Some Fixed Point Theorems for Hardy-Rogers Multi-valued Mappings with non-constant coefficients

Nadjeh Redjel and Abdelkader Dehici

Laboratory of Informatics and Mathematics

University of Souk-Ahras, P.O.Box 1553, Souk-Ahras 41000, Algeria

E-mails: dehicikader@yahoo.fr, najehredjel@yahoo.fr

Abstract

In this paper, we study some properties of boundedly orbitally multi-valued mappings on complete metric spaces. These results are explored to give an investigation of the existence of fixed points concerning Hardy-Rogers mappings involving some non-constant coefficients which extend and improve many known contributions in the literature.

1 Introduction

It is well known that the Banach contraction principle (1922) was the starting point of the fixed point theory. This result was extended to the case of multi-valued mappings by J. Nadler (1969) as follows:

If (X, d) is a complete metric space and T is a multi-valued mapping of X into the family $CB(X)$ of all nonempty closed bounded subsets of X and let H be the Hausdorff metric with respect to d. Then if there exists $\alpha \in [0, 1)$ such that

$$H(Tx, Ty) \leq \alpha d(x, y), \quad \forall x, y \in X.$$

Then T has a fixed point, in other words, there exists $z \in X$ such that $z \in Tz$.

This contribution is of great importance due to the fact that this class of mappings plays a central role in applied sciences (Optimization, equilibrium problem, games theory, differential and partial differential equations involving integral inclusions, ...). For a good reading concerning theses mappings and their impact in applied sciences, we can quote for example [1].

Nadler's result was generalized by S. Reich for mappings with non-constant coefficients taking values in the family $K(X)$ of all nonempty compact subsets of X satisfying that

$$H(Tx, Ty) \leq \alpha((d(x, y))d(x, y), \quad \forall x, y \in X.$$

where α is a function of $(0, \infty)$ to $[0, 1)$ such that $\limsup\limits_{r \longrightarrow t^+} \alpha(r) < 1$ for every $t \in (0, \infty)$

The same author asked the question asked whether this contribution holds true if T takes its values into $CB(X)$ instead $K(X)$. This question was answered positively by Mizoguchi and Takahashi [13] in almost complete form by replacing the condition on α by the following stronger condition

$$\limsup_{r \longrightarrow t^+} \alpha(r) < 1 \text{ for every } t \in [0, \infty)$$

The analysis of Mizoguchi-Takahashi was improved and generalized by [4]. In (2011) Gordji et al [8] gave a positive answer to the Reich conjecture for generalized multi-valued version of Hardy-Rogers mappings [9] as follows:

Theorem 1.1 [8] Let (X, d) be a complete metric space and let $T : X \longrightarrow CB(X)$ be a multi-valued mapping satisfying that

$$\begin{aligned} H(Tx, Ty) &\leq \alpha(d(x, y))d(x, y) + \beta(d(x, y))(d(x, Tx) + d(y, Ty)) \\ &\quad + \gamma(d(x, y))(d(x, Ty) + d(y, Tx)), \end{aligned}$$

for all $x, y \in X$ where α, β, γ are mappings from $[0, +\infty)$ into $[0, 1)$ such that $\alpha(t) + 2\beta(t) + 2\gamma(t) < 1$ and

$$\limsup_{r \longrightarrow t^+} \frac{\alpha(s) + \beta(s) + \gamma(s)}{1 - (\beta(s) + \gamma(s))} < 1 \text{ for every } t \in [0, \infty)$$

Then there exists $z \in X$ such that $z \in Tz$.

Single valued mappings or multi-valued contraction mappings are uniformly continuous hence continuous which is not the case of other generalized contractions. This implies that the function $x \longrightarrow d(x, Tx)$ is continuous, this fact fails for the other generalized contractions, by this reason the authors added the lower semicontinuity of the function $x \longrightarrow d(x, Tx)$ to obtain some fixed point results via Caristi's fixed point theorem which is also another extension to Nadler's result (for this subject, we can see [6, 12, 20]). Also, one of the advantage of contraction or generalized contraction mappings with constant coefficients is that the principle to study and the investigation of their fixed points is the same consisting to obtain this fixed point as a limit of a Cauchy sequence (x_n) satisfying that $x_{n+1} \in Tx_n$. Unfortunately, when the coefficients are not constant, many constraints appeared and thus we will have to go through other arguments including the behavior of the mapping on its orbits.

The organization of our paper is as follows: First of all, we give some preliminaries and notations which will be used in the sequel. In the section 2, we study some properties of strictly and uniformly orbitally bounded mappings and their relation with the property $\mathcal{P}$ defined in this section. Finally, in section 3, these results are explored to obtain fixed point results for generalized Hardy-Rogers mappings involving some functions as coefficients by reducing the problem on the case of an equivalent problem on bounded complete metric spaces.

2 Main Results

Let (X, d) be a complete metric space. Then, for $x \in X$ and $A \subseteq X, d(x, A) = \inf\{d(x, y) : y \in A\}$. The Hausdorff metric with respect to d is defined by

$$H(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{z \in B} d(z, A)\},$$

The following properties satisfied by the metric H are used in the sequel.

1. $(CB(X), H)$ is a complete metric space (see [10]),

2. For arbitrary bounded sets A_1 and A_2 of X, we have $H(\overline{A_1}, \overline{A_2}) = H(A_1, A_2)$ ($\overline{A}$ denotes the closure of the set A in the metric space $(CB(X), H)$),

3. If A_1 and A_2 are bounded subsets of X. Then,

$$d(x, A_2) \leq H(A_1, A_2) \text{ for all } x \in A_1.$$

4. If A_1 and A_2 are bounded subsets of X. Then,

$$d(x, A_1) \leq d(x, A_2) + H(A_1, A_2).$$

For $x \in X$, we denote $\mathcal{C}_x(T) = \{T^n x\}_{n \geq 0} = \{\{x\}, Tx, T^2x, ...\}$ ($T^0 x = \{x\}$) and $\mathfrak{C}(T) = \{T^n x\}_{n \geq 0, x \in X}$.

The orbit of T at x is given by

$$\mathcal{O}_T(x) = \bigcup_{n=0}^{\infty} T^n x \text{ where } T^n x = \bigcup_{\omega \in T^{n-1}x} T\omega.$$

Definition 2.1 Let (X, d) be a metric space and let $T : X \longrightarrow CB(X)$ be a multi-valued mapping. T is said to be closely orbitally continuous at $x_0 \in X$ if for any point $x \in \overline{\mathcal{O}_T(x_0)}$ and any sequence $\{x_n\} \subset \mathcal{O}_T(x_0)$ such that $x_n \longrightarrow x$ as $n \longrightarrow \infty$ in (X, d) we have $Tx_n \longrightarrow Tx$ as $n \longrightarrow \infty$ in $(CB(X), H)$. T is said to be closely orbitally continuous on X if it is closely orbitally continuous at any point x_0 of X.

***Remark* 2.1** It is to be noted that a continuous multi-valued mapping is closely orbitally continuous but the converse is not necessarily true.

Definition 2.2 Let (X, d) be a metric space and let $T : X \longrightarrow CB(X)$ be a multi-valued mapping. T is said to be strictly orbitally bounded at $x \in X$ if for any integer $n \geq 2$, the sets $T^n x$ are bounded. T is said to be orbitally bounded if it is strictly orbitally bounded at any point x of X.

***Remark* 2.2** Let (X, d) be a metric space and let $T : X \longrightarrow CB(X)$ be a multi-valued mapping, hence for all $x \in X$, $\{x\}$ and Tx are bounded sets. By this reason the previous definition is focused to the case of $n \geq 2$.

Definition 2.3 Let (X,d) be a metric space and let $T : X \longrightarrow CB(X)$ be a multi-valued mapping. T is said to be uniformly orbitally bounded at $x \in X$ if the orbit $\mathcal{O}_T(x)$ is a bounded set or its diameter is finite, in other words if

$$\delta(\mathcal{O}_T(x)) = \sup\{d(u_1,u_2)/u_1,u_2 \in \mathcal{O}_T(x)\} < \infty.$$

T is said to be uniformly orbitally bounded if it is uniformly orbitally bounded at any point $x \in X$.

***Remark* 2.3** Let (X,d) be a metric space and $T : X \longrightarrow CB(X)$. Let $x \in X$, it is easy to observe that if T is uniformly orbitally bounded at x, then T is strictly orbitally bounded at the same point but the converse is in general not true since in general an infinite union of bounded sets is not necessarily bounded.

Definition 2.4 Let (X,d) be a metric space and $T : X \longrightarrow CB(X)$. We say that T has the property $\mathcal{P}$ if there exists $\varphi : \mathfrak{C}(T) \longrightarrow [0,+\infty[$ such that $H(Z,TZ) \leq \varphi(Z) - \varphi(TZ)$ for all $Z \in \mathfrak{C}(T)$.

Theorem 2.1 Let (X,d) be a metric space and $T : X \longrightarrow CB(X)$ be a strictly orbitally bounded multi-valued mapping. Then T has the property $\mathcal{P}$ if and only if for all $z \in X$ the series $\displaystyle\sum_{n=0}^{\infty} H(T^{n+1}z, T^n z)$ are convergent.

Proof. Assume that there exists $\varphi : \mathfrak{C}(T) \longrightarrow [0,+\infty[$ such that $H(Z,TZ) \leq \varphi(Z) - \varphi(TZ)$ for all $Z \in \mathfrak{C}(T)$, then

$$\varphi(Z) \geq \varphi(TZ) \geq \varphi(T^2Z) \geq \varphi(T^3Z) \geq ....$$

It follows that for all $z \in X$, we have

$$H(T^n z, T^{n+1}z) \leq \varphi(T^n z) - \varphi(T^{n+1}z).$$

By summation, we infer that for all integer $m \geq 1$

$$\sum_{n=0}^{m} H(T^{n+1}z, T^n z) \leq \varphi(Tz) - \varphi(T^{m+1}z) \leq \varphi(Tz).$$

By letting $m \longrightarrow \infty$ in the previous inequality, we obtain the convergence of the series $\displaystyle\sum_{n=0}^{\infty} H(T^{n+1}z, T^n z)$.

Conversely, assume that $\displaystyle\sum_{n=0}^{\infty} H(T^{n+1}x, T^n x) < \infty$ for all $x \in X$. If we put

$$\varphi : \mathfrak{C}(T) \longrightarrow [0,+\infty[.$$

defined by $\varphi(Z) = \displaystyle\sum_{n=0}^{\infty} H(T^{n+1}Z, T^n Z)$, $Z \in \mathfrak{C}(T)$. Hence φ has the desired property.

Theorem 2.2 Let (X, d) be a complete metric space and $T : X \longrightarrow CB(X)$ be a strictly orbitally bounded multi-valued mapping having the property $\mathcal{P}$. Then T is a uniformly orbitally bounded multi-valued mapping.

Proof. First of all, we will prove that for all $x \in X, \overline{\{T^n x\}_n}$ is a Cauchy sequence in $(CB(X), H)$. Indeed, from our assumptions together with Theorem 2.1, we deduce that $\sum_{n=0}^{\infty} H(T^{n+1}x, T^n x) < \infty$ for all $x \in X$. Now, let z an arbitrary element of X. For all integers $n, m (n < m)$, we have

$$\begin{aligned} H(\overline{T^n z}, \overline{T^m z}) = H(T^n z, T^m z) &\leq H(T^n z, T^{n+1} z) + ...H(T^{m-1} z, T^m z) \\ &\leq \sum_{k=n}^{\infty} H(T^n z, T^{n+1} z). \end{aligned}$$

the Cauchy criterion concerning convergent series shows that for all $\epsilon > 0$ there exists an integer $n_0 \geq 1$ for which $\sum_{k=n}^{\infty} H(T^n z, T^{n+1} z) < \epsilon$ for $n \geq n_0$. This proves that $\{\overline{T^n z}\}_n$ is a Cauchy sequence in $(CB(X), H)$ and consequently there exists $A \in CB(X)$ such that $\overline{T^n z} \longrightarrow A$ with respect to the metric H. Fix $\epsilon > 0$, then there exists an integer $n_1 \geq 1$ such that $H(\overline{T^n z}, A) < \frac{\epsilon}{2}$ for all $n \geq n_1$. Hence

$$d(u, A) \leq H(\overline{T^n z}, A) = H(T^n z, A) < \tfrac{\epsilon}{2} \text{ for all } u \in T^n z$$

Let $n \geq n_1$ and let $x_1, x_2 \in T^n z$, then

$$d(x_1, A) \leq H(\overline{T^n z}, A) = H(T^n z, A) < \frac{\epsilon}{2};$$

and

$$d(x_2, A) \leq H(\overline{T^n z}, A) = H(T^n z, A) < \frac{\epsilon}{2};$$

By the definition of the infimum, we obtain the existence of $v_1, v_2 \in A$ such that

$$d(x_1, v_1) \leq H(T^n z, A) + \frac{\epsilon}{2};$$

and

$$d(x_2, v_2) \leq H(T^n z, A) + \frac{\epsilon}{2};$$

Thus

$$\begin{aligned} d(x_1, x_2) &\leq d(x_1, v_1) + d(v_1, v_2) + d(v_2, x_2) \\ &\leq 2\epsilon + d(v_1, v_2) \\ &\leq 2\epsilon + \delta(A). \end{aligned}$$

This gives that

$$\delta(T^n z) = \sup_{x_1, x_2 \in T^n z} \{d(x_1, x_2)\} \leq 2\epsilon + \delta(A);$$

Moreover, the fact that T is strictly orbitally bounded implies that the sets $\{T^n z\}_{n\in\{0,1,\dots,n_1-1\}}$ are bounded. Consequently, by the reasoning given above, we deduce that the set $\mathcal{O}_T(x)$ is bounded which gives the desired result.

Proposition 2.1 Let (X,d) be a complete metric space and $T : X \longrightarrow CB(X)$ be a closely continuously orbitally multi-valued mapping at $y \in X$. Then $\overline{\mathcal{O}_T(y)}$ is invariant under T.

Proof. Let $z \in \overline{\mathcal{O}_T(y)}$ then there exists a sequence $(z_n)_n \subset \mathcal{O}_T(y)$ such that $z_n \longrightarrow z$ in (X,d). Moreover, the fact that T is strictly continuously orbitally bounded shows that $Tz_n \longrightarrow Tz$ in $(CB(X),H)$. Now if $y \in Tz$, then

$$d(y,Tz_n) \le d(y,Tz) + H(Tz,Tz_n);$$

Since $H(Tz,Tz_n) \longrightarrow 0$ $(n \longrightarrow \infty)$ and $y \in Tz$ we get $d(y,Tz_n) \longrightarrow 0$ $(n \longrightarrow \infty)$. The definition of the infimum shows that for any integer $n \ge 1$, there exists $y_n \in Tz_n$ for which

$$d(y,y_n) \le d(y,Tz_n) + \tfrac{1}{n};$$

which implies that $d(y,y_n) \longrightarrow 0 (n \longrightarrow \infty)$. On the other hand since $(y_n)_n \subset \mathcal{O}_T(y)$, then $Tz_n \subset \mathcal{O}_T(y)$. This proves that $Tz \subset \overline{\mathcal{O}_T(y)}$ and gives the desired result.

3 Fixed Point Results

3.1 Caristi fixed point theorem and some consequences

It is well known that Caristi's fixed point theorem is one of the powerfull result existing in the fixed point theory. The multivalued version of this theorem was established by Mizoguchi-Takahashi [13] who proved that this version is equivalent to Eukeland's principle. Also, this theorem was used by [4, 13] to obtain some fixed point results and weakly inwardness property for some generalized contractions.

Theorem 3.1 [12] Let (X,d) be a complete metric space and let $T : X \longrightarrow CB(X)$ a multi-valued mapping such that for each $x \in X$, there exists $y \in Tx$ satisfying

$$\psi(y) + d(x,y) \le \psi(x),$$

where ψ is a proper, bounded below and lower semicontinuous function of X into $(-\infty,+\infty)$. Then T has a fixed point, that is there exists $z \in X$ such that $z \in Tz$.

In the next proposition, we will prove that multi-valued mapppings having the property $\mathcal{P}$ have fixed points if some additional assumptions on the function φ are required. More precisely, let (X,d) be a complete metric space and $T : X \longrightarrow CB(X)$ a multi-valued mapping having the property $\mathcal{P}$, we denote by $\widetilde{\varphi} : X \longrightarrow [0,+\infty[$ the function defined by

$$\widetilde{\varphi}(x) = \varphi(\{x\});$$

Thus we have

Proposition 3.1 Let (X,d) be a complete metric space and let $T : X \longrightarrow CB(X)$ a multi-valued mapping having the property $\mathcal{P}$ for which $\widetilde{\varphi}$ is lower semicontinuous and satisfies that for each $x \in X$ there exists $y \in Tx$ satisfying that $\varphi(Tx) \geq \widetilde{\varphi}(y)$. Then T has a fixed point, that is there exists $z \in X$ such that $z \in Tz$.

Proof. For each $x \in X$, we have

$$\begin{aligned} H(\{x\}, Tx) &\leq \varphi(\{x\}) - \varphi(Tx) \\ &\leq \widetilde{\varphi}(x) - \widetilde{\varphi}(y). \end{aligned}$$

On the other hand since $y \in Tx$, we have

$$d(x,y) \leq H(\{x\}, Tx) = \max\{d(x,Tx), \sup_{z \in Tx} d(z,x)\};$$

Hence, we get

$$d(x,y) \leq \widetilde{\varphi}(x) - \widetilde{\varphi}(y);$$

Now, the result follows immediately from Theorem 3.1.

3.2 Fixed point theorems for Hardy-Rogers mappings

First of all, we prove that Hardy-Rogers mappings are strictly orbitally bounded.

Theorem 3.2 Let (X,d) be a complete metric space and $T : X \longrightarrow CB(X)$ be such that

$$H(Tx,Ty) \leq k_1 d(x,y) + k_2(d(x,Tx) + d(y,Ty)) + k_3(d(x,Ty) + d(y,Tx)),$$

for $k_i \geq 0$ $(i = 1,2,3)$ such that $k_1 + 2k_2 + 2k_3 < 1$ and all $x, y \in X$. Then T is strictly orbitally bounded.

Proof. Let $x \in X$. It suffices to prove that T^2x is bounded, the case for $n > 2$ can be deduced by induction. Let $u_1 \in T^2x$, then there exists $w_1 \in Tx$ such that $u_1 \in Tw_1$. Now, if $u_2 \in T^2x$, then $u_2 \in Tw_2$ for some $w_2 \in Tx$.

Let $\epsilon > 0$, thus there exists $v_1 \in Tw_2$ such that

$$d(u_1,v_1) \leq H(Tw_1,Tw_2) + \epsilon.$$

Hence

$$\begin{aligned} d(u_1,u_2) &\leq d(u_1,v_1) + d(v_1,u_2) \\ &\leq H(Tw_1,Tw_2) + \epsilon + d(v_1,u_2) \\ &\leq k_1 d(w_1,w_2) + k_2(d(w_1,Tw_1) + d(w_2,Tw_2)) \\ &+ k_3(d(w_1,Tw_2) + d(w_2,Tw_1)) + \epsilon + d(v_1,u_2). \qquad (\star) \end{aligned}$$

Now, since $w_1 \in Tx$, we have

$$
\begin{aligned}
d(w_1,Tw_1) &\leq H(Tx,Tw_1)\\
&\leq k_1 d(x,w_1) + k_2(d(x,Tx) + d(w_1,Tw_1)) + k_3(d(x,Tw_1) + d(w_1,Tx))\\
&\leq k_1(d(x,Tx) + d(w_1,Tx)) + k_2(d(x,Tx) + d(w_1,Tw_1))\\
&+ k_3(d(x,Tw_1) + d(w_1,Tx))\\
&= k_1 d(x,Tx) + k_2(d(x,Tx) + d(w_1,Tw_1)) + k_3 d(x,Tw_1)\\
&\leq k_1 d(x,Tx) + k_2(d(x,Tx) + d(w_1,Tw_1) + k_3(d(x,Tx) + d(w_1,Tw_1)).
\end{aligned}
$$

Therefore

$$d(w_1,Tw_1) \leq \frac{k_1+k_2+k_3}{1-k_2-k_3} d(x,Tx)$$

By a similar way, we can show that

$$d(w_2,Tw_2) \leq \frac{k_1+k_2+k_3}{1-k_2-k_3} d(x,Tx)$$

By substituting in $(\star)$, we obtain that

$$
\begin{aligned}
d(u_1,u_2) &\leq k_1 d(w_1,w_2) + k_2(d(w_1,Tw_1) + d(w_2,Tw_2))\\
&+ k_3(d(w_1,Tw_2) + d(w_2,Tw_1)) + \epsilon + d(v_1,u_2)\\
&\leq (k_1+2k_3)\delta(Tx) + 2(k_2+k_3)(\frac{k_1+k_2+k_3}{1-k_2-k_3})d(x,Tx) + \epsilon + \delta(Tw_2).
\end{aligned}
$$

Since u_2 is arbitrary in T^2x, this shows that T^2x is included in the closed ball of center u_2 and radius $r = (k_1+2k_3)\delta(T(x)) + 2(k_2+k_3)(\frac{k_1+k_2+k_3}{1-k_2-k_3})d(x,Tx) + \epsilon + \delta(Tw_2)$ which gives the result.

Lemma 3.1 Let (X,d) be a complete metric space and $h_1,h_2,h_3 : X \times X \longrightarrow [0,\infty)$ three functions satisfying that

$$\sup\{h_1(x,y) + 2h_2(x,y) + 2h_3(x,y) : a \leq d(x,y) \leq b\} < 1,$$

for each finite closed interval $[a,b] \subset (0,\infty)$. Assume that if $(x_n,y_n) \in X \times X$ such that $\lim\limits_{n\longrightarrow\infty} d(x_n,y_n) = 0$ then $\lim\limits_{n\longrightarrow\infty}(h_1(x_n,y_n) + 2h_2(x_n,y_n) + 2h_3(x_n,y_n)) = \eta$ for some $\eta \in [0,1)$. Then

$$\sup\{h_1(x,y) + 2h_2(x,y) + 2h_3(x,y) : 0 \leq d(x,y) \leq b\} < 1.$$

Proof. Assume that $\sup\{h_1(x,y) + 2h_2(x,y) + 2h_3(x,y) : 0 \leq d(x,y) \leq b\} = 1$. Thus there exists $(x_n,y_n) \in X \times X$ such that

$$h_1(x_n,y_n) + 2h_2(x_n,y_n) + 2h_3(x_n,y_n) \longrightarrow 1.$$

Necessarily (x_n,y_n) has a subsequence (x_{n_k},y_{n_k}) such that $d(x_{n_k},y_{n_k}) \longrightarrow 0$, hence by assumption

$$\lim_{k\longrightarrow\infty}(h_1(x_{n_k},y_{n_k}) + 2h_2(x_{n_k},y_{n_k}) + 2h_3(x_{n_k},y_{n_k})) = \eta \text{ for some } \eta \in [0,1).$$

which is a contradiction.

Theorem 3.3 Let (X, d) be a complete metric space and let $T : X \longrightarrow CB(X)$ satisfying that

$$H(Tx, Ty) \leq h_1(x, y)d(x, y)+h_2(x, y)(d(x, Tx)+d(y, Ty))+h_3(x, y)(d(x, Ty)+d(y, Tx))$$

where $h_1, h_2, h_3 : X \times X \longrightarrow [0, \infty)$ three functions with

$$\sup\{h_1(x, y) + 2h_2(x, y) + 2h_3(x, y) : 0 \leq d(x, y) \leq b\} < 1,$$

Then T has the property $\mathcal{P}$.

Proof. Let $x \in X$. Following Theorem 2.1, it suffices to show that the series $\sum_{n=0}^{\infty} H(T^{n+1}x, T^n x)$ is convergent. First, we prove that the sequence $H(T^{n+1}x, T^n x)$ is decreasing.
Let $u_1 \in T^n x = \bigcup_{z \in T^{n-1}x} Tz$ and $T^{n+1}x = \bigcup_{v \in T^n x} Tv$. Then

$$\begin{aligned}
d(u_1, T^{n+1}x) &= \inf_{y \in T^{n+1}x} d(u_1, y) \\
&\leq d(u_1, Tv) \text{ for each } v \in T^n x \\
&\leq H(Tu, Tv) \text{ for some } u \in T^{n-1}x \\
&\leq h_1(u, v)d(u, v) + h_2(u, v)(d(u, Tu) + d(v, Tv)) + h_3(u, v)(d(v, Tu) + d(u, Tv)) \\
&\leq h_1(u, v)(d(u, T^n x) + d(v, T^n x)) + h_2(u, v)(d(u, T^n x) + d(Tu, T^n x)) \\
&+ d(v, T^n x) + d(Tv, T^n x)) + h_3(u, v)(d(v, T^n x) + d(Tu, T^n x) \\
&+ d(u, T^n x) + d(Tv, T^n x)) \\
&\leq h_1(u, v)d(u, T^n x) + h_2(u, v)(d(u, T^n x) + H(T^{n+1}x, T^n x)) \\
&+ h_3(u, v)(H(T^{n+1}x, T^n x) + d(u, T^n x)) \\
&\leq h_1(u, v)H(T^{n-1}x, T^n x) + h_2(u, v)(H(T^{n-1}x, T^n x)+ \\
&H(T^{n+1}x, T^n x)) + h_3(u, v)(H(T^{n+1}x, T^n x) + H(T^{n-1}x, T^n x)) \\
&\leq (h_1(u, v) + h_2(u, v) + h_3(u, v))H(T^{n-1}x, T^n x) \\
&+ (h_2(u, v) + h_3(u, v))H(T^{n+1}x, T^n x).
\end{aligned}$$

By a same way, we can prove that

$$\begin{aligned}
d(z, T^n x) \leq{} & (h_1(u, v) + h_2(u, v) + h_3(u, v))H(T^{n-1}x, T^n x) \\
& + (h_2(u, v) + h_3(u, v))H(T^{n+1}x, T^n x),
\end{aligned}$$

for all $z \in T^{n+1}x$. Hence

$$\begin{aligned}
H(T^n x, T^{n+1}x) \leq{} & (h_1(u, v) + h_2(u, v) + h_3(u, v))H(T^{n-1}x, T^n x) \\
& + (h_2(u, v) + h_3(u, v))H(T^{n+1}x, T^n x),
\end{aligned}$$

This gives that

$$H(T^n x, T^{n+1}x) \leq \frac{h_1(u, v) + h_2(u, v) + h_3(u, v)}{1 - h_2(u, v) - h_3(u, v))} H(T^{n-1}x, T^n x) \leq H(T^{n-1}x, T^n x),$$

Now, let $\epsilon > 0$, thus for every integer $n \geq 1$ and $y_n \in T^{n-1}x$, we can choose $z_n \in T^n x$ such that

$$d(y_n, z_n) \leq d(y_n, T^n x) + \epsilon, \tag{$\star\star$}$$

Hence

$$d(y_n, z_n) \leq H(T^{n-1}x, T^n x) + \epsilon,$$

Since ($\star\star$) is true for every $y_n \in T^{n-1}x$ and the fact that the sequence $H(T^n x, T^{n+1}x)$ is decreasing implies that the sequence $\{d(y_n, z_n)\}$ is bounded for all n.

Let $r_0 = \lim_{n \longrightarrow \infty} H(T^n x, T^{n+1}x)$. Since $\sup\{h_1(x,y)+2h_2(x,y)+2h_3(x,y) : 0 \leq d(x,y) \leq r_0+\epsilon\} < 1$, there exists $k \in [0,1)$ such that $\sup\{\dfrac{h_1(y_n,z_n)+h_2(y_n,z_n)+h_3(y_n,z_n)}{1-h_2(y_n,z_n)-h_3(y_n,z_n)}\} = k$.

It follows that for every $u_1 \in T^n x$,

$$\begin{aligned}
d(u_1, T^{n+1}x) &\leq h_1(y_n,z_n)d(y_n,z_n) + h_2(y_n,z_n)(d(y_n,Ty_n) + d(z_n,Tz_n)) \\
&+ h_3(y_n,z_n)(d(y_n,Tz_n) + d(z_n,Ty_n)) \\
&\leq h_1(y_n,z_n)(d(y_n,T^n x) + d(z_n,T^n x)) + h_2(y_n,z_n)(d(y_n,T^n x) \\
&+ d(T^n x,Ty_n) + d(z_n,T^n x) + d(Tz_n,T^n x)) + h_3(y_n,z_n)(d(y_n,T^n x) \\
&+ d(T^n x,Tz_n) + d(z_n,T^n x) + d(Ty_n,T^n x)) \\
&\leq h_1(y_n,z_n)H(T^{n-1}x,T^n x) + h_2(y_n,z_n)(H(T^{n-1}x,T^n x) \\
&+ H(T^n x,T^{n+1}x) + H(T^{n+1}x,T^n x)) + h_3(y_n,z_n)(H(T^{n-1}x,T^n x) \\
&+ H(T^n x,T^{n+1}x) + H(T^{n+1}x),T^n x)).
\end{aligned}$$

Similarly, we can prove that for each $u_2 \in T^{n+1}x$, we have

$$\begin{aligned}
d(u_2, T^n x) &\leq h_1(y_n,z_n)H(T^{n-1}x,T^n x) + h_2(y_n,z_n)(H(T^{n-1}x,T^n x) \\
&+ H(T^n x,T^{n+1}x) + H(T^{n+1}x,T^n x)) + h_3(y_n,z_n)(H(T^{n-1}x,T^n x) \\
&+ H(T^n x,T^{n+1}x) + H(T^{n+1}x,T^n x)).
\end{aligned}$$

Consequently, we get

$$\begin{aligned}
H(T^n x, T^{n+1}x) &\leq h_1(y_n,z_n)H(T^{n-1}x,T^n x) + h_2(y_n,z_n)(H(T^{n-1}x,T^n x) \\
&+ H(T^n x,T^{n+1}x) + H(T^{n+1}x,T^n x)) + h_3(y_n,z_n)(H(T^{n-1}x,T^n x) \\
&+ H(T^n x,T^{n+1}x) + H(T^{n+1}x,T^n x)).
\end{aligned}$$

Thus,

$$H(T^n x, T^{n+1}x) \leq kH(T^{n-1}x, T^n x)$$

and by induction, we deduce that

$$H(T^n x, T^{n+1}x) \leq k^n H(\{x\}, Tx)$$

and shows the convergence of the series $\sum_{n=0}^{\infty} H(T^n x, T^{n+1}x)$ which achieves the proof.

In the following result we will prove that the fixed point result for Hardy-Rogers orbitally continuous mappings is equivalent to the same result in the case of bounded complete metric spaces.

Theorem 3.4 [11, 7] Let (Y, d) be a complete metric space and $T : Y \longrightarrow CB(Y)$ be a closely orbitally continuous multi-valued mapping such that

$$H(Tx, Ty) \leq k_1 d(x, y) + k_2(d(x, Tx) + d(y, Ty)) + k_3(d(x, Ty) + d(y, Tx)),$$

for $k_i \geq 0$ $(i = 1, 2, 3)$ with $k_1 + 2k_2 + 2k_3 < 1$ and all $x, y \in Y$. Then there exists $z \in X$ such that $z \in Tz$.

Assume that the boundedness assumption on the metric space does not weaken the previous theorem. We give a proof of this equivalence after the following theorem

Theorem 3.5 Let (Y, d) be a bounded complete metric space and $T : Y \longrightarrow CB(Y)$ be a closely orbitally continuous multivalued mapping such that

$$H(Tx, Ty) \leq k_1 d(x, y) + k_2(d(x, Tx) + d(y, Ty)) + k_3(d(x, Ty) + d(y, Tx)),$$

for $k_i \geq 0$ $(i = 1, 2, 3)$ with $k_1 + 2k_2 + 2k_3 < 1$ and all $x, y \in Y$. Then there exists $z \in X$ such that $z \in Tz$.

Proof. It is easy to observe that Theorem 3.4 implies Theorem 3.5. For the converse, since T is closely orbitally continuous multivalued mapping, then Theorem 3.3 and 2.2 together with Proposition 2.1 show that the set $\overline{\mathcal{O}_x(T)}$ is bounded and invariant. Now, by taking $Y = \overline{\mathcal{O}_x(T)}$ in Theorem 3.5, we get the result.

***Remark* 3.1** It is be noted that generalized contractions with non constant coefficients are not in general Nadler's contractions (see Example 1 in [8]).

Corollary 3.1 Let (X, d) be a complete metric space and let $T : X \longrightarrow CB(X)$ be a closely orbitally continuous multi-valued mapping such that

$$\begin{aligned} H(Tx, Ty) \leq\ & h_1(x, y)d(x, y) + h_2(x, y)(d(x, Tx) + d(y, Ty)) \\ & + h_3(x, y)(d(x, Ty) + d(y, Tx)), \end{aligned}$$

such that $h_1, h_2, h_3 : X \times X \longrightarrow [0, \infty)$ are three functions satisfying that

$$\sup\{h_1(x, y) + 2h_2(x, y) + 2h_3(x, y) : a \leq d(x, y) \leq b\} < 1.$$

for each finite closed interval $[a, b] \subset (0, \infty)$. Assume that if $(x_n, y_n) \in X \times X$ is such that $\lim_{n\longrightarrow\infty} d(x_n, y_n) = 0$, then $\lim_{n\longrightarrow\infty} (h_1(x_n, y_n) + 2h_2(x_n, y_n) + 2h_3(x_n, y_n)) = k$ for some $k \in [0, 1)$. Then T has a fixed point in X.

Proof. From Lemma 3.1, we have

$$\sup\{h_1(x,y) + 2h_2(x,y) + 2h_3(x,y) : 0 \leq d(x,y) \leq b\} < 1$$

Since T is closely orbitally continuous, Proposition 2.1 shows that $\overline{\mathcal{O}_T(x)}$ is invariant under T for $x \in X$. Moreover, by Theorem 3.3, it is a bounded set. Thus T restricted to $\overline{\mathcal{O}_T(x)}$ reduces to a Hardy-Rogers multivalued mapping. Then following Theorem 3.5, T has a fixed point in X.

We note that our results extend and generalize in a certain sense those of [5]. Also, the results of this subsection can be adopted to obtain a same results concerning Kannan's and Reich's multivalued mappings [15] and as a corollaries of theses results, we can obtain

Corollary 3.2 Let (X,d) be a complete metric space and let $T : X \longrightarrow CB(X)$ be a closely orbitally continuous multi-valued mapping such that

$H(Tx,Ty) \leq h_1(x,y)d(x,y) + h_2(x,y)(d(x,Tx) + d(y,Ty)),$

where $h_1, h_2, h_3 : X \times X \longrightarrow [0,\infty)$ are two functions satisfying that

$$\sup\{h_1(x,y) + 2h_2(x,y) : a \leq d(x,y) \leq b\} < 1.$$

for each finite closed interval $[a,b] \subset (0,\infty)$. Assume that if $(x_n,y_n) \in X \times X$ is such that $\lim\limits_{n\longrightarrow\infty} d(x_n,y_n) = 0$, then $\lim\limits_{n\longrightarrow\infty}(h_1(x_n,y_n) + 2h_2(x_n,y_n)) = k$ for some $k \in [0,1)$. Then T has a fixed point in X.

Corollary 3.3 Let (X,d) be a complete metric space and let $T : X \longrightarrow CB(X)$ be a closely orbitally continuous multi-valued mapping such that

$H(Tx,Ty) \leq h_1(x,y)d(x,y) + h_2(x,y)(d(x,Ty) + d(y,Tx)),$

where $h_1, h_2 : X \times X \longrightarrow [0,\infty)$ are two functions satisfying that

$$\sup\{h_1(x,y) + 2h_2(x,y) : a \leq d(x,y) \leq b\} < 1.$$

for each finite closed interval $[a,b] \subset (0,\infty)$. Assume that if $(x_n,y_n) \in X \times X$ is such that $\lim\limits_{n\longrightarrow\infty} d(x_n,y_n) = 0$, then $\lim\limits_{n\longrightarrow\infty}(h_1(x_n,y_n) + 2h_2(x_n,y_n)) = k$ for some $k \in [0,1)$. Then T has a fixed point in X.

Competing Interests

The authors declare that there is no conflict of interest regarding the publication of this paper.

References

[1] J. P. Aubin, *Optima and Equilibria. An introduction to Nonlinear Analysis, Grad. Texts in Math. Springer-Verlag, Berlin,* 1998.

[2] J. Caristi, *Fixed points theorems for mappings satisfying inwardness conditions,* Trans. Amer. Math. Soc., 215 (1976), 241-251.

[3] T. H. Chang, *Common fixed point theorems for multi-valued mappings,* Math. Jap., 41 (1995), 311-320.

[4] P. Daffer and H. Kaneko, *Fixed points theorems of generalized contractive multi-valued mappings,* J. Math. Anal. Appl., 192 (1995), 655-666.

[5] A. A. Eldred, J. Anuradha and P. Veeramani, *On equivalence of generalized multi-valued contractions and Nadler's fixed point theorem,* J. Math. Anal. Appl., 336 (2007), 751-757.

[6] Y. Feng and S. Liu, *Fixed point theorems for multi-valued contractive mappings and multi-valued Caristi type mappings,* J. Math. Anal. Appl., 317 (2006), 103-112.

[7] M. E. Gordji, H. Baghani and M. Ramezani, *A generalization of Nadler's fixed point theorem,* The. Journ. Nonlinear. Sciences. Appl., 3 (2010), 148-151.

[8] M. E. Gordji, H. Baghani and M. Ramezani and H. Khodaei, *An extension of the Mizoguchi-Takahashi fixed point theorem,* Miskole. Mathematical. Notes., 13 (2) (2012), 317-323.

[9] G. E. Hardy and T. D. Rogers, *A generalization of a fixed point theorem of Reich,* Can. Math. Bull., 16 (1973), 201-206.

[10] K. Kuratowski, *Topology, 1,* PWN, Warszawa, 1966.

[11] K. Iseki, *Multi-valued contraction mappings in complete metric spaces,* Rendiconti. del. Semin. Mat. della. Univ. Padova., 53 (1975), 15-19.

[12] J. R. Jachymski, *Caristi's fixed point theorem and selections of set-valued contractions,* J. Math. Anal. Appl. 227 (1998), 55-67.

[13] N. Mizoguchi and W. Takahachi, *Fixed point theorems for multivalued mappingson complete metric spaces,* J. Math. Anal. Appl. 141 (1989), 177-188.

[14] S. B. Nadler Jr, *Multi-valued contraction mappings,* Pacific. J. Math., 30 (1969), 475-487.

[15] S. Reich, *Kannan's fixed point theorem,* Boll. Un. Mat. Ital., (4) (1971), 1-11.

[16] S. Reich, *Fixed points of contractive functions,* Boll. Un. Mat. Ital., (4) 5 (1972), 26-42.

[17] P. V. Semenov, *Fixed points of multivalued contractions,* Func. Anal. Appl., (36) 2 (2002), 159-161.

[18] T. Suzuki, *Mizoguchi-Takahashi's fixed point theorem is a real generalization of Nadler's,* J. Math. Anal. Appl., (340) 1 (2008), 752-755.

[19] J. D. Wetson, *A characterization of metric completeness,* Proc. Amer. Math. Soc., (64) 1 (1977), 186-188.

[20] C. Zhong, J. Zhu and P. Zhao, *An extension of multi-valued contraction mappings and fixed points,* Proc. Amer. Math. Soc., (8) (1999), 2439-2444.