On Some Fixed Point Results and Properties of $C_{\lambda_1,\lambda_2,.....,\lambda_n}$ Mappings

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Abstract

In this paper, we introduce the notion of $C_{\lambda_1,\lambda_2,.....,\lambda_n}$ mappings as an extension of Suzuki mappings. Some results concerning fixed points of theses mappings and the the convergence of Kirk's Process associated to them are studied.

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1 Introduction

It's well known that the works of W. A. Kirk, D. Gohde and F. E. Browder (1965-1966) [2, 8, 12] on the existence of fixed points for nonexpansive mappings have been of great impact on fixed point theory, by involving the geometry of Banach spaces in the study which was new since the famous Banach contraction principle (1922). The problem that a nonexpansive mapping on a convex bounded subset of an arbitrary Banach space has or not at least a fixed point is in general delicate, it needs a good knowledge of the geometry of the space or its closed bounded subsets as was the case of Hilbert spaces which is reflexive having the normal structure or others characterized by certain quantities linked to the norm of the space. The question of wether nonexpansive mappings defined on the closed bounded convex subsets of Banach spaces have or not fixed points is known under the abbreviation FPP (fixed point property). After the works mentioned above, L. A. Karlovitz (1976) [9] established the FPP property for spaces without normal structure, after, B. Maurey [15] showed that the space c_0 and each reflexive subspace of $L_1([0,1])$ have the property FPP, in 1981, D. Alpasch [1] gave a counter-example of a fixed point free nonexpansive mapping on a closed bounded convex subset of $L_1([0,1])$, notice that his example is an isometry and the problem is always open concerning the contractive case. In 1997, P. L. Dowling and C. J. Lennard [3] proved that in $L_1([0,1])$ the reflexivity of subspaces is a necessary and sufficient condition to have the property FPP. Notice that Goebel-Karlovitz [5, 6] Lemma and ultra-product spaces [10, 11] are a power full tools which have helped to overcome many problems in the theory. For a good read on this subject, we can quote [5, 6, 7, 10, 11, 14] and the references therein.

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In (2008), T. Suzuki [16] introduced the notion of $C_\lambda(\lambda \in (0,1))$-mappings as an extension of nonexpansive mappings. He showed that this class of mappings is wider than that of nonexpansive ones and he established some remarkable properties. In this work, we observe that the reasoning of T. Suzuki can be generalized to the case of $C_{\lambda_1,\lambda_2,\ldots,\lambda_n}$-mappings with $\sum_{i=1}^{n} \lambda_i \leq 1$ by noticing that the Krasnoselskii process used by Suzuki can be replaced by Kirk's process more general. Our results are extensions of some one given respectively in [13, 16].

2 Notations and Preliminaries

Definition 2.1 Let T be a mapping on a subset C of a Banach space X and $\lambda \in (0,1)$. T is said to satisfy condition C_λ if

$$\lambda\|x - Tx\| \leq \|x - y\| \Longrightarrow \|Tx - Ty\| \leq \|x - y\|$$

For $\lambda = \dfrac{1}{2}$, T is said to satisfy condition C or T is said to be Suzuki mapping. These classes of mappings are introduced by T. Suzuki as an extension of nonexpansive mappings and it is shown that we can construct a lot of C_λ mappings which are not nonexpansive. On the other hand is clear that if $\lambda_1 \leq \lambda_2$ thus C_{λ_1} implies C_{λ_2}. On the other hand, if C is convex and T satisfies condition C_λ for $\lambda \in (0,1)$, then for every $\alpha \in (\lambda, 1)$ the mapping $T_\alpha : C \longrightarrow C$ defined by $T_\alpha x = \alpha Tx + (1-\alpha)x$ satisfies condition $(C_{\frac{\lambda}{\alpha}})$. Also, notice that it is possible that the mapping is not C_λ but one of its powers satisfy this property as the following example indicated in [4].

***Example* 2.1** Define the mapping T on $[0,3]$ by

$$T(x) = \begin{cases} 0 & \text{if x } \neq 3 \\ 2 & \text{if x } = 3. \end{cases}$$

T does not satisfy the condition C, but it is easy to observe that $T^2 \equiv 0$ and hence T^2 satisfies trivially this condition.

Definition 2.2 Let $T : X \longrightarrow X$ be a mapping acting on a metric space (X, d) and let (x_n) be a sequence in X. (x_n) is said to be an approximate fixed point sequence for T if

$$\lim_{n \longrightarrow \infty} d(x_n, T(x_n)) = 0.$$

Definition 2.3 Let $T : X \longrightarrow X$ be a mapping acting on a metric space (X, d). T is said to be asymptotically regular if for every $x_0 \in C$, the sequence $x_n = T^n(x_0)$ is an approximate fixed point sequence for T.

Lemma 2.1 Let C be a bounded convex subset of a Banach space X. Assume that $T : C \longrightarrow C$ satisfies condition C_λ for $\lambda \in (0,1)$. For $\alpha \in (\lambda, 1)$ define a sequence (x_n) in C by taking $x_1 \in C$ and

$$x_{n+1} = \alpha T x_n + (1-\alpha)x_n, \text{for all } n \geq 1$$

Then (x_n) is an approximate fixed point sequence.

Recall that for $\alpha = \dfrac{1}{2}$, the sequence $x_{n+1} = \frac{1}{2}Tx_n + \frac{1}{2}x_n$ is called the Krasnoselskii process associated to T and (x_n).

***Remark* 2.1** Let C be a bounded convex subset of a Banach space X and let (x_n) be a sequence in C. If $\lambda \in (0,1)$, one of the advantages of Krasnoselskii process is that (x_n) is an approximate fixed point sequence for T if and only if (x_n) is an approximate fixed point sequence for $S = \lambda I + (1-\lambda)T$ (I is the identity mapping on C). Indeed, this fact follows immediately from the formula

$$\|S(x_n) - x_n\| = (1-\lambda)\|T(x_n) - x_n\|$$

which is not true for other processes.

Definition 2.4 A uniformly convex Banach space X is a Banach space such that for every $0 < \epsilon \leq 2$ there is some $0 < \delta$ such that for any two vectors x, y with $\|x\| = \|y\| = 1$, the condition $\|x - y\| \geq \epsilon$ implies $\dfrac{\|x+y\|}{2} \leq 1 - \delta$.

A strictly convex Banach space X is a Banach space such that for every $x, y \in X$, if $x \neq 0, y \neq 0$ and $\|x+y\| = \|x\| + \|y\|$ then necessarily we obtain that $x = cy$ for some $c > 0$.

It is known that every uniformly convex Banach space is strictly convex while the converse is not true in general (see [6]). Also uniformly Banach spaces are reflexive.

Definition 2.5 Let $T : C \longrightarrow C$ be a mapping acting on a convex subset C of a Banach space X. Let $\lambda_0, ..., \lambda_n \in (0,1)$ such that $\lambda_1 > 0$ and $\displaystyle\sum_{i=0}^{n} \lambda_i = 1$. The sequence $(x_m) \subset C$ defined by $x_1 \in C$ and

$$x_{m+1} = \lambda_0 x_m + \lambda_1 x_m + ... + \lambda_n T^n(x_m).$$

is called Kirk's process associated to the sequence (x_n) and the mapping T.

Definition 2.6 Let T be a mapping on a subset C of a Banach space X and $\lambda_1, ..., \lambda_n \in (0,1)$ such that $\displaystyle\sum_{i=1}^{n} \lambda_i < 1$ with $\lambda_1 > 0$. T is said to be $C_{\lambda_1, \lambda_2, ...\lambda_n}$ mapping if

$$\sum_{i=1}^{n} \lambda_i \|x - T^i x\| \leq \|x - y\| \Longrightarrow \sum_{i=0}^{n} \lambda_i \|T^i x - T^i y\| \leq \|x - y\|$$

where $\lambda_0 \in (0,1)$ is such that $\displaystyle\sum_{i=0}^{n} \lambda_i = 1$.

***Remark* 2.2** $C_{\lambda_1, \lambda_2, ..., \lambda_n}$ can be seen as an extension of $(\alpha = (\alpha_1, ..., \alpha_n))$ nonexpansive mappings introduced by M. J. Pineda for $\lambda_0 = 0$ (for more details, see [7]).

3 Main Results

Lemma 3.1 For $n = 1$ the previous definition is reduced to the case of C_λ mappings.

Proof. Indeed, if for $\lambda_1 \in (0, 1)$ we have

$$\lambda_1\|x - Tx\| \leq \|x - y\| \Longrightarrow \sum_{i=0}^{1} \lambda_i\|T^i x - T^i y\| \leq \|x - y\|$$

Then if $\lambda_0 \in (0, 1)$ for which $\lambda_0 + \lambda_1 = 1$ we obtain that

$$\lambda_0\|x - y\| + \lambda_1\|Tx - Ty\| \leq \|x - y\| = \lambda_0\|x - y\| + \lambda_1\|x - y\|$$

It follows that

$$\lambda_1\|Tx - Ty\| \leq \lambda_1\|x - y\|$$

Since, $\lambda_1 \in (0, 1)$ then $\lambda_1 \neq 0$, we get

$$\|Tx - Ty\| \leq \|x - y\|$$

which is the desired result.

Proposition 3.1 Let T be a mapping on a subset C of a Banach space X and let $\lambda_1, ..., \lambda_n \in (0, 1)$ for which $\lambda_1 > 0$ and $\sum_{i=1}^{n} \lambda_i < 1$. Assume that for each $i = 1, ...n$, T^i is a C_{λ_i} mapping. Then T is a $C_{\lambda_1,\lambda_2,.....,\lambda_n}$ mapping.

Proof. Assume that

$$\sum_{i=1}^{n} \lambda_i\|x - T^i x\| \leq \|x - y\|$$

For every $i = 1, ..., n$, this implies that

$$\lambda_i\|x - T^i x\| \leq \|x - y\|$$

By hypothesis, the fact that T^i is a C_{λ_i} mapping implies that

$$\|T^i x - T^i y\| \leq \|x - y\|$$

and hence

$$\lambda_i\|T^i x - T^i y\| \leq \lambda_i\|x - y\| \text{ for all } i = 1, ..., n$$

Consequently, by summation and using the fact that $\sum_{i=0}^{n} \lambda_i = 1$, we infer that

$$\sum_{i=0}^{n} \lambda_i\|T^i x - T^i y\| \leq \|x - y\|$$

which is the desired result.

Corollary 3.1 Let T be a nonexpansive mapping on a subset C of a Banach space X. Then T is a $C_{\lambda_1,\lambda_2,.....,\lambda_n}$ mapping for every $\lambda_1, ..., \lambda_n \in (0,1)$ with $\lambda_1 > 0$ and $\sum_{i=1}^{n} \lambda_i < 1$.

Proof. For every integer $i \geq 1$ we have

$$\|T^i x - T^i y\| \leq ...... \leq \|x - y\|$$

which implies that T^i is a nonexpansive mapping for all integer $i \geq 1$. Hence for all $i = 1, ..., n$, T^i ia a C_{λ_i} mapping. Now the result is an immediate consequence of Proposition 1.1.

Lemma 3.2 *(see [16], Lemma 5)* Let T be mapping on a subset C of a Banach space X. Assume that T is a C mapping. Then for every $x, y \in C$, the following hold.

($\imath$) $\|Tx - T^2x\| \leq \|x - Tx\|$.

($\imath\imath$) Either $\frac{1}{2}\|x - Tx\| \leq \|x - y\|$ or $\frac{1}{2}\|Tx - T^2x\| \leq \|Tx - y\|$ holds.

($\imath\imath\imath$) Either $\|Tx - Ty\| \leq \|x - y\|$ or $\|T^2x - Ty\| \leq \|Tx - y\|$.

As an extension of Lemma 1.3, we have the following result concerning $C_{\lambda_1,......,\lambda_n}$ mappings.

Lemma 3.3 Let T be mapping on a subset C of a Banach space X and let $\lambda_0, ..., \lambda_n \in (0,1)$ and $\sum_{i=0}^{n} \lambda_i = 1$. Assume that T is a $C_{\lambda_1,\lambda_2,.....,\lambda_n}$ mapping. Then for every $x, y \in C$, the following hold.

($\imath$) $\sum_{i=0}^{n} \lambda_i \|T^i x - T^{i+n(x)}x\| \leq \|x - T^{n(x)}x\|$ where $n(x)$ is the integer such that $\|x - T^n(x)\| = \max\{\|x - T(x)\|, ..., \|x - T^n(x)\|\}$.

($\imath\imath$) If $\lambda_0 \geq \frac{1}{2}$. Then, either $\sum_{i=1}^{n} \lambda_i \|x - T^i x\| \leq \|x - y\|$ or $\sum_{i=1}^{n} \lambda_i \|T^i x - T^{i+n(x)}x\| \leq \|T^{n(x)}x - y\|$

($\imath\imath\imath$) If $\lambda_0 \geq \frac{1}{2}$. Then either $\sum_{i=0}^{n} \lambda_i \|T^i x - T^i y\| \leq \|x - y\|$ or $\sum_{i=0}^{n} \lambda_i \|T^i x - T^{i+n(x)}y\| \leq \|T^{n(x)}x - y\| + \sum_{i=0}^{n} \lambda_i \|T^{i+n(x)}x - T^{i+n(x)}y\|$.

Proof. ($\imath$) Let $x \in C$. Assume that $n(x) \geq 1$ is the integer for which $\|x - T^{n(x)}(x)\| = \max\{\|x - T(x)\|, ..., \|x - T^n(x)\|\}$. Thus from the inequality

$$\sum_{i=1}^{n} \lambda_i \|x - T^i x\| \leq \|x - T^{n(x)}x\|$$

it follows that

$$\sum_{i=0}^{n} \lambda_i \|T^i x - T^{i+n(x)} x\| \leq \|x - T^{n(x)} x\|$$

$(\imath\imath)$ Arguing by contradiction and assume that

$$\sum_{i=1}^{n} \lambda_i \|x - T^i x\| > \|x - y\| \text{ and } \sum_{i=1}^{n} \lambda_i \|T^i x - T^{i+n(x)} x\| > \|T^{n(x)} x - y\|$$

Hence, it follows that

$$\|x - T^{n(x)} x\| \leq \|x - y\| + \|T^{n(x)} x - y\|$$
$$< \sum_{i=1}^{n} \lambda_i \|x - T^i x\| + \sum_{i=1}^{n} \lambda_i \|T^i x - T^{i+n(x)} x\|$$

By using $(\imath)$, it follows that

$$\|x - T^{n(x)} x\| < 2(1 - \lambda_0) \|x - T^{n(x)} x\|$$

Since $\lambda_0 \geq \frac{1}{2}$ we get a contradiction.

$(\imath\imath\imath)$ follows immediately from $(\imath\imath)$. Indeed, if $\sum_{i=1}^{n} \lambda_i \|x - T^i x\| \leq \|x - y\|$, the fact that T is a $C_{\lambda_1, \lambda_2, \ldots\ldots, \lambda_n}$ mapping, we get that $\sum_{i=0}^{n} \lambda_i \|T^i x - T^i y\| \leq \|x - y\|$.

Now if $\sum_{i=1}^{n} \lambda_i \|T^i x - T^{i+n(x)} x\| \leq \|T^{n(x)} x - y\|$, by the triangular inequality and using $(\imath)$ we obtain that

$$\sum_{i=0}^{n} \lambda_i \|T^i x - T^{i+n(x)} y\| \leq \sum_{i=0}^{n} \lambda_i \|T^i x - T^{i+n(x)} x\| + \sum_{i=0}^{n} \lambda_i \|T^{i+n(x)} x - T^{i+n(x)} y\|$$
$$\leq \|x - T^{n(x)} x\| + \sum_{i=0}^{n} \lambda_i \|T^{i+n(x)} x - T^{i+n(x)} y\|$$

***Remark* 3.1** As it was indicated above, Lemma 1.3 is a particular case of Lemma 1.4 by taking $n = 1$ and $n(x) = 1$ for all $x \in C$.

T be a mapping on a subset C of a Banach space X. We denote by $F(T)$ the set of fixed points of T.

Lemma 3.4 Let T be mapping on a subset C of a Banach space X and let $\lambda_0, \ldots\ldots, \lambda_n \in (0, 1)$ with $\lambda_1 > 0$ and $\sum_{i=0}^{n} \lambda_i = 1$. Assume that T is a C_λ mapping. Let P is the real polynomial given by

$$P(x) = \lambda_0 + \ldots. + \lambda_n x^n.$$

Assume that for every $x_0 \in F(P(T))$, the first alternative in $(\imath\imath\imath)$ of Lemma 1.3 holds for all $x, y \in O(x_0)$. Then $F(T) = F(P(T))$.

Proof. If $x_0 \in F(T)$, it's clear that $x_0 \in F(P(T))$. Now assume that $x_0 \in F(P(T))$ and we denote by

$$\delta = \max\{\|T^i x - T^j x\|, i, j = 0, ..., n, i \neq j\}.$$

The fact that the first alternative of the assertion $(\imath\imath)$ in Lemma 1.3 holds for every $x, y \in O(x_0)$ implies the existence of a smallest integer $m_0 \geq 1$ such that $\delta = \|x_0 - T^{m_0}x_0\|$. Assume that $\delta > 0$. It is easy to write x_0 under the form

$$x_0 = \alpha T x_0 + (1-\alpha)z.$$

where $z \in conv\{T^2 x_0, ..., T^n x_0\}$. It follows that

$$\delta = \|x_0 - T^{m_0}x_0\| \leq \|\alpha T x_0 + (1-\alpha)z - T^{m_0}x_0\|.$$
$$\leq \alpha\|Tx_0 - T^{m_0}x_0\| + (1-\alpha)\|z - T^{m_0}x_0\|.$$

This proves that $\delta \leq \|Tx_0 - T^{m_0}x_0\| \leq \|x_0 - T^{m_0-1}x_0\|$ which is a contradiction if $m_0 > 1$. Hence $m_0 = 1$, also here we obtain a contraction since we get $\delta \leq \|Tx_0 - Tx_0\| = 0$. Hence necessarily $\delta = 0$ and $Tx_0 = x_0$ which proves that $x_0 \in F(T)$ which is the desired result.

***Remark* 3.2** Let T be mapping on a subset C of a Banach space X. Assume that T is a C_λ mapping. If for every $x_0 \in C$ and let $m_0, m_1 (m_0 < m_1)$ the smallest integers such that $\delta = \|T^{m_0}x - T^{m_1}x\| = \max\{\|T^j x - T^k x\|\}, j, k = 0, 1, ...n$. Then by $(\imath)$ of Lemma 1.3, necessarily $m_1 \neq m_0 + 1$.

Proposition 3.2 Let T be mapping on a subset C of a Banach space X and let $\lambda_0, ..., \lambda_n \in (0,1)$ with $\lambda_1 > 0$ and $\sum_{i=0}^{n} \lambda_i = 1$. Assume that T is a $C_{\lambda_1,\lambda_2,.....,\lambda_n}$ mapping which has a fixed point. Then the mapping $\sum_{i=0}^{n} \lambda_i T^i$ is quasinonexpansive.

Proof. If x_0 a fixed point for T, then for every integer $i \geq 1$, x_0 is a fixed point for T^i, then for $x \in C$, since

$$\sum_{i=1}^{n} \lambda_i \|x_0 - T^i x_0\| = 0 \leq \|x_0 - x\|$$

The fact that T is a $C_{\lambda_1,\lambda_2,.....,\lambda_n}$ mapping gives that

$$\|x_0 - \sum_{i=0}^{n} \lambda_i T^i x\| = \|\sum_{i=0}^{n} \lambda_i T^i x_0 - \sum_{i=0}^{n} \lambda_i T^i x\|$$
$$\leq \|x - x_0\|$$

which is the desired result.

Proposition 3.3 Let T be mapping on a subset C of a Banach space X and let $\lambda_0, ..., \lambda_n \in (0,1)$ with $\lambda_1 > 0$ and $\sum_{i=0}^{n} \lambda_i = 1$. Assume that T is a $C_{\lambda_1,\lambda_2,.....,\lambda_n}$ mapping.

($\imath$) Then the set $F(T)$ is closed.

($\imath\imath$) Assume that C is convex and the Banach space X is strictly convex. If $F(T) = F(P(T))$, then $F(T)$ is convex.

$$P(x) = \lambda_0 + ... + \lambda_n x^n.$$

Proof. Let (x_n) be a sequence in $F(T)$ converging to some $x_0 \in C$. Thus $(x_n) \subseteq F(T^k)$ for all integer $k = 2, ..., n$. Since $\sum_{i=1}^{n} \lambda_i \|x_n - T^i x_n\| = 0 \leq \|x_n - x_0\|$ for $n \in \mathbb{N}$. The fact that T is a $C_{\lambda_1,\lambda_2,.....,\lambda_n}$ mapping implies that

Then

$$\limsup_{m \longrightarrow \infty} \sum_{i=0}^{n} \lambda_i \|T^i x_m - T^i x_0\| \leq \limsup_{m \longrightarrow \infty} \|x_m - x_0\| = 0.$$

Since $\lambda_1 > 0$, it follows that for $i = 1$, we get

$$\limsup_{m \longrightarrow \infty} \|x_m - T x_0\| = 0.$$

Hence, the sequence $\{x_n\}$ converges to Tx_0 which gives that $Tx_0 = x_0$.

($\imath\imath$) Let $x, y \in F(T) = F(P(T))$ with $x \neq y$, put $z = \lambda_1 x + (1 - \lambda_1) y \in C$. Thus we have

$$\begin{aligned} \|x - y\| &\leq \|x - P(T)(z)\| + \|P(T)(z) - y\| \\ &\leq \sum_{i=0}^{n} \lambda_i \|x - T^i(z)\| + \sum_{i=0}^{n} \lambda_i \|y - T^i(z)\|. \\ &= \sum_{i=0}^{n} \lambda_i \|T^i x - T^i(z)\| + \sum_{i=0}^{n} \lambda_i \|T^i y - T^i(z)\|. \end{aligned}$$

Since T is a $C_{\lambda_1,....,\lambda_n}$ mapping and $\sum_{i=1}^{n} \lambda_i \|x - T^i(x)\| = \sum_{i=1}^{n} \lambda_i \|y - T^i(y)\| = 0$. It follows that

$$\sum_{i=0}^{n} \lambda_i \|T^i x - T^i(z)\| \leq \|x - z\| \quad \text{and} \quad \sum_{i=0}^{n} \lambda_i \|T^i y - T^i(z)\| \leq \|y - z\|.$$

This implies that

$$\|x - y\| \leq \|x - P(T)(z)\| + \|P(T)(z) - y\| \leq \|x - z\| + \|y - z\|.$$

Since X is strict convex, there exists $\lambda_2 \in [0, 1]$ such that $P(T(z)) = \lambda_2 x + (1 - \lambda_2) y$. A similar argument as above shows that

$$(1 - \lambda_1)\|x - y\| = \|P(T)(x) - P(T)(y)\| \leq (1 - \lambda_2)\|x - y\|.$$

and

$$\lambda_2 \|x - y\| = \|P(T)(x) - P(T)(y)\| \leq \lambda_1 \|x - y\|.$$

So we obtain $1 - \lambda_2 \leq 1 - \lambda_1$ and $\lambda_2 \leq \lambda_1$ which gives that $\lambda_1 = \lambda_2$. Consequently, $z \in F(P(T)) = F(T)$ which is the desired result.

4 Convergence of Kirk's process

This section is devoted to the study of Kirk's process and to extend some of results given in [13].

Lemma 4.1 Let T be mapping on a convex subset C of a uniformly convex Banach space X and let $\lambda_0, ..., \lambda_n \in (0,1)$ with $\lambda_1 > 0$ and $\sum_{i=0}^{n} \lambda_i = 1$. If T has at least a fixed point in C and T^i is a C_{λ_i} mapping for every integer $i \geq 1$. Then for every $x_0 \in C$, the sequence $\{x_n\}_{n\geq 1}$ defined by Kirk's process is an approximate fixed point sequence for the mapping $S = P(T) = \sum_{i=0}^{n} \lambda_i T^i$.

Proof. For $x_0 \in C$. Define the sequence $\{x_n\}$ by $x_{n+1} = \sum_{i=0}^{n} \lambda_i T^i(x_0)$. Assume that z is a fixed point for T in C. Then z is a fixed point for the mappings T^i for every integer $i = 2, ..., n$. Thus

$$\|x_{n+1} - z\| = \| \sum_{i=0}^{n} \lambda_i T^i(x_n) - \sum_{i=0}^{n} \lambda_i z\|$$

On the other hand, since for every integer $i \geq 1$, we have

$$\lambda_i \|T^i(z) - z\| = 0 \leq \|z - x_n\|$$

It follows that

$$\|T^i(x_n) - T^i z\| \leq \|z - x_n\|$$

Hence

$$\sum_{i=0}^{n} \lambda_i \|T^i(x_n) - T^i z\| \leq \sum_{i=0}^{n} \lambda_i \|x_n - z\| = \|x_n - z\|.$$

which proves that $\{\|x_n - z\|\}$ is a decreasing sequence. Then $\lim_{n\longrightarrow\infty} \|x_n - z\| = l \geq 0$.

Thus

$$S(x_n) - z = \sum_{i=0}^{n} \lambda_i T^i(x_n) - z.$$

$$= \lambda_0(x_n - z) + (1 - \lambda_0) y_n.$$

such that

$$y_n = \frac{1}{(1-\lambda_0)} \sum_{i=1}^{n} \lambda_i (T^i(x_n) - z).$$

Since

$$\|T^i(x_n) - z\| = \|T^i(x_n) - T^i(z)\| \leq \|x_n - z\|.$$

Moreover, since $\sum_{i=0}^{n} \lambda_i = 1$, we get $\limsup \|z_n\| \leq l$. Afterwards, the fact that $\lim_{n\longrightarrow\infty} \|x_n - d\| = l$, $\lim_{n\longrightarrow\infty} \|x_{n+1} - d\| = l$.

The uniform convexity of the space X implies that

$$\lim_{n\longrightarrow\infty} \|x_n - z - y_n\| = 0.$$

Since $S(x_n) - x_n = x_{n+1} - x_n = (1-\lambda_0)(x_n - z - y_n)$ and $\lim_{n\longrightarrow\infty} \|x_n - z - y_n\| = 0.$, we obtain

$$\lim_{n\longrightarrow\infty} \|x_{n+1} - x_n\| = 0.$$

which is the desired result.

Lemma 4.2 Let T be compact continuous mapping on a uniformly convex Banach space X and let $\lambda_0, ..., \lambda_n \in (0,1)$ with $\lambda_1 > 0$ and $\sum_{i=0}^{n} \lambda_i = 1$ and T^i is a C_{λ_i} mapping for every integer $i \geq 1$. Assume that the following holds:

$(\imath)$ T has at least one fixed point,

$(\imath\imath)$ For every $z_0 \in F(P(T))$, the first alternative in $(\imath\imath\imath)$ of Lemma 1.3 holds for all $x, y \in O(z_0)$.

Then for each $x_0 \in X$ the sequence $\{S^n(x_0)\}$ converges to a fixed point of T.

Proof. Lemma 1.5 shows that $F(T) = F(P(T))$. Moreover, from Lemma 2.1, we deduce that S is asymptotically regular, now the rest of the proof is the same as that given in Corollary of [13].

Definition 4.1 A mapping $T : C \longrightarrow C$ is said to be demiclosed at $y \in C$ if $Tx = y$ whenever $(x_n) \subset C$ with x_n converges weakly to x and $Tx_n \longrightarrow y$. T is said to be demiclosed if T is demiclosed at any point of C.

By adapting the same techniques in the proof of Theorem 3 in [13] we obtain the following result.

Proposition 4.1 Let $T : C \longrightarrow C$ be a mapping on a closed bounded convex subset of a uniformly convex Banach space X and let $\lambda_0, ..., \lambda_n \in (0,1)$ with $\lambda_1 > 0$ and $\sum_{i=0}^{n} \lambda_i = 1$ and T^i is a C_{λ_i} mapping for every integer $i \geq 1$. Assume that the following holds:

$(\imath)$ T has at most one fixed point $z_0 \in C$,

$(\imath\imath)$ the mapping $I - S$ is demiclosed,

$(\imath\imath\imath)$ For every $z_0 \in F(P(T))$, the first alternative in $(\imath\imath\imath)$ of Lemma 1.3 holds for all $x, y \in O(z_0)$.

Then for each $x_0 \in C$ the sequence $\{S^n(x_0)\}$ converges weakly to $z_0 \in C$.

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