On the structure of Riesz and polynomially Riesz operators on the Schlumprecht space S and the space of Gowers-Maurey X_{GM}

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Abstract

In this paper, we prove that Riesz operators on the arbitrary distortable Schlumprecht's Banach space S and the hereditarily indecomposable Banach space of Gowers-Maurey X_{GM} have West decomposition. By using these results, the structure of polynomially Riesz operators on these spaces is established.

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1 Introduction and Preliminaries

In (1966), T. West [23] showed that if X is a Hilbert space, then every Riesz operator R can be written under the form $R = K + Q$ where K is a compact operator and Q is quasinilpotent (its spectrum is reduced to the set $\{0\}$), twenty years after, K. Davidson and D. Herrero (1986) [6] proved this decomposition for such operators on $l_p(1 \leq p < \infty)$ spaces and c_0 or more generally on the spaces having the finite dimensional p-Block decomposition (FDPBD) written as an infinite direct sum of finite dimensional spaces. In (1988), H. Zhong [25] gave an affirmative answer to this problem if $X = L_p(\mu)$ $(1 < p < \infty)$ and extended his results to the case of B-convex Banach space. Finally in (1995), the last author proved that the result holds for the case of local strong subprojective Banach spaces, in particular, he studied it's validity in the case of Tsirelson space. Since every Banach space X has a Rademacher's type $p(X) \in [1, 2]$ and the fact that $1 < p(X)$ is equivalent to the fact that X is a B-convex Banach space implies the necessity to investigate the West decomposition of Riesz operators on Banach spaces with type 1. For a good read on the subject concerning the problem of the West decomposition of Riesz operators, we can quote for example([6, 7, 12, 15, 21, 24, 25]) and the references therein.

In this work, based on the principal theorem of H. Zhong [24], we show that this decomposition is true on the arbitray distortable Banach space of Schlumprecht S and

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the space of Gowers-Maurey X_{GM} which have Rademacher's type 1. Afterwards, by using these results and inspired by those established in [13], we give a characterization of polynomially Riesz operators on Banach spaces S and X_{GM}. Finally, the work is achieved by some interesting comments and questions.

If X is a complex Banach space, we denote respectively by $\mathcal{L}(X), \mathcal{K}(X), \mathcal{R}(X)$ and $\mathcal{Q}(X)$, the space of bounded linear operators on X, the space of compact linear operators on X, the set of Riesz operators on X and the set of the quasinilpotent operators on X.

For $1 \leq p \leq \infty$ and an arbitrary integer $n \geq 1$, the space l_p^n designates the finite dimensional normed space $(\mathbb{R}^n, \|.\|_p)$ where $\|(a_i)_{i=1}^n\|_p = (\sum_{i=1}^{n} |a_i|^p)^{\frac{1}{p}}$ $(1 \leq p < \infty)$ and $\|(a_i)_{i=1}^n\|_\infty = \max_{1 \leq i \leq n} (|a_i|)$.

Definition 1.1 A Banach space X is said to be of Rademacher's type p if there exists a positive constant $M < \infty$ such that, for every choice of vectors $\{x_i\}_{i=1}^n$ in X, we have

$$(\int_0^1 (\| \sum_{i=1}^{n} r_i(t) x_i \|)^2 dt)^{\frac{1}{2}} \leq M (\sum_{i=1}^{n} \|x_i\|^p)^{\frac{1}{p}},$$

where $\{r_i\}_{i=1}^\infty$ denote the Rademacher functions.

***Remark* 1.1** It is well known that every Banach space X has a Rademacher's type $p(X) \in [1, 2]$. Hilbert spaces have Rademacher's type 2 and for the $L_r(\mu)$ spaces, we have $p(L_r(\mu)) = r$ if $1 \leq r \leq 2$ and $p(L_r(\mu)) = 2$ if $2 \leq r < \infty$. Also, if $r = \infty$, we have $p(L_\infty(\mu)) = 1$. For more details on the Rademacher's type of Banach spaces and other quantities corresponding to it, see ([19], [22]).

Definition 1.2 Let X be a Banach space and Y is a closed subspace of X. Y is said to be complemented in X (resp. c complemented in X) if there exists a bounded linear projection P from X onto Y (resp. $\|P\| \leq c$).

Definition 1.3 $(\imath)$ A sequence $(x_i)_1^\infty$ of elements in a Banach space X is called a Schauder basis of X if for every $x \in X$ there exists a unique sequence $(\lambda_i)_1^\infty$ such that $x = \sum_{i=1}^{\infty} \lambda_i x_i$.

$(\imath\imath)$ The basis projections $P_n : X \longrightarrow X$ defined by $P_n(\sum_{i=1}^{\infty} \lambda_i x_i) = \sum_{i=1}^{n} \lambda_i x_i$ are uniformly bounded and the number $K = \sup_n(\|P_n\|)$ is called the basis constant of $(x_i)_1^\infty$.

$(\imath\imath\imath)$ A sequence $(z_i)_1^\infty$ which is a basis of its closed linear span denoted by $[z_i]$ is called a basic sequence.

$(\imath v)$ Bases $(x_i)_1^\infty$ of a Banach space X and $(y_i)_1^\infty$ of a Banach space Y are said to be $c-$equivalent $(1 \leq c < \infty)$ if there exists an isomorphism $S : X \longrightarrow Y$ with $S(x_i) = y_i, i \in \mathbb{N}$ and $\|S\|.\|S^{-1}\| \leq c$. It is said that two basic sequences $(x_n)_1^\infty$ and $(y_n)_1^\infty$ are equivalent if they are c-equivalent for some $c \geq 1$.

Definition 1.4 Two Banach spaces X and Y are called c-isomorphic if there exists an isomorphism $T : X \longrightarrow Y$ such that $\|T\|\|T^{-1}\| \leq c$.

Theorem 1.1 (The principle of small perturbations, Theorem 4.7, p.41 in [3]): Let $(x_i)_n$ be a normalized basic sequence in a Banach space X with basis constant K and suppose that (y_i) is a sequence in X with $\sum_{i=1}^{\infty} \|x_i - y_i\| = \delta$.

($\imath$) If $2K\delta < 1$ then (y_i) is a basic sequence equivalent to (x_i).

($\imath\imath$) If $[x_i]$ is complemented by a bounded linear projection $P : X \longrightarrow X$ and if $8\delta K\|P\| < 1$, then $[y_i]$ is also complemented in X.

Corollary 1.1 Let X be a Banach space with a Schauder basis $\{e_i\}_i$. If Y is an infinite dimensional closed subspace of X, then Y contains an infinite dimensional closed subspace Z with a Schauder basis $\{b_i^Y\}_{i=1}^{\infty}$ which is equivalent to a block basic sequence $\{z_i^Y\}_{i=1}^{\infty}$ of $\{e_i\}$.

Definition 1.5 A Banach space X is said to be saturated by uniform copies of l_p^n $(1 \leq p \leq \infty)$ if there exists $C > 1$ such that for every closed infinite dimensional subspace Y of X and for all integer $n \geq 1$, Y contains a subspace which is C-isomorphic to l_p^n.

Remark **1.2** Let X be a complex Banach space and Y a closed subspace of X. It is easy to observe that if Y is c-complemented in X, then Y is c'-complemented for all $c' > c$. Also, if Z_1 and Z_2 are c-isomorphic closed subspaces of X then Z_1 and Z_2 are c'-isomorphic for all $c' > c$.

Definition 1.6 A Banach space X is said to be strong subprojective if every infinite-dimensional closed subspace Y of X, there exists a closed infinite-dimensional subspace $Y_0 \subseteq Y$ such that Y_0 is complemented in X and isomorphic to l_p $(1 \leq p < \infty)$ or c_0.

Example **1.1** The following Banach spaces are strong subprojective (see Proposition 2.4 in [9]).

1. The spaces l_p for $1 \leq p < \infty$ and c_0.
2. The James spaces J.
3. The Lorentz sequence spaces $d(w, p)$ for $1 \leq p < \infty$ and $w = (w_n)$ a non-increasing null sequence with $\sum_{n=1}^{\infty} w_n$ is divergent.
4. The Baernstein spaces B_p for $1 < p < \infty$.
5. L_p-spaces for $2 \leq p < \infty$.

6. The function spaces $L_p(0,\infty)\bigcap L_2(0,\infty)$-spaces for $1 \leq p \leq 2$.

7. The Lorentz spaces $\Lambda_{W,p}(0,1)$, $L_{p,q}(0,\infty)$ and $L_{p,q}(0,1)$ for $2 < p < \infty$ and $1 \leq q < \infty$.

8. The space of continuous functions $C(K)$ with K a scattered compact.

Definition 1.7 A Banach space X is said to be local strong subprojective if for any infinite-dimensional closed subspace Y of X, there exists a constant $c_Y \geq 1$ (depending on the subspace Y) such that for any infinite-dimensional subspace $Y_0 \subseteq Y$ and any integer $n \geq 1$, Y_0 contains an n-dimensional subspace which is c_Y-isomorphic to some $l_p^n (1 \leq p \leq \infty)$ and is c_Y-complemented in X. Moreover, if the constant c_Y is independent of the choice of Y, then X is called c-local strong subprojective.

***Remark* 1.3** It is easy to observe that every strong subprojective Banach space is local strong subprojective but the converse is in general false as the Tsirelson space shows (see [24]). Thus all Banach spaces given in Example 1.1 are local strong subprojective and by Theorem 2.1 of [24], West decomposition of Riesz operators is valid on them.

2 Main results

2.1 West decomposition of Riesz operators on S and X_{GM}

Our main result in this work is given by

Theorem 2.1 Assume that X is a Banach space with Schauder basis $\{e_i\}_{i=1}^{\infty}$ satisfying that there exists $C > 1$ and $1 \leq p \leq \infty$ such that for all block basis $\{z_k\}_{k=1}^{\infty}$ of $\{e_i\}_{i=1}^{\infty}$ and for all scalars $\{a_i\}_{i=1}^{n}$, we have

$$\frac{1}{C}\|(a_i)_{i=1}^n\|_p \leq \|\sum_{i=1}^n a_i z_i\| \leq C\|(a_i)_{i=1}^n\|_p \qquad (\star)$$

Thus X is saturated by uniform copies of l_p^n. Moreover, under the notations of Corollary 1.1, if the closed subspace $[z_k^Y]_{k=1}^{\infty}$ is complemented in X for all infinite dimensional closed subspace Y of X, then X is local subprojective space and hence for all $R \in \mathcal{R}(X)$, there exist $K \in \mathcal{K}(X)$ and $Q \in \mathcal{Q}(X)$ such that $R = K + Q$.

Proof. The fact that X is l_p^n saturated follows directly by combining $(\star)$ and Corollary 1.1. Let Y be an arbitrary infinite dimensional closed subspace of X. By assumption, the closed subspace $[z_k^Y]_{k=1}^{\infty}$ is complemented in X, thus there exists a bounded linear projection $P_Y : X \longrightarrow [z_k^Y]_{k=1}^{\infty}$. Afterwards, since $\{z_k^Y\}_{k=1}^{\infty}$ is equivalent to the basis $\{e_k^Y\}_{k=1}^{\infty}$, we obtain the existence of $\lambda \geq 1$ such that the subspaces $[z_k^Y]_{k=1}^{\infty}$ and $[e_k^Y]_{k=1}^{\infty}$ are λ-isomorphic. Now if n is a nonzero integer, the following factorisation

$$X \underbrace{\longrightarrow}_{P_Y} [z_k^Y]_{k=1}^{\infty} \underbrace{\longrightarrow}_{S} [e_k^Y]_{k=1}^{\infty} \underbrace{\longrightarrow}_{S^{-1}} [z_k^Y]_{k=1}^{\infty} \underbrace{\longrightarrow}_{P_r^n} [z_k^Y]_{k=1}^{n} \underbrace{\longrightarrow}_{A_Y} [e_k^Y]_{k=1}^{n}$$

where $S:[z_k^Y]_{k=1}^{\infty}\longrightarrow[e_k^Y]_{k=1}^{\infty}$ is an isomorphism such that $\|S\|.\|S^{-1}\|\leq\lambda$, $P_r^n:[z_k^Y]_{k=1}^{\infty}\longrightarrow[z_k^Y]_{k=1}^{n}$ is the canonical projection defined by $P_r^n(\sum_{i=1}^{\infty}a_iz_i^Y)=\sum_{i=1}^{n}a_iz_i^Y$ and $A_Y:[z_k^Y]_{k=1}^{n}\longrightarrow[e_k^Y]_{k=1}^{n}$ is an isomorphism for which $\|A_Y\|\leq C$. If we denote by $B_Y:X\longrightarrow[e_k^Y]_{k=1}^{n}$ the bounded linear operator $A_YP_r^nS^{-1}SP_Y$, it is easy to observe that $\|B_Y\|\leq C\lambda\|P_Y\|$, which gives that the n-dimensional subspace $[e_k^Y]_{k=1}^{n}$ is $C\lambda\|P_\lambda\|$ complemented in X. On the other hand, from $(\star)$ and the fact that the spaces $[z_k^Y]_{k=1}^{\infty}$ and $[e_k^Y]_{k=1}^{\infty}$ are λ-isomorphic shows that $[e_k^Y]_{k=1}^{n}$ and l_p^n are $C\lambda$ isomorphic. If $c_Y=\max\{C\lambda,C\lambda\|P_Y\|\}$, then following Remark 1.2, $[e_k^Y]_{k=1}^{n}$ is c_Y-isomorphic to l_p^n and is c_Y complemented in X which gives the desired result.

Proposition 2.1 Let X be a Banach space. Under the assumptions of Theorem 2.1, if moreover, the closed subspace $[e_k^Y]_{k=1}^{\infty}$ is complemented for all closed infinite dimensional subspace Y, then X is local strong subprojective for $c_Y=\max\{C\lambda,C\lambda\|P_Y\|\|\widetilde{P}_Y\|\}$ where $\widetilde{P}_Y:X\longrightarrow[e_k^Y]_{k=1}^{\infty}$ is a bounded linear projection. On the other hand if there exists $M>0$ such that $c_Y\leq M$ for all infinite dimensional closed subspace Y, then X is M-local strong subprojective space.

Proof. By the same idea given above, we infer that the finite dimensional subspaces $[e_k^Y]_{k=1}^{n}$ and l_p^n are $C\lambda$-isomorphic. Now, let be the following factorisation

$$X\underbrace{\longrightarrow}_{\widetilde{P}_Y}[e_k^Y]_{k=1}^{\infty}\underbrace{\longrightarrow}_{S^{-1}}[z_k^Y]_{k=1}^{\infty}\underbrace{\longrightarrow}_{S}[e_k^Y]_{k=1}^{\infty}\underbrace{\longrightarrow}_{\widetilde{\widetilde{P}}_Y}[z_k^Y]_{k=1}^{\infty}\underbrace{\longrightarrow}_{P_r^n}[z_k^Y]_{k=1}^{n}\underbrace{\longrightarrow}_{A_Y}[e_k^Y]_{k=1}^{n}$$

where $\widetilde{\widetilde{P}}_Y=P_{Y|[e_k^Y]_{k=1}^{\infty}}$ (the restriction of the bounded linear projection P_Y to the closed subspace $[e_k^Y]_{k=1}^{\infty}$) and let $B_Y=AP_r^n\widetilde{\widetilde{P}}_YSS^{-1}\widetilde{P}_Y$. Since $\|\widetilde{\widetilde{P}}_Y\|=\|P_{Y|[e_k^Y]_{k=1}^{\infty}}\|\leq\|P_Y\|$, we get $\|B_Y\|\leq C\lambda\|P_Y\|\|\widetilde{P}_Y\|$. The fact that if there exists $M>0$ such that $c_Y\leq M$ (independently of the choice of Y) implies that X is M-local strong subprojective follows from Remark 1.2.

***Example* 2.1** If X is the Tsirelson space which is of Rademacher's type 1, then $C=2,p=1$ and $M=216\|\widetilde{P}_Y\|$ where $\|\widetilde{P}_Y\|$ is independent of the choice of the subspace Y in this case (see Theorem 1.1 in [24]).

***Remark* 2.1** It is easy to observe that under the assumptions of Theorem 2.1, a sufficient condition for the subspace $[e_i^Y]_{i=1}^{\infty}$ to be complemented in X is that the two basic sequences $\{z_i^Y\}_{i=1}^{\infty}$ and $\{e_i^Y\}_{i=1}^{\infty}$ satisfy $(\imath\imath)$ of Theorem 1.1.

Let (e_n) be the standard vector basis of c_{00}, the space of eventually null sequences of scalars. The support of a vector $x=\sum_i x_ie_i$ in c_{00} is denoted by $suppx$ is defined by $suppx=\{i\in\mathbb{N}:x_i\neq0\}$ and the range of x is the interval of integers $ranx=[\min(suppx),\max(suppx)]$ or $\emptyset$ if $x=0$. If $x=\sum_i x_ie_i\in c_{00}$ and $E=[m,n]$ is an interval of integers, then Ex denotes the vector $x=\sum_{i=m}^{n}x_ie_i$. The norm of the arbitrary distortable Schlumprecht space S ([14, 20]) is defined by the implicit equation on c_{00}:

$$\|x\|_S = \|x\|_\infty \vee \sup_{n\geq 2, E_1<E_2.......<E_n} \frac{1}{f(n)} \sum_{k=1}^{n} \|E_k x\|_S,$$

where $f(x) = log_2(x+1)$ and $E_1, ..., E_n$ are successive intervals of integers.

The construction of the hereditarily indecomposable Banach space of Gowers-Maurey denoted by X_{GM} is more complicated, it's based on that of the space S, the general idea is to add a third term to the two terms given above as follows: Let $J \subset \mathbb{N}$ be a set such that if $m < n$ and $m, n \in J$, then $logloglogn \geq 4m^2$. Let us write J in increasing order as $\{j_1, j_2, ...\}$ and assume that $f(j_1) \geq 256$. We denote $K = \{j_1, j_3, j_5, ...\}$. Thus the norm of the space X_{GM} is defined by it's implicit formula:

$$\|x\|_{GM} = \|x\|_{c_0} \vee \sup_{n\geq 2, E_1<E_2.......<E_n} \frac{1}{f(n)} \sum_{k=1}^{n} \|E_k x\|_{GM}$$
$$\vee \sup\{|g(Ex)| : k \in K, g \in B_k^\star(X), E \subset \mathbb{N}\},$$

where $B_k^\star(X)$ is a set of some special functionals. For more details on the spaces S, X_{GM}, their construction and other properties, we quote for example ([1], [5], [11], [14], [16], [20]).

Theorem 2.2 Let X one of the Banach spaces S or X_{GM}. Then X is c-local strong subprojective and consequently Riesz operators on X have West decomposition.

Proof. By combining the results of [1] and [14], we deduce that X is saturated by uniform copies of l_∞^n (see also [4]). Now, let Y be a closed infinite dimensional subspace of X, thus there exists $C > 1$ such that for all integer $n \geq 1$, Y contains a finite dimensional subspace F_Y^n which is C-isomorphic to l_∞^n, this implies the existence of an isomorphism $S : F_Y^n \longrightarrow l_\infty^n$ such that $\|S\|.\|S^{-1}\| \leq C$. By Hahn-Banach theorem, we obtain the existence of bounded linear operator $T : X \longrightarrow l_\infty^n$ for which $\|T\| = \|S\|$ (for more details, see [3] Theorem 4.7, p. 41). Denote $B = S^{-1}T : X \longrightarrow F_Y^n$, then $\|B\| = \|S^{-1}T\| \leq \|T\|.\|S^{-1}\| = \|S\|.\|S^{-1}\| \leq C$ which proves that F_Y^n is C-complemented in X. Thus X is c-local strong subprojective space by taking $c_Y = C$ for all closed infinite dimensional subspace Y which is the desired result.

***Remark* 2.2** It is known that the set $\mathcal{R}(X_{GM})$ is nothing else but the ideal of strictly singular operators $\mathcal{S}(X_{GM})$ on this space (see [11]).

Corollary 2.1 Riesz operators on the spaces $S^\star$ and $X_{GM}^\star$ (the dual spaces of the Banach spaces S and X_{GM}) have West decomposition.

Proof. Let X be one of the Banach spaces S or X_{GM} and $R \in \mathcal{R}(X^\star)$. The reflexivity of Banach spaces S and X_{GM} (see [5] and [11]) shows that $R^\star \in \mathcal{R}(X)$, thus there exist $K \in \mathcal{K}(X)$ and $Q \in \mathcal{Q}(X)$ such that $R^\star = K + Q$. Again by duality, we get $R = K^\star + Q^\star$ where $K^\star \in \mathcal{K}(X^\star)$ and $Q \in \mathcal{Q}(X^\star)$ which is the desired result.

***Remark* 2.3** We don't know if the West decomposition is true on all complex Banach spaces which have type 1.

2.2 The case of Polynomially Riesz operators on S and X_{GM}

Let X be a Banach space and $T \in \mathcal{L}(X)$. We denote by $\sigma(T), \rho(T) = \mathbb{C}\backslash\sigma(T), Ker(T)$ and $R(T)$ the spectrum, the resolvent set, the null space and the range of T respectively. We say that T is a Fredholm operator if $dim(Ker(T))$ and $codim(R(T))$ are finite integer numbers. In this case, the index number of T denoted by $ind(T)$ and is given by $ind(T) = dim(Ker(T)) - codim(R(T))$. $T \in \mathcal{L}(X)$ is called Weyl operator if T is a Fredholm operator with index 0. The essential spectrum $\sigma_e(T)$ and the Weyl essential spectrum $\sigma_\omega(T)$ are defined by

$$\sigma_e(T) = \{\lambda \in \mathbb{C}/\lambda I - T \text{ is not Fredholm}\}$$

$$\sigma_\omega(T) = \{\lambda \in \mathbb{C}/\lambda I - T \text{ is not Weyl}\}$$

It is well known that for all $T \in \mathcal{L}(X)$, the sets $\sigma_e(T)$ and $\sigma_\omega(T)$ are compacts in the complex plane and we have

$$\sigma_e(T) \subseteq \sigma_\omega(T) \subseteq \widehat{\sigma_e(T)}$$

(where $\widehat{\sigma_e(T)}$ denotes the polynomially convex hull of $\sigma_e(T)$).

Definition 2.1 An operator $T \in \mathcal{L}(X)$ is polynomially Riesz if there exists a nonzero complex polynomial P such that $P(T) \in \mathcal{R}(X)$.

The subject of polynomially Riesz operators as an extension of polynomially compact operators has attracted the interest of some authors (see for example [8, 13]), it was proved that the following lemma characterized this class of operators by means of the structures of essential and Weyl essential spectra.

Lemma 2.1 (see [7]) Let X be a Banach space and let $T \in \mathcal{L}(X)$. Then

$$\text{T is polynomially Riesz} \iff \sigma_e(T) \text{ is a finite set.}$$

Remark **2.4** For $T \in \mathcal{L}(X)$. If $\sigma_e(T)$ is a finite set, it is easy to deduce that $\sigma_e(T) = \sigma_\omega(T)$. More precisely, this equality holds for the case of bounded linear operators T for which the set $\rho(T) = \mathbb{C}\backslash\sigma_e(T)$ is connected.

Our first result in this section is given by a characterization of polynomially Riesz operators on S.

Theorem 2.3 Let $T \in \mathcal{L}(S)$ be a polynomially Riesz operator. Then T can be decomposed into the form

$$T = \bigoplus_{i=1}^{n}(K_i + B_i + \lambda_i I)$$

where

($\imath$) K_i are compact operators on S;

($\imath\imath$) B_i are quasinilpotent operators;

(*iii*) $\sigma_e(T) = \sigma_\omega(T) = \{\lambda_1, ..., \lambda_n\}$.

Proof. The proof is based on the techniques used by [13] to prove Lemma 3 in their paper together with the fact that West decomposition of Riesz operators holds on S by using Theorem 2.2.

Corollary 2.2 For all $T \in \mathcal{L}(X_{GM})$, T is polynomially Riesz.

Proof. Let $T \in \mathcal{L}(X_{GM})$. Then, there exists $\lambda \in \mathbb{C}$ and $S \in \mathcal{S}(X_{GM})$ such that $T = \lambda I + S$. Consequently, $\sigma_e(T) = \sigma_\omega(T) = \{\lambda\}$ and the result follows directly by Lemma 2.1.

***Remark* 2.5** For all $T = \lambda I + S \in \mathcal{L}(X_{GM})$ ($\lambda \in \mathbb{C}$ and $S \in \mathcal{S}(X_{GM})$). Thus by using the complex polynomial $P(z) = z - \lambda$, it is easy to observe that $P(T) = \lambda I + S - \lambda I = S \in \mathcal{S}(X_{GM}) = \mathcal{R}(X_{GM})$ which gives another proof to Corollary 2.2.

3 Some comments and interesting questions:

As it's indicated in Remark 2.3, the problem of West decomposition of Riesz operators remains open in arbitrary Banach space since it is related directly to their geometry as the case of relevant problems in operator theory and functional analysis. Notice that this problem can be seen as a particular case of the problem of Salinas (see [7]) but we don't know if these two problems are equivalent.

Here, we give some questions connected to this problem which can develop this subject.

Question 1: Following the results given in [6], we deduce that Riesz operators have West decomposition on the space l^1. Does this problem holds on the Banach space $L^1([0, 1])$. More precisely, what is the topological property of the locally compact group G such that this decomposition remains true on $L^1(G)$?

In (1970), B. S. Mityagin studied the contractibility problem of the group of invertible linear bounded operators denoted by $GL(X)$ on some Banach spaces. It was proved in particular that this group is contractible for Banach spaces $l^p, c_0, L^p([0, 1]) (1 \leq p < \infty)$.

Question 2: Is there exists a relation between the problem of West decomposition of Riesz operators and that of the contractibility of the group of linear bounded invertible (or Fredholm) operators on Banach spaces?

One of the fascinate subject in operator theory is related to the structure of compact power operators by means of compact and nilpotent operators. In the case of Hilbert spaces, this problem is solved in [13]. More precisely, if $\mathcal{H}$ is a Hilbert space then for all $T \in \mathcal{H}$.

T^n *is compact* ($n \in \mathbb{N}$) $\Longleftrightarrow T = K + R$ *where* K *is compact and* R *is nilpotent.*

Notice that the result given above is based on a crucial lemma established by C. L.Olsen [18] which asserts that if $\mathcal{H}$ is a Hilbert space then for all $A_1, A_2 \in \mathcal{L}(\mathcal{H})$:

If $A_1A_2 \in \mathcal{K}(\mathcal{H})$ *then there exists a projection* $P \in \mathcal{L}(\mathcal{H})$ *such that* A_1P *and* $(I - P)A_2$ *are compact.*

But we don't know if Olsen's lemma holds for arbitrary Banach spaces. So, we can ask the following questions:

Question 3: Let $T \in \mathcal{L}(\mathcal{S})$ or $T \in \mathcal{S}(X_{GM})$, is it true that

T^n *is compact* $(n \in \mathbb{N}) \Longleftrightarrow T = K + R$ *where* K *is compact and* R *is nilpotent*?

Also, notice that on X_{GM}, it was conjectured [1] that we can construct $S \in \mathcal{S}(X_{GM})$ for which all of it's powers $T^n (n \in \mathbb{N})$ are not compacts. By the West decomposition result (see Theorem 2.2), this result implies that on X_{GM} we can find examples of quasinilpotent operators which are not nilpotent. In (2014), S. A. Argyros and P. Motakis [2] have constructed a famous reflexive hereditarily indecomposable Banach space denoted by X_{AM} such that for all $T \in \mathcal{S}(X_{AM})$ we have T^3 is compact.

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