., x_{n+1} \in X$ such that $x_i \neq x_j$ for all $i, j = 1, 2, ..., n+1, i \neq j$. Assume that

$$0 < r < \frac{n}{2(n+1)}.$$

If there exist bounded linear operators $(S_i)_{i=1}^n$ on X satisfying assumptions $(\imath), (\imath\imath)$ and $(\imath\imath\imath)$ of Theorem 3.1 and if each $\overline{B}(x_i, r)$ is invariant by a nonexpansive mapping T for each $i = 1, ..., n+1$ with

$$\bigcap_{i=1}^{n+1} \overline{B}(x_i, r) \neq \emptyset.$$

Then T has a fixed point in $\bigcap_{i=1}^{n+1} \overline{B}(x_i, r)$.

Proof. In this case, for all $i \in \{1, 2, ..., n+1\}$, each $\overline{B}(x_i, r)$ is weakly compact since X is reflexive. Moreover, here we have $diam(\overline{B}(x_i, r)) = 2r$ for all $i \in \{1, 2, ..., n+1\}$. Now, the result follows from Theorem 3.1.

Also, as an immediate consequence of Theorem 3.2, we have

Corollary 3.4. Let X be a reflexive Banach space and let $x_1, x_2, ..., x_{n+1} \in X$ $(n \geq 2)$ such that $x_i \neq x_j$ for all $i, j = 1, 2, ..., n+1, i \neq j$. Assume that

$$r < \frac{n}{(\alpha_2 + 1) + n(\alpha_1 + 1)},$$

where $\alpha_1 > 1$ and $\alpha_2 > 0$.

Assume that there exist bounded linear operators $(S_i)_{i=1}^n$ on X satisfying $(\imath), (\imath\imath)$ and $(\imath\imath\imath)$ of Theorem 3.2. If each $\overline{B}(x_i, r)$ is invariant under a nonexpansive mapping T for each $i = 1, 2, ..., n+1$ with

$$\bigcap_{i=1}^{n+1} \overline{B}(x_i, r) \neq \emptyset.$$

Then T has a fixed point in $\bigcap_{i=1}^{n+1} \overline{B}(x_i, r)$.

In 2008, T. Suzuki [23] has introduced C_λ-mappings as an extension of nonexpansive mappings.

Definition 3.5. Let K be a nonempty subset of a Banach space X. A selfmapping $T : K \longrightarrow K$ is said to be C_λ-mapping if for some $\lambda \in (0, 1)$ and all $x, y \in K$,

$$\lambda \|x - Tx\| \leq \|x - y\| \Longrightarrow \|Tx - Ty\| \leq \|x - y\|.$$

If $\lambda = \frac{1}{2}$, T is said to be C-mapping.

Remark 3.6. If we denote $\widetilde{S_x}$ the set defined by

$$\widetilde{S_x} = \{y \in K : \lambda\|x - Tx\| \leq \|x - y\|\}.$$

We observe that $Tx \in \widetilde{S_x}$ and thus $\widetilde{S_x}$ is nonempty.

Remark 3.7. It is easy to observe that every nonexpansive mapping is C_λ-mapping for every $\lambda \in (0, 1)$ but the converse is not true as the following example shows:

Example 3.8. Let $T : [0, 3] \longrightarrow [0, 3]$ defined by

$$Tx = \begin{cases} 0 & \text{if } x \neq 3 \\ 1 & \text{if } x = 3. \end{cases}$$

Then T is a C-mapping on $[0, 3]$ but T fails to be nonexpansive since it is not continuous at $x_0 = 3$ (for more details, see [5]).

Remark 3.9. Let K be a nonempty bounded and convex subset of a Banach space X and $\lambda \in (0, 1)$. Assume that T is a continuous C_λ-selfmapping on K. Then T has an approximate fixed point sequence $(x_n)_n \subset K$ (see [3]).

Lemma 3.10. (see [3], Lemma 2.4) Let T be continuous C_λ-mapping ($\lambda \in (0, 1)$) defined on a minimal weakly compact and convex set K and let $(x_n)_n \subset K$ be an approximate fixed point sequence for T. Then there exists $\rho > 0$ such that for all $x \in K$, we have

$$\lim_{n \longrightarrow +\infty} \|x_n - x\| = \rho.$$

and if $\lambda = \frac{1}{2}$, then the continuity assumption can be dropped.

Remark 3.11. By combining Remark 3.9 and Lemma 3.10, we conclude that Theorems 3.1, 3.2 and Corollaries 3.3, 3.4 hold for the case of continuous C_λ-mappings ($\lambda \in (0, 1)$).

4. Conclusion

In this paper, fixed point results are established for nonexpansive mappings defined on a finite intersection of bounded, closed and convex subsets of an arbitrary Banach space X. These results are obtained independently of the geometrical properties of theses subsets but they are related directly to their diameters and the structure of the Banach algebra $\mathcal{L}(X)$ of bounded linear operators on X.

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Competing interests

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Authors contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the manuscript.

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