Orthogonality and Some Fixed Point Results For Generalized Nonexpansive Mappings

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Abstract

In this note, we study some fixed point results for generalized nonexpansive mappings containing in particular C_λ mappings (called also Suzuki mappings) by means of the notion of orthogonality in Banach spaces.

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1 Introduction

Let X be a Banach space, we say that X has the property FPP (fixed point property) if for every convex weakly compact subset C of X and every nonexpansive selfmapping T on C, T has a fixed point in C. The first works in this direction were established by [5, 10, 15] and they have been of great benefit showing the close link between the study of existence of fixed points and the geometry of Banach spaces. Since, the subject has attracted the attention of several mathematicians who contributed to establish pertinent results, we can quote for example, Goebel-Karlovitz Lemma [8, 9, 12, 13] proving that the existence of an approximatively fixed point sequence for a nonexpansive selfmapping on a convex weakly compact subset C implies that every point of C is diametral, this result is a crucial tool in the theory on which are based many well known results. The fact that the space $L_1([0,1])$ has not the property FPP proved by D. Alspach [4] is a striking result, it appeared one year after another remarkable result established by B. Maurey [17] claiming that every closed reflexive subspace of $L_1([0,1])$ has the property FPP. In 1985, P. K. Lin [16] showed that if X has an unconditional basis with a basic constant less than $\dfrac{\sqrt{33}-3}{2}$ then X has the property FPP. In 1997, P. N. Dowling and C. J. Lennard [6] proved that the reflexivity is necessary and sufficient for a closed subspace of $L_1([0,1])$ to have the property FPP. In 2008, T. Suzuki [7, 18] has defined the notion of C_λ mappings and he showed that this class contains strictly that of nonexpansive mappings. This class of mappings was explored by several authors (for example see [1] and the references therein). Moreover,

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it was shown that many results which hold for nonexpansive mappings can be extended to the case of C_λ mappings. In this paper, we work in this direction and we prove in particular that the results of L. A. Karlovitz [12] hold for the case of continuous C_λ mappings.

2 Notations and Preliminaries

Definition 2.1 Let X be a normed space and $x_1, x_2 \in X$. We say that x_1 is orthogonal to x_2 and we denote $x_1 \perp x_2$ if $\|x_1\| \leq \|x_1 + \lambda x_2\|$ for all scalars λ.

In the following, we denote by $D(0,\epsilon)$ the open disc with center 0 and radius ϵ in the complex plane $\mathbb{C}$.

Definition 2.2 Let X be a normed space and S_X its unit sphere. Let $x_1, x_2 \in X$; we say that the relation $\perp$ is approximatively symmetric if for each $x \in X$ and each $\epsilon > 0$ there exists a finite codimensional subspace $V_{x,\epsilon}$ of X (which depends on x and ϵ) such that

$$\|v\| \leq \|v + \lambda x\| \qquad \forall v \in V_{x,\epsilon} \bigcap S_X \text{ and } \forall \lambda \notin D(0,\epsilon). \tag{$\star$}$$

Definition 2.3 Let X be a dual space, in other words, there exists a normed space Y such that $Y = X^\star$. We say that the relation $\perp$ is weak* approximatively symmetric if $V_{x,\epsilon}$ in Definition 2.2 can be chosen weak* closed.

Definition 2.4 Let X be a normed space.

($\imath$) We say that the relation $\perp$ is uniformly approximatively symmetric if it is approximatively symmetric and ($\star$) is replaced by the following:

$$\|v\| \leq \|v + \lambda x\| - \delta, \text{ for some } \delta = \delta(x,\epsilon) > 0, \qquad \forall v \in V_{x,\epsilon} \bigcap S_X \text{ and } \forall \lambda \notin D(0,\epsilon). \tag{$\star\star$}$$

($\imath\imath$) If X is a dual space. Then $\perp$ is said to be uniformly weak* approximatively symmetric if it is weak* approximatively symmetric and ($\star\star$) is satisfied.

***Example* 2.1** As examples of Banach spaces satisfying the previous properties, we have

1. If X is one of the following Banach spaces, then the relation $\perp$ is uniformly approximately symmetric.

   (a) Hilbert spaces.

   (b) l_p spaces ($1 < p < \infty$).

2. If X is one of the following Banach spaces, then the relation $\perp$ is weak* uniformly approximately symmetric.

(c) James space.

(d) l_1 space.

3. In L_p spaces $p \neq 2$ and c_0, the relation $\perp$ fails to be approximatively symmetric.

For more details on these notions of orthogonality, we can see for example [3, 11].

Definition 2.5 Let T be a mapping on a subset C of a Banach space X and $\lambda \in (0,1)$. T is said to satisfy condition C_λ if

$$\lambda\|x - Tx\| \leq \|x - y\| \Longrightarrow \|Tx - Ty\| \leq \|x - y\|$$

For $\lambda = \dfrac{1}{2}$, T is said to satisfy condition C or T is said to be Suzuki mapping. These classes of mappings are introduced by T. Suzuki [18] as an extension of nonexpansive mappings and it is shown that we can construct a lot of C_λ mappings which are not nonexpansive. It is clear that if $\lambda_1 \leq \lambda_2$ thus C_{λ_1} implies C_{λ_2}. On the other hand, if C is convex and T satisfies condition C_λ for $\lambda \in (0,1)$, then for every $\alpha \in (\lambda, 1)$ the mapping $T_\alpha : C \longrightarrow C$ defined by $T_\alpha x = \alpha Tx + (1-\alpha)x$ satisfies condition $(C_{\frac{\lambda}{\alpha}})$.

Definition 2.6 Let $T : X \longrightarrow X$ be a mapping acting on a metric space (X, d) and let (x_n) be a sequence in X. (x_n) is said to be an approximate fixed point sequence for T if

$$\lim_{n \longrightarrow \infty} d(x_n, T(x_n)) = 0.$$

Definition 2.7 Let $T : X \longrightarrow X$ be a mapping acting on a metric space (X, d). T is said to be asymptotically regular if for every $x_0 \in C$, the sequence $x_n = T^n(x_0)$ is an approximate fixed point sequence for T.

Lemma 2.1 *(see [1, 2])* Let C be a bounded convex subset of a Banach space X. Assume that $T : C \longrightarrow C$ satisfies condition C_λ for $\lambda \in (0,1)$. For $\alpha \in (\lambda, 1)$ define a sequence (x_n) in C by taking $x_1 \in C$ and

$$x_{n+1} = \alpha T x_n + (1-\alpha)x_n, \text{for all } n \geq 1$$

Then (x_n) is an approximate fixed point sequence.

Lemma 2.2 *(see [1, 2, 7])* Let C be a nonempty convex weakly compact subset of a Banach space X which is minimal and invariant under the mappaing $T : C \longrightarrow C$. If T is continuous and C_λ mapping for some $\lambda \in (0,1)$, then there exists $r \geq 0$ such that for any approximate fixed point sequence for T and every $x \in C$ we have

$$\lim_{n \longrightarrow \infty} \|x_n - x\| = r.$$

In the case $\lambda = \dfrac{1}{2}$, the continuity assumption can be dropped.

Definition 2.8 Let C be a nonempty subset of a Banach space X. We say that $T : C \longrightarrow C$ satisfy condition E_μ on C if there exists $\mu \geq 1$ such that for all $x, y \in C$, we have

$$\|x - Ty\| \leq \mu\|x - Tx\| + \|x - y\|.$$

T is said to be satisfy the condition E on C if there exists a certain $\mu \geq 1$ such that T satisfies E_μ.

Remark **2.1** It is easy to show that every nonexpansive mapping satisfies condition E_1 but the converse is not true. Moreover, every $C_{\frac{1}{2}}$ mapping satisfies condition E_3 (for more details, see Definition 2 in [7]).

Definition 2.9 A Banach space X is said to satisfy the Opial property whenever for every sequence (x_n) with x_n converges weakly to z (denoted by $x_n \rightharpoonup z$) we have

$$\liminf_{n \longrightarrow \infty} \|x_n - z\| < \liminf_{n \longrightarrow \infty} \|x_n - x\|$$

whenever $x \neq z$.

Example **2.2** Hilbert spaces $l_p(1 \leq p < \infty)$ satisfy Opial property. On the other hand, it is known that every separable Banach space can be renormed to satisfy Opial property (see [19]).

3 Main Results

We start this section by the following Lemma which will be used in the rest of the paper.

Lemma 3.1 Let T be mapping on a subset C of a Banach space X. Assume that T is a C_λ mapping ($\lambda \in (0, 1)$). Then for every $x, y \in C$, the following hold.

($\imath$) $\|Tx - T^2x\| \leq \|x - Tx\|$.

($\imath\imath$) Either $\lambda\|x - Tx\| \leq \|x - y\|$ or $(1 - \lambda)\|Tx - T^2x\| \leq \|Tx - y\|$ holds.

($\imath\imath\imath$) If moreover T is $C_{1-\lambda}$ mapping. Then, either $\|Tx - Ty\| \leq \|x - y\|$ or $\|T^2x - Ty\| \leq \|Tx - y\|$.

Proof.

($\imath$) For $\lambda \in (0, 1)$ we have $\lambda\|x - Tx\| \leq \|x - Tx\|$. Since T is C_λ mapping we get $\|Tx - T^2x\| \leq \|x - Tx\|$.

($\imath\imath$) Assume that $\lambda\|x - Tx\| > \|x - y\|$ and $(1 - \lambda)\|Tx - T^2x\| > \|Tx - y\|$. Thus

$$\begin{aligned}\|x - Tx\| \leq& \|x - y\| + \|y - Tx\| \\ <& \lambda\|x - Tx\| + (1 - \lambda)\|T^2x - Tx\|.\end{aligned}$$

By $(\imath)$ it follows that

$$\begin{aligned}\|x-Tx\| <&\lambda\|x-Tx\|+(1-\lambda)\|x-Tx\| \\ =&\|x-Tx\|\end{aligned}$$

which is a contradiction.

$(\imath\imath\imath)$ Follows directly from $(\imath\imath)$.

***Remark* 3.1** By taking $\lambda = \dfrac{1}{2}$ in Lemma 2.1, Lemma 5 in [18] can be deduced. On the other hand for each $\lambda \in (0, \dfrac{1}{2}]$ then if T is C_λ mapping then necessarily T is $C_{1-\lambda}$ mapping.

Lemma 3.2 Let T be mapping on a subset C of a Banach space X. Assume that T is a C_λ and $C_{1-\lambda}$ mapping ($\lambda \in (0,1)$). Then T satisfies condition E_3.

Proof. The proof of this lemma can be adapted from that given in Lemma 7 of [18].

Theorem 3.1 Let C be a weakly compact convex subset of a Banach space X. Assume that the relation $\perp$ is uniformly approximately symmetric in X. If $T : C \longrightarrow C$ is a continuous C_λ and $C_{1-\lambda}$ mapping, then T has a fixed point.

Proof. Assume that T is a free fixed point mapping and define

$$\Xi = \{K \subset C, K \neq \emptyset, \text{closed convex and } TK \subset K\}$$

Using Zorn's Lemma, it follows that the family Ξ has a minimal element (see [14]). Let K_0 one these minimal elements, since C is weakly compact, then K_0 is a bounded convex subset of X and by Lemma 1.1, T has an approximate fixed point sequence (x_n). On the other hand K_0 is weakly compact, thus from x_n we can extract a subsequence x_{n_k} such that $\lim\limits_{k\longrightarrow\infty} \|Tx_{n_k} - x_{n_k}\| = 0$ and $x_{n_k} \rightharpoonup z$. Afterwards, Lemma 2.1 implies the existence of a positive number r such that $\lim\limits_{k\longrightarrow\infty} \|x_{n_k} - z\| = r$. Let $\gamma = Tz - z$. If $\gamma = 0$ or $r = 0$ then the proof is finished. Now, assume that $r > 0$ and $\gamma \neq 0$. By a same argument given in the proof of Theorem 1 in [12], it follows that for all integer $k \geq 1$, we have

$$x_{n_k} - z = \lambda_{n_k}\gamma + v_{n_k} + v'_{n_k}.$$

and for all integer $k \geq 1$ we have

$$\|x_{n'_k} - Tz\| \geq \|v_{n'_k}\|(1+\delta) - \|v'_{n'_k}\| \qquad \text{for some } \delta > 0,$$

where $\|v_{n'_k}\| \longrightarrow r$ and $\|v'_{n'_k}\| \longrightarrow 0$ for some subsequences $(v_{n'_k})_k$ and $(v'_{n'_k})_k$ of $(v_{n_k})_k$ and $(v'_{n_k})_k$ respectively.

Afterwards, by using Lemma 2.2 and the triangular inequality, we get

$$\|x_{n'_k} - Tz\| \leq 3\|x_{n'_k} - Tx_{n'_k}\| + \|x_{n'_k} - z\|.$$

By taking $k \longrightarrow \infty$ and using the inequality above, it follows that

$$r \geq (1+\delta)r.$$

which is a contradiction. Hence necessarily $r = 0$ which achieves the proof.

By the same reasoning given in Theorem 2.1 we can prove the following result.

Corollary 3.1 Let C be a weak* closed convex bounded subset of l_1 or the James space J_0. If $T : C \longrightarrow C$ is a continuous C_λ and $C_{1-\lambda}$ mapping, then T has a fixed point.

Theorem 3.2 Let C be a bounded closed convex subset of a reflexive separable Banach space X. Assume that the relation $\perp$ is uniformly approximately symmetric in X. If $T : C \longrightarrow C$ is a continuous asymptotically regular C_λ and $C_{1-\lambda}$ mapping, then for each $x \in C$ the sequence $\{T^n x\}$ converges weakly to some fixed point z of T.

Proof. First of all, the reflexivity of X implies that C is weakly compact. Let $x_0 \in C$ arbitrary. Taking $x_n = T^n x_0$ $(n \geq 1)$. The fact that T is asymptotically regular shows that $\lim\limits_{n\longrightarrow\infty} \|T(x_n) - x_n\| = 0$. Hence by the same argument given in the proof of Theorem 2.1, (x_n) has a subsequence (x_{n_k}) such that $x_{n_k} \rightharpoonup z$ with $Tz = z$. On the other hand since z is a fixed point for T, it follows that

$$\lambda\|Tz - z\| = 0 \leq \|z - x_n\|$$

Since T is C_λ mapping, we get

$$\|x_{n_k+1} - z\| = \|Tx_{n_k} - Tz\| \leq \|z - x_{n_k}\|$$

which proves that the sequence $\|z - x_{n_k}\|$ is decreasing and hence there exists a positive number r such that $\lim\limits_{n\longrightarrow\infty} \|x_{n_k} - z\| = r$. By using Theorem 2 in [12], X satisfies Opial condition and consequently $\liminf\limits_{n\longrightarrow\infty} \|x_{n_k} - z'\| > r$ for $z' \neq z$. Now if there exists a subsequence $\{x_{n'_k}\}$ such that $x_{n'_k} \rightharpoonup z' \neq z$. The previous argument repeated for the subsequence $x_{n'_k}$ together with Opial condition leads to

$$\liminf_{n\longrightarrow\infty} \|x_{n_k} - z'\| = \lim_{n\longrightarrow\infty} \|x_n - z'\| < \liminf_{n\longrightarrow\infty} \|x_{n'_k} - z\| = \lim_{n\longrightarrow\infty} \|x_{n'_k} - z\| = r.$$

which is a contradiction. This achieves the proof.

By the same reasoning as in the proof of Theorem 2.2, the following result can be established.

Corollary 3.2 Let C be a bounded closed convex subset of a dual of separable Banach space X. Assume that the relation $\perp$ is weak* uniformly approximately symmetric in X. If $T : C \longrightarrow C$ is a continuous asymptotically regular C_λ and $C_{1-\lambda}$ mapping, then for each $x \in C$ the sequence $\{T^n x\}$ converges weakly* to some fixed point z of T.

***Remark* 3.2** By Remark 2.1, the assumption that T is $C_{1-\lambda}$ in Theorem 2.1, Theorem 2.2, Corollary 2.1 and Corollary 2.2 can be dropped for $\lambda \in (0, \frac{1}{2}]$.

***Remark* 3.3** Notice that Lemma 3.2, Theorems 3.1, Theorem 3.2, Corollaries 3.1 and 3.2 extend those established in [12] for the case of nonexpansive mappings.

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