The strong convex property and C_0-semigroups on some hereditarily indecomposable Banach spaces

Abdelkader Dehici

Abstract

In this paper, we study the strong convex compactness property on the hereditarily indecomposable Banach spaces denoted respectively by X_{GM} and X_{AM} constructed by T. Gowers and B. Maurey (1993) and Argyros-Motakis (2013), we prove in particular that the Bochner integral of a bounded family of strictly singular operators is a strictly singular operator. Also, some properties of c_0-semigroups defined on these Banach spaces are given.

Mathematics Subject Classification (2010): 47A15 46B06 46B03

1 Introduction and Preliminaries

It is known that the space of compact linear operators between Banach spaces X and Y (denoted by $\mathcal{K}(X, Y)$) is not complete for the strong topology defined on $\mathcal{L}(X, Y)$ (the space of all bounded linear operators from X into Y). Unfortunately, this closed subspace has a completeness property namely the strict convex compact property. The first author who has introduced this notion was L. Weis (1988), in order to study the time asymptotic behavior of solutions of neutronic transport equations. Noting that there exists other closed subspaces of $\mathcal{L}(X, Y)$ which have this property, but not all of them [23]. We note that Banach spaces on which the strict convex property was studied are sufficiently rich, in other words, having unconditional basis (for example $X = L_p(\mu)(1 < p < \infty)$) or they posses closed subspaces which have unconditional basis (for example $X = L_1(\mu)$ or $X = C(K)$, K compact). Since the appearance of the works of T. Gowers and B. Maurey (1993) solving the famous problem of unconditional basic problem (see [12]), another category of Banach spaces was discovered, called exotic Banach spaces or sufficiently poor Banach spaces giving a dichotomy to the structure of Banach spaces (see [11]). It was shown that the space of Gowers-Maurey X_{GM} is an hereditarily indecomposable Banach space (H.I), i.e., no infinite dimensional subspace

Keywords: *Hereditarily indecomposable Banach space, Strictly singular operator, spreading model, strong convex compactness property; Banach space of Gowers-Maurey X_{GM}, Banach space of Argyros-Motakis X_{AM}, C_0-semigroup.*

Corresponding author address: Laboratory of Informatics and Mathematics
University of Souk-Ahras,P.O.Box 1553, Souk-Ahras 41000, Algeria
`E-mails: dehicikader@yahoo.fr`

of X_{GM} can be decomposed into a direct sum of two further infinite dimensional closed subspaces, moreover, every bounded linear operator on X_{GM} can be written under the for $\lambda I + S$, where $\lambda \in \mathbb{C}$ and S is a strictly singular operator. On the other hand, in [1], the authors showed that the ideal of strictly singular operators on X_{GM} contains strictly that of compact ones by constructing a strictly singular not compact operator (with infinite dimensional kernel). In (2007), A. Dehici [6] showed that this operator is quasinilpotent (its spectrum is reduced to the set $\{0\}$). By inspiring the techniques of [20] on the construction of a strictly singular operator without non-trivial invariant subspace, the authors in [1] conjectured that on X_{GM}, we can construct a strictly singular operator none of whose powers is compact. Notice that the construction of X_{GM} is based on the arbitrary distortable (complementably minimal) Banach space of T. Schlumprecht [21] whose the source of the construction is the Tsirelson space [22] which does not contain any copy of l_p or c_0. The experts in geometry of Banach spaces asked the following question: Is there exists Banach space X such that every bounded linear operator on X can be written under the form $\lambda I + K$, where $\lambda \in \mathbb{C}$ and K is compact operator? It took fifteen years after the works of T. Gowers and B. Maurey to solve this open complicated problem. Indeed, in [3], the authors were able to construct a non reflexive Banach hereditarily indecomposable Banach space X_{AH} having this property, giving in particular a positive answer to the invariant subspace problem, since on this space, every bounded linear operator which is not scalar-identity is compact or it commutes with a non-zero compact operator [14]. The secret of the construction of the space X_{AH} is a combination of HI techniques and the Bourgain-Delbaen construction [5]. After, Argyros-Motakis [4] have constructed a HI Banach space denoted by X_{AM} having the property of invariant subspace problem and strictly singular operators on this spaces are not necessarily compact but have compact powers, also the philosophy of the construction of this space is different from that concerning the space X_{GM}. The properties enjoyed by the space X_{AM} are indicated in the following theorem.

Theorem 1.1 (see [4]) There exists a reflexive Banach space X_{AM} having a Schauder basis with the following other properties.

($\imath$) The space X_{AM} is hereditarily indecomposable.

($\imath\imath$) Every seminormalized weakly null sequence $\{x_n\}_n$ has a subsequence generating either l_1 or c_0 as a spreading model. Moreover, every infinite dimensional subspace Y of X_{AM} admits both l_1 and c_0 as spreading models.

($\imath\imath\imath$) For every Y infinite dimensional closed subspace of X_{AM} and every $T \in \mathcal{L}(X, X_{AM})$, $T = \lambda I_{Y,X_{AM}} + S$ with S strictly singular.

($\imath v$) For every closed infinite dimensional subspace Y of X_{AM}, the ideal $\mathcal{S}(Y)$ of the strictly singular operators is non separable.

(v) For every closed infinite dimensional subspace Y of X_{AM} and every $T_1, T_2, T_3 \in \mathcal{S}(Y)$ the operator $T_1T_2T_3$ is compact.

($v\imath$) For every closed infinite dimensional subspace Y of X_{AM} and every $T \in \mathcal{L}(Y)$, T admits a non-trivial closed invariant subspace.

***Remark* 1.1** By taking $Y = X$ in assertion $(\imath\imath\imath)$, we obtain that for every $A \in \mathcal{L}(X_{AM})$, there exists λ such that $A = \lambda I + S$ where S is strictly singular on X_{AM}, this shows that every bounded linear operator on X_{AM}, or it's a Fredholm operator (with index 0) or it's strictly singular. Notice that this property is satisfied also for the case of bounded linear operators defined on X_{GM} as it was indicated in the introduction.

In this work, we study the strict convex compactness property on the Banach spaces X_{GM} and X_{AM} for $E =$ the ideal of strictly singular operators on X_{GM} and X_{AM}. This property will arise from a general inequality involving a convenable semi-norm which will be defined on these spaces. Moreover, we will give some properties of c_0-semigroups defined on them and we end this work by some interesting questions.

Definition 1.1 Let E be a closed (with respect to the operator norm topology) subspace of $\mathcal{L}(X,Y)$. E is said to be have the strong convex property if the following holds: for any finite measure space (Ω, Σ, μ) and any bounded function $U : \Omega \longrightarrow E$ which is strongly measurable (i.e., U(.)x is measurable for all $x \in X$), the strong integral $\int_\Omega U(\omega) d\mu$ (which is an element of $\mathcal{L}(X,Y)$) defined by

$$\int_\Omega U d\mu x = \int_\Omega U(\omega) x d\mu(\omega), x \in X$$

belongs to E.

***Example* 1.1** The following closed subspaces have the strong convex property

1. The space $\mathcal{K}(X,Y)$ of compact linear operators [23, 24].
2. The space $\mathcal{V}(X,Y)$ of completely continuous operators [23].
3. The space $\mathcal{W}(X,Y)$ of weakly compact linear operators [21].
4. The space $\mathcal{U}(X,Y)$ of unconditionally summing operators, $\{T \in \mathcal{L}(X,Y);$ for all sequences $(x_n) \subset X$ such that $\sum_n | < x_n, x' > | < \infty$ for all $x^\star \in X^\star$, the series $\sum_n T(x_n)$ is convergent$\}$ [23].
5. The space of Dieudonné operators, $\{T \in \mathcal{L}(X,Y);$ for all weak Cauchy sequences $(x_n) \subset X$ the sequence $T(x_n)$ is weakly convergent$\}$ [23].
6. The space $\mathcal{S}(X)$ of strictly singular linear operators, if $X = Y = L_p(\mu), 1 \leq p \leq \infty$ [24],

Let X, Y be two complex Banach spaces and let $A \in \mathcal{L}(X,Y)$, we define the measure of noncompactness of A by

$$\|A\|_m = \inf\{\|A_{|M}\|, M \text{ closed subspace of } X \text{ with finite codimension}\}$$

It is shown [13] that $\|.\|_m$ is a multiplicative semi-norm on $\mathcal{L}(X,Y)$ with the following property

$$\|A\|_m = 0 \text{ if and only if } A \text{ is compact.}$$

In the following theorem, we mention estimates of the measure of noncompactness of strong integrals.

Theorem 1.2 (see [8, 24]) Let X, Y be two Banach spaces, (Ω, μ) be a finite measure space and

$$U : \Omega \longrightarrow \mathcal{L}(X, Y) \text{ be bounded and strongly measurable}$$

$(\imath)$ If $X^\star$ is separable, then

$$\omega \in \Omega \longrightarrow \|U(\omega)\|_m \text{ is measurable}$$

and

$$\|\int_\Omega U(\omega)d\mu(\omega)\|_m \leq \int_\Omega \|U(\omega)\|_m d\mu(\omega) \tag{2.1}$$

$(\imath\imath)$ If X is separable, then

$$\|\int_\Omega U(\omega)d\mu(\omega)\|_m \leq 2\overline{\int_\Omega} \|U(\omega)\|_m d\mu(\omega) \tag{2.2}$$

$(\imath\imath\imath)$ In general, we have

$$\|\int_\Omega U(\omega)d\mu(\omega)\|_m \leq 4\overline{\int_\Omega} \|U(\omega)\|_m d\mu(\omega) \tag{2.3}$$

where $\overline{\int_\Omega}$ is the upper integral.

***Remark* 1.2** Noting that the strict convex property of $\mathcal{K}(X, Y)$ is an immediate consequence of the inequality (2.1) by taking $\|U(\omega)\|_m$ equal to 0.

2 Strict convex property for the ideal of strictly singular operators on X_{GM} and X_{AM}

Recall that a bounded linear operator T from a Banach space X to a Banach space Y is called strictly singular if its restriction to any infinite dimensional subspace is not an isomorphism which is equivalent to the fact that for every infinite dimensional closed subspace Z of X and for every $\epsilon > 0$, there exists $z \in Z$ such that $\|T(z)\| < \epsilon\|z\|$.

2.1 The case of the space X_{GM}

Let $y \in X_{GM}$ and $n \geq 1$, the positive scalar $\|y\|_{(n)}$ is defined by

$$\|y\|_{(n)} = \sup \sum_{i=1}^{n} \|E_i y\|$$

where the sup is extended to all families $E_1 < E_2 ...... < E_n$ of successive intervals (for more details, see [12]). It is known that, we can define on $\mathcal{L}(X_{GM})$ a seminorm as follows:

Let $\mathcal{K}_{X_{GM}}$ be the set of sequences $(x_n)_n$ of almost successive vectors in X_{GM} such that $\limsup \|x_n\|_{(n)} \leq 1$. Let $T \in \mathcal{L}(X_{GM})$, we define $|||T|||_{GM}$ by

$$|||T|||_{GM} = \sup_{x=(x_n)\in\mathcal{K}_{X_{GM}}} \limsup \|T(x_n)\|$$

Proposition 2.1 For every $T \in \mathcal{L}(X_{GM})$, the following assertions are equivalent:

($\imath$) T is strictly singular.

($\imath\imath$) $|||T|||_{GM} = 0$.

Proof. For the case ($\imath\imath$) implies ($\imath$) (see [11], Lemma 11.2, p. 68). Now, if T is strictly singular, then there exists $\lambda \in \mathbb{C}$ such that $|||T - \lambda I|||_{GM} = 0$ (see [15], p. 81 (the construction of H.I space)), which gives by the first implication that $T - \lambda I$ is strictly singular, but if $\lambda \neq 0$, it follows that T is a Fredholm operator, which is a contradiction, hence we get necessarily that $\lambda = 0$ and consequently, we obtain $|||T|||_{GM} = 0$ which gives the second implication and achieves the proof.

Theorem 2.1 Let (Ω, μ) be a measure space and $w \in \Omega \longrightarrow U(\omega) \in \mathcal{L}(X_{GM})$ be a strongly integrable function, i.e.,

$$Ux = \int_{\Omega} U(\omega) x d\mu(\omega)$$

exists for exists for all $x \in X_{AM}$ as a Bochner integral and

$$\int_{\Omega} ||U(\omega)|| d\mu(\omega) < \infty$$

Then

$$|||U|||_{GM} \leq \int_{\Omega} |||U(\omega)|||_{GM} d\mu(\omega) \tag{2.4}$$

Proof. We have by definition

$$|||U|||_{GM} = \sup_{x=(x_n) \in \mathcal{K}_{X_{GM}}} \limsup \| \int_{\Omega} U(\omega) x_n d\mu(\omega) \|$$

A classical inequality satisfied by the norm gives that

$$\sup_{x=(x_n) \in \mathcal{K}_{X_{GM}}} \limsup \| \int_{\Omega} U(\omega) x_n d\mu(\omega) \| \leq \sup_{x=(x_n) \in \mathcal{K}_{X_{GM}}} \limsup \int_{\Omega} \|U(\omega) x_n\| d\mu(\omega)$$

On the other hand, the functions $w \in \Omega \longrightarrow \|U(\omega)(x_n)\|$ are measurable. The fact that

$$\int_{\Omega} ||U(\omega)|| d\mu(\omega) < \infty$$

enable us (by using the Lebesgue's convergence theorem) to obtain that

$$\limsup \int_{\Omega} \|U(\omega) x_n\| d\mu(\omega) \leq \int_{\Omega} \limsup \|U(\omega) x_n\| d\mu(\omega)$$

Hence

$$\limsup \int_{\Omega} \|U(\omega) x_n\| d\mu(\omega) \leq \int_{\Omega} \sup_{x=(x_n) \in \mathcal{K}_{X_{GM}}} \limsup \|U(\omega) x_n\| d\mu(\omega) = \int_{\Omega} |||U(\omega)||| d\mu(\omega)$$

which gives that

$$|||U|||_{GM} \leq \int_{\Omega} \sup_{x=(x_n) \in \mathcal{K}_{X_{GM}}} \limsup \|U(\omega) x_n\| d\mu(\omega) = \int_{\Omega} |||U(\omega)|||_{GM} d\mu(\omega)$$

and achieves the proof.

Corollary 2.1 Let (Ω, μ) be a measure space and $w \in \Omega \longrightarrow U(\omega) \in \mathcal{L}(X_{GM})$ be a strongly integrable function, i.e.,

$$Ux = \int_{\Omega} U(\omega)x d\mu(\omega)$$

exists for all $x \in X_{GM}$ as a Bochner integral and

$$\int_{\Omega} ||U(\omega)|| d\mu(\omega) < \infty$$

Then

if $U(\omega)$ is strictly singular for almost every $\omega \in \Omega$ on X_{GM}, then U is strictly singular.

Proof. The result is an immediate consequence of Theorem 2.1 by taking $|||U(\omega)|||_{GM} = 0$ for almost every $\omega \in \Omega$.

2.2 The case of the space X_{AM}

Definition 2.1 Let X be a Banach space. Two basic sequences (x_n) and (y_n) are called C-equivalent for some $C > 1$, denoted by $(x_n) \approx (y_n)$, if for every $(a_n) \in c_{00}$ (the vector space of all scalar sequences of finite support), we have that

$$\frac{1}{C} \| \sum a_n x_n \| \leq \| \sum a_n y_n \| \leq C \| \sum a_n x_n \|$$

Two basic sequences (x_n) and (y_n) are called equivalent, denoted by $(x_n) \approx (y_n)$, if they are C-equivalent for some $C > 1$.

Let X be a Banach space and let $(y_i)_i$ a seminormalized basic sequence in X and (ϵ_n) a decreasing sequence of positive real numbers. It is known that there exists a subsequence (x_i) of (y_i) and a seminormalized basic sequence $(\widetilde{x_i})_i$ (in other Banach space) such that for all $n \in \mathbb{N}, (a_i)_{i=1}^n \in [-1, 1]^n$ and $n \leq k_1 < ..... < k_n$, one has

$$||| \sum_{i=1}^n a_i x_{k_i} \| - \| \sum_{i=1}^n a_i \widetilde{x_i} ||| < \epsilon_n.$$

The sequence $(\widetilde{x_i})$ is called the spreading model of (x_i). Spreading models of weakly null seminormalized basic sequences have been studied by [2].

The following proposition gives a characterization of strictly singular operators on the space of Argyros-Motakis X_{AM}.

Proposition 2.2 (see [4], Proposition 5.7) Let Y be an infinite dimensional closed subspace of X_{AM} and $T : Y \longrightarrow X_{AM}$ be a linear bounded operator. Then the following assertions are equivalent:

($\imath$) T is not strictly singular.

($\imath\imath$) There exists a sequence $(x_k)_{k \in \mathbb{N}}$ in Y generating a c_0 spreading model such that $(T(x_k))_{k \in \mathbb{N}}$ is not norm convergent to 0.

Let $\mathcal{M}_X$ be the set of sequences $(x_n)_{n=1}^{\infty}$ generating c_0 models in X_{AM}. Now, given $T \in \mathcal{L}(X_{AM})$ and let

$$|||T|||_{AM} = \sup_{x=(x_n)\in\mathcal{M}_X} \limsup \|T(x_n)\|$$

Our first result in this section is the following

Proposition 2.3 $|||T|||_{AM}$ is a seminorm on X_{AM} satisfying that $|||T|||_{AM} = 0$ if and only if T is strictly singular.

Proof. It suffices to prove the second assertion. Let T be strictly singular and assume that $|||T|||_{AM} \neq 0$. By the definition given below, there exists a sequence $(y_n)_n$ generating a c_0 model in X_{AM} such that $\limsup \|T(y_n)\| \longrightarrow l \neq 0$, thus $(y_n)_n$ has a subsequence $(y_{n_k})_k$ generating a c_0 spreading model (see [4]) for which $\|T(y_{n_k})_k)\| \nrightarrow 0$ which contradicts the fact that T is strictly singular (see Proposition 2.2). Conversely, if $|||T|||_{AM} = 0$, then $\limsup \|T(y_n)\| \longrightarrow 0$ for all sequence $(y_n)_{n=1}^{\infty}$ generating a c_0 model in X_{AM} which implies that $\|T(y_n)\| \longrightarrow 0$, again by using Proposition 2.2, we get that T is strictly singular which achieves the proof.

By the same spirit of the techniques in the proofs concerning the case of the Banach space X_{GM}, we can obtain the following results.

Theorem 2.2 Let (Ω, μ) be a measure space and $w \in \Omega \longrightarrow U(\omega) \in \mathcal{L}(X_{AM})$ be a strongly integrable function, i.e.,

$$Ux = \int_\Omega U(\omega)x d\mu(\omega)$$

exists for exists for all $x \in X_{AM}$ as a Bochner integral and

$$\int_\Omega ||U(\omega)||d\mu(\omega) < \infty$$

Then

$$|||U|||_{AM} \leq \int_\Omega |||U(\omega)|||_{AM} d\mu(\omega) \tag{2.5}$$

Corollary 2.2 Let (Ω, μ) be a measure space and $w \in \Omega \longrightarrow U(\omega) \in \mathcal{L}(X_{AM})$ be a strongly integrable function, i.e.,

$$Ux = \int_\Omega U(\omega)x d\mu(\omega)$$

exists for exists for all $x \in X_{AM}$ as a Bochner integral and

$$\int_\Omega ||U(\omega)||d\mu(\omega) < \infty$$

Then

if $U(\omega)$ is strictly singular for almost every $\omega \in \Omega$ on X_{AM}, then U is strictly singular.

3 C_0-semigroups on X_{GM} and X_{AM}

In this section, we give a characterization together with some properties of c_0-semigroups defined on X_{AM}. First of all, we start by giving some basic notions concerning c_0-semigroups which will be used in the following.

Definition 3.1 Let X be a Banach space. A family $(U(t))_{t\geq 0}$ of bounded linear operators from X into X is a semigroup of bounded linear operators on X if

($\imath$) $U(0) = I$, (I is the identity operators on X).

($\imath\imath$) $U(t+s) = U(t)U(s)$ for all $t, s \geq 0$.

Moreover, if

$$\lim_{t\longrightarrow 0} U(t)x = x \text{ for every } x \in X$$

$(U(t))_{t\geq 0}$ is called a C_0-semigroup.

Definition 3.2 Let X be a Banach space and let $(U(t))_{t\geq 0}$ be a c_0-semigroup on X with infinitesimal generator T. We define the type ω of $(U(t))_{t\geq 0}$ by

$$\omega = \lim_{t\longrightarrow\infty} \log\frac{\|U(t)\|}{t}.$$

The spectral bound of T is given by

$$s(T) = \sup_{\lambda\in\sigma(T)} Re\lambda. \text{ (where } \sigma(T) \text{ denotes the spectrum of } T)$$

It is known that $s(T) \leq \omega$ and the resolvent of T denoted by $(\lambda - T)^{-1}$ is given as a Laplace transform of $(U(t))_{t\geq 0}$ by the formula

$$(\lambda - T)^{-1} = \int_0^\infty e^{-\lambda t} U(t)dt; \qquad Re\lambda > \omega.$$

For more details on the theory of c_0-semigroups on Banach spaces, we can quote for example [17].

Theorem 3.1 Let $(U(t))_{t\geq 0}$ be a c_0-semigroup on X_{AM} with infinitesimal generator T. Thus we have the following three possibilities.

($\imath$) $\sigma(T) = \emptyset$;

($\imath\imath$) $(\lambda - T)^{-1}$ is compact for all $\lambda > s(T)$;

($\imath\imath\imath$) $T \in \mathcal{L}(X_{AM})$.

Proof. Assume that there exists $t_0 > 0$ such that $U(t_0)$ is strictly singular, then by Theorem 1.1 (v), if $U(3t_0) = [U(t_0)]^3 = 0$, we get that for all $n \geq 3, U(nt_0) = 0$ and consequently $U(t) = 0$ for all $t \geq 3t_0$. Hence, in this case we get that $\omega = -\infty$ and consequently $\sigma(T) = \emptyset$.

Now, assume that for all $t_0 > 0$, $U(t_0)$ is strictly singular and $U(t_0) \neq 0$, also by Theorem 1.1 (v), $U(t_0) = [U(\frac{t_0}{3})]^3$ is compact (here $U(t_0)$ can not be a Fredholm operator since the composition of Fredholm operators is Fredholm). Thus, for all $t_0 > 0$, $U(t_0)$ is compact on X_{AM}. On the other hand, the fact that $\mathcal{K}(X_{AM})$ has strict convex compactness property shows that $\int_0^N e^{-\lambda t} U(t)dt$ is compact for all $N \geq 1$ and $\lambda > \omega$, hence by limit in $\mathcal{L}(X_{AM})$, we get

$$\int_0^N e^{-\lambda t}U(t)dt \longrightarrow \int_0^\infty e^{-\lambda t}U(t)dt = (\lambda - T)^{-1}; \qquad \lambda > \omega.$$

which implies that $(\lambda - T)^{-1}$ is compact for $\lambda > \omega$. The compactness for $\lambda > s(T)$ follows from analyticity argument.

Finally, if $U(t_0)$ is a Fredholm operator for $t_0 > 0$, thus $(U(t))_{t\geq 0}$ can be embedded into a c_0-group of $\mathcal{L}(X_{AM})$ (see Lemma 1 in [10]). Now, the boundedness of T follows from Theorem 2.3 in [18].

By using the reflexivity of the space X_{GM} and X_{AM} together with the Proposition 4.1 in [18], we can obtain the following result concerning the generators of uniformly bounded C_0-semigroups.

Corollary 3.1 Let T be the generator of a uniformly bounded C_0-semigroup $(U(t))_{t\geq 0}$ on X_{GM} or X_{AM}. Then the following statements are equivalent.

($\imath$) Ker$T$ is infinite dimensional.

($\imath\imath$) T is bounded and a finite rank operator.

Proposition 3.1 Let T be the generator of a uniformly bounded C_0-semigroup $(U(t))_{t\geq 0}$ on X_{GM} or X_{AM}. Then $\sigma(T)\bigcap i\mathbb{R}$ is a finite set.

Proof. Assume that $\sigma(T)\bigcap i\mathbb{R}$ is an infinite set. The first part in the proof of Proposition 4.2 in [18] gives that T is bounded. On the other hand, the spectral mapping theorem implies that for t close enough to 1, the bounded linear operator $U(t)$ has an infinite eigenvalues on the unit circle $\{z \in \mathbb{C}/|z| = 1\}$, now by using Proposition 6.3 in [7], we get that for every $\epsilon > 0$, the space X_{GM} (resp. X_{AM}) has a $(1+\epsilon)$-unconditional basic sequence which is a contradiction. Hence the set $\sigma(T)\bigcap i\mathbb{R}$ must be finite.

***Remark* 3.1** Notice that the two papers of F. Räbiger and W.J. Ricker [18, 19] are a pioneer works on the subject concerning C_0-semigroups on hereditarily indecomposable Banach spaces, many of their results were improved and refined by [7].

4 Some interesting questions

In finalizing this study, the first question that may come to our mind is the following

Question 1: Is it true that $\mathcal{S}(X,Y)$ has the strong convex property for every two Banach spaces X and Y? otherwise, what are the structures of Banach spaces such that this property holds?

The same question can be asked for the case of the closed subspace of Fredholm perturbations denoted by $\mathcal{F}(X,Y)$ (for more details, see [9]). More precisely

Question 2: Is it true that $\mathcal{F}(X,Y)$ has the strong convex property for every two Banach spaces X and Y? otherwise, what are the structures of Banach spaces such that this property holds?

Acknowledgements: The author would like to thank the Professors P. Motakis (University of Athens, Greece) and V. Ferenczi (University of Sao Paolo, Brazil) for the interesting discussions and helpful comments.

References

[1] G. Androulakis and T. Schlumprecht, *Strictly singular, non-compact operator exist on the space of Gowers and Maurey,* J. Lond. Math. Soc., 2 (64) (2001), 655-674.

[2] G. Androulakis, E. Odell, T. Schlumprecht and N. T. Jaegermann, *On the structure of the spreading models of a Banach space,* Canadian. J. Math., (57) 4 (2005), 673-707.

[3] S. A. Argyros and R. G. Haydon, *A hereditarily indecomposable $\mathcal{L}_\infty$ space that solves the scalar-plus-compact problem,* Acta. Math., (206) (2011), n.1, 1-54.

[4] S. A. Argyros and P. Motakis, *Reflexive HI space with hereditarily invariant subspace property,* to appear in the Proceeding of the London Mathematical Society.

[5] J. Bourgain and F. Delbaen, *A class of special $\mathcal{L}_\infty$ spaces,* Acta. Math., (145) (1980), n 3-4, 155-176.

[6] A. Dehici, *Quasinilpotence et perturbations de Fredholm,* Séminaire de la société mathématique de Tunisie, 19-22 Mars, 2007.

[7] V. Ferenczi and C. Rosendal, *On isometry groups and maximal symmetry,* Duck. Math. J., (162) (2013), n 10, 1771-1831.

[8] M. Kunze and G. Schluchtermann, *Strongly generated Banach spaces and measures of non-compactness,* Math. Nachr., (191) (1998), 197-214.

[9] I. C. Gohberg, A. S. Markus and I. A. Feldmann, *Normally solvable operators and ideals associated with them,* Amer. Math. Soc. Trans., 61 (1967 (1960)), 63-84.

[10] S. Goldberg and C. H. Smith, *Strongly continuous semigroups of semi-Fredhom operators,* J. Math. Anal. Appl., 64 (1978), 407-420.

[11] T. W. Gowers, *A new dichotomy for Banach spaces,* Geom. Func. Anal., 6 (1996), 1083-1093.

[12] T. W. Gowers and B. Maurey, *The unconditional basic sequence problem,* J. Amer. Math. Soc., 6 (1993), 851-874.

[13] A. Lebow and M. Schechter, *Semi-groups of operators and measures of non-compactness,* J. Funct. Anal., 7 (1971), 1-26.

[14] V. I. Lomonosov, *Invariant subspaces of the family of operators that commute with a completely continuous operator,* Funkcional. Anal. i Priložen., 7 (1973), no.3, 55-56.

[15] B. Maurey, *Operator theory and exotic Banach spaces,* Online lecture notes, université de jussieu, 1994.

[16] M. Mokhtar-Kharroubi, *Mathematical topics in neutron transport theory,* New-Aspects, Series on Adv. Math. Appl. Sci. (Vol 46 World Scientific), 1997.

[17] A. Pazy, *Semigroups of linear operators and applications to partial differential equations,* Springer-Verlag, New-York Inc, 1983.

[18] F. Räbiger and W.J. Ricker, *C_0-groups and C_0-semigroups on hereditarily indecomposable Banach spaces,* Arch. Math., 66(1996), 60-70.

[19] F. Räbiger and W.J. Ricker, *C_0-semigroups and cosine families of linear operators in hereditarily indecomposable Banach spaces,* Acta. Sci. Math (Szeged)., 64 (1998), 697-706.

[20] C. J. Read, *Strictly singular operators and the invariant subspace problem,* Stud. Math., 132(1991), 203-226.

[21] T. Schlumprecht, *An arbitrary distortable Banach space,* Israel. J. Math., 76 (1) (1991), 81-95.

[22] B.S. Tsirelson, *Not every Banach space contain an embedding of l_p or c_0,* Funct. Anal. Appl., 8 (1974), 138-141.

[23] J. Voigt, *On the convex compactness property for the strong operator topology,* Note. di. Math., Vol XII(1992), 259-269.

[24] L. Weis, *A generalization of the Vidav-Jorgens perturbation theorem for semigroups and its application to transport theory,* J. Math. Anal. Appl., 129 (1988), 6-23.