Common Fixed Point Theorems For Semigroup Actions Of Kannan type On Strictly Convex Banach spaces

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Abstract

Let C be a weakly compact convex subset of a strictly convex Banach space X. Let S be a semitopological semigroup which acts on C so that the action is weakly separately continuous of weakly continuous Kannan mappings with some additional conditions for which the functions $s \in S \longrightarrow f_x(s) = f(sx)$ and $s \in S \longrightarrow f_x(s) = f(xs)$ belongs to Z a closed linear subspace of $l^\infty(S)$ containing constants and invariant under translations for every $f \in C(S)$. We prove that if Z has a left invariant mean then C has a common fixed point of S.

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1 Introduction

Let S be a semitopological semigroup, in other words a semigroup with a Hausdorff topology such that the mappings $s \in S \longrightarrow st$ and $s \in S \longrightarrow ts$ are continous from S into S for each $t \in S$. Let $l^\infty(S)$ be the Banach space of bounded real-valued functions on S with the supremum norm. For $s \in S$ and $g \in l^\infty(S)$, the left and right translations of g in $l^\infty(S)$ are defined by

$$l_s g(t) = g(st) \text{ and } r_s g(t) = g(ts)$$

for all $t \in S$.

Let X be a closed linear subspace of $l^\infty(S)$ containing constants and invariant under translations, i.e., $l_s(X) \subset X$ and $r_s(X) \subset X$. A linear functional $\mu \in X^\star$ is called a left invariant mean on X if $\|\mu\| = \mu(1) = 1$ and $\mu(l_s g) = \mu(g)$ for each $s \in S$ and $g \in X$. By a same way, we can define a right invariant mean. Let $C(S)$ the closed subalgebra consisting of all continuous functions on S and let $LUC(S)$ be the space of left uniformly continuous functions on S, i.e. all functions $f \in C(S)$ such that the mappings $s \in S \longrightarrow l_s f$ from S to $C(S)$ is continuous if $C(S)$ is equipped with the sup norm topology.

Keywords: *strictly convex Banach space, weakly compact convex subset, Kannan semigroup action, invariant mean, weakly almost periodic function, common fixed point.*

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A function $f \in C(S)$ is strongly almost periodic if the set $\{l_a f; a \in S\}$ is relatively compact in $C(S)$ equipped with the sup norm topology. We denote by $AP(S)$, the space of strongly almost periodic functions on S which is a sup norm closed translation invariant subalgebra of $C(S)$ containing constants. Also, it is known that $f \in AP(S)$ if and only if the set $\{r_a f; a \in S\}$ is relatively compact in $C(S)$ equipped with the sup norm topology.

A function $f \in C(S)$ is strongly weakly almost periodic if the set $\{l_a f; a \in S\}$ (equivalently, $\{r_a f; a \in S\}$) is relatively weakly compact in $C(S)$ equipped with the sup norm topology. We denote by $WAP(S)$, the space of strongly weakly almost periodic functions on S. In general, we have

$$AP(S) \subset WAP(S) \bigcap LUC(S)$$

and if S is a discrete set, we obtain the following inclusions

$$AP(S) \subset WAP(S) \subset LUS(S) = l^{\infty}(S)$$

Let K be a subset of a Banach space X with a norm $\|.\|$. An action of S on K is a mapping of the set $S \times K$ into K, denoted by $(s, x) \longrightarrow s.x = T_s x$ for which $(s_1 s_2).x = s_1(s_2.x)$ for all $s_1, s_2 \in S, x \in K$. A point $x \in K$ is a common fixed point of S with respect to this action if $s.x = T_s x = x$ for all $s \in S$. An action of S on K is called of Kannan's type if it satisfies that:

$$\begin{aligned} \|sx - sy\| &= \|T_s x - T_s y\| \\ &\leq \tfrac{1}{2}(\|x - sx\| + \|y - sy\|) \text{ for all } x, y \in K. \end{aligned}$$

A strictly convex Banach space X is a Banach space such that for every $x, y \in X$, if $x \neq 0, y \neq 0$ and $\|x + y\| = \|x\| + \|y\|$ then necessarily we obtain that $x = cy$ for some $c > 0$. This is equivalently to the fact that, if for any $x_1, x_2, x_3 \in X$ and $\|x_1 - x_2\| + \|x_2 - x_3\| = \|x_1 - x_3\|$, then necessarily x_2 belongs to the segment $[x_1, x_3] = \{tx_1 + (1-t)x_3; t \in [0, 1]\}$.

***Remark* 1.1** It is well known that Hilbert spaces and $L_p([0, 1](1 < p < \infty)$ are strictly convex. Also, every unifomly convex space is strictly convex but the converse is in general not true (see [2]).

2 Main Results

First of all, we give the following basic lemma.

Lemma 2.1 Let S be a semitopological semigroup and Z a closed linear subspace of $l^{\infty}(S)$ containing constants and invariant under translations. Suppose that S acts on a weakly compact subset K of a Banach space X so that the action is weakly separately continuous and there exists $x \in K$ such that for every $h \in C(K)$, the function $s \in S \longrightarrow h_y(s) = h(s.y)$ belongs to Z. If Z has a left mean, then there exists a nonempty weakly compact and norm-separable subset $F_0 \subset K$ such that $sF_0 = \{s.y : y \in F_0\} = F_0$ for every $s \in S$.

***Remark* 2.1** The existence of F_0 given in the previous lemma is connected to the properties of probability Radon measures on K corresponding to some functionals associated to left means on Z (see Lemma 3.5 in [7]).

Our main result in this work is given by the following theorem for which the proof is based essentially of many techniques used by [3, 7].

Theorem 2.1 Let S be a semitopological semigroup and Z a closed linear subspace of $l^\infty(S)$ containing constants and invariant under translations. Assume that S acts on a weakly compact subset D of a strictly convex Banach space X such that the action is weakly separately continuous of weakly continuous Kannan mappings satisfying the following assumptions:

$(\imath)$ For every neighborhood $\widetilde{V}$ of 0 in X, there exists $\delta > 0$ with $N_\delta \subset \widetilde{V}$ such that for every $z \in D$, there exists a sequence $\{t_j\}_{j=1}^\infty \subset S$ and $\{x_j\}_{j=1}^\infty \subset D$ for which

$$T_{s_j}((x_j + N_\delta)\bigcap D) \subseteq (z + N_\delta + \widetilde{V})\bigcap D$$

where $s_j = t_j t_{j-1} \ldots . t_1$ for all $j \geq 1$.

$(\imath\imath)$ For every $\epsilon > 0$ and for every integers $j, p \geq 1 (j \leq p)$, we have

$$T_{t_p t_{p-1} \ldots . t_{j+1}}((z + N_\delta)\bigcap D) \subseteq T_{t_p t_{p-1} \ldots . t_{j+1}}(z) + N_\epsilon$$

Assume that the function $s \in S \longrightarrow h_y(s) = h(s.y)$ belongs to Z for every $y \in D$ and every $h \in C(D)$. If Z has a left invariant mean then there is a common fixed point of S in D.

Proof. We recall that a subset $Y \subseteq D$ is called invariant if $sy \in Y$ for all $s \in S, y \in Y$. So, by Kuratowski-Zorn's lemma, there exists a nonempty minimal weakly compact and convex subset C of D which is invariant under S. Again, by the same lemma, there exists a nonempty minimal weakly compact subset K of D which is invariant under S. Fix $x \in K$ then if $h \in C(K)$ we have $h_x = \widetilde{h}_x \in Z$, where $\widetilde{h}_x : D \longrightarrow \mathbb{C}$ is an extension of h_x to the set D. Applying lemma 2.1 to ensure the existence of a weakly compact and norm-separable subset F_0 of K such that $sF_0 = F_0$ for every $s \in S$. From minimality of K, we obtain $F_0 = K$ is separable and $\{T_s.x = x : s \in S\}$ is weakly dense in K for every $x \in K$. Now, by taking account to assumptions $(\imath)$ and $(\imath\imath)$ and the proof of Lemma 5.2 in [4], we get that K is norm-compact. If $diam(K) = 0$, then K is a singleton, then the point in this set is a common fixed point of S. Assume then that K has at least two points. Let $\delta(K) = diam(K) > 0$. Now, let $s \in S$, by the Schauder-Tychonoff fixed point theorem, we have $sw_s = w_s$ for some $w_s \in K$. Since K is compact, then there exists $y_0 \in K$ such that $\|y_0 - w_s\| = \sup\{\|x - w_s\| : x \in K\}$. Also, the fact that $y_0 = s\tilde{y_0}$ for some $\tilde{y_0} \in K$ gives that

$$\|w_s - y_0\| = \|sw_s - s\tilde{y_0}\| \leq \tfrac{1}{2}\|\tilde{y_0} - s\tilde{y_0}\| \leq \tfrac{1}{2}\delta(K).$$

By the choice of $\tilde{y_0}$, it follows that

$$\|w_s - x\| \leq \tfrac{1}{2}\delta(K) \quad \text{for all } x \in K. \tag{2.1}$$

Also, the compactness of K implies the existence of $x_1, x_2 \in K$ such that $\|x_1 - x_2\| = \delta(K)$. Hence

$$\delta(K) = \|x_1 - x_2\| \leq \|x_1 - w_s\| + \|x_2 - w_s\| \leq \delta(K). \qquad (2.2)$$

Consequently, we get

$$\|x_1 - x_2\| = \|x_1 - w_s\| + \|x_2 - w_s\|.$$

Since X est strictly convex, it follows that, $w_s = t_0 x_1 + (1 - t_0) x_2$ ($t_0 \in]0, 1[$). Now, by (2.1), we prove easily that $t_0 = \frac{1}{2}$ and shows that $w_s = \frac{1}{2}(x_1 + x_2)$ for all $s \in S$. If we denote by $w_0 = \frac{1}{2}(x_1 + x_2)$, it follows that w_0 is a unique common fixed point of S.

***Remark* 2.2** For Kannan's action, the uniqueness of the common fixed point is ensured which is not the case for nonexpansive actions.

Theorem 2.2 Let S be a semitopological semigroup. Assume that S acts on a weakly compact subset D of a strictly convex Banach space X such that the action is weakly separately continuous (i.e., separately continuous when D is equipped with the weak topology), weakly quasi-equicontinuous of weakly continuous Kannan mappings satisfying the following assumptions:

($\imath$) For every neighborhood $\widetilde{V}$ of 0 in X, there exists $\delta > 0$ such that $N_\delta \subset \widetilde{V}$ such that for every $z \in D$, there exists a sequence $\{t_j\}_{j=1}^{\infty} \subset S$ and $\{x_j\}_{j=1}^{\infty} \subset D$ for which

$$T_{s_j}((x_j + N_\delta) \bigcap D) \subseteq (z + N_\delta + \widetilde{V}) \bigcap D)$$

where $s_j = t_j t_{j-1} .....t_1$ for all $j \geq 1$.

($\imath\imath$) For every $\epsilon > 0$ and for very integers $j, p \geq 1 (j \leq p)$, we have

$$T_{t_p t_{p-1} .....t_{j+1}}((z + N_\delta) \bigcap D) \subseteq T_{t_p t_{p-1} .....t_{j+1}}(z) + N_\epsilon$$

Assume that $WAP(S)$ has a left invariant mean, then S has a common fixed point on S.

Proof. If $WAP(S)$ has a left invariant mean, the fact that the action is weakly separately continuous and weakly quasi-equicontinuous, by using ([5], Lemma 3.2), we get that $h_x(s) = h(s.x), s \in S$ belongs to $WAP(S)$ for every $h \in C(D)$ and $x \in D$. Now, the assumptions of Theorem 2.1 are satisfied with $Z = WAP(S)$ and consequently, we obtain the common fixed point of S in D.

Theorem 2.3 Let S be a semitopological semigroup. Assume that S acts on a weakly compact subset D of a strictly convex Banach space X such that the action is weakly separately continuous (i.e., separately continuous when D is equipped with the weak topology), weakly quasi-equicontinuous of weakly continuous Kannan mappings satisfying the following assumptions:

($\imath$) For every neighborhood $\widetilde{V}$ of 0 in X, there exists $\delta > 0$ such that $N_\delta \subset \widetilde{V}$ such that for every $z \in D$, there exists a sequence $\{t_j\}_{j=1}^{\infty} \subset S$ and $\{x_j\}_{j=1}^{\infty} \subset D$ for which

$$T_{s_j}((x_j + N_\delta) \bigcap D) \subseteq (z + N_\delta + \widetilde{V}) \bigcap D)$$

where $s_j = t_j t_{j-1} .....t_1$ for all $j \geq 1$.

($\imath\imath$) For every $\epsilon > 0$ and for very integers $j, p \geq 1 (j \leq p)$, we have

$$T_{t_p t_{p-1}.....t_{j+1}}((z+N_\delta)\bigcap D) \subseteq T_{t_p t_{p-1}.....t_{j+1}}(z) + N_\epsilon$$

Assume that $AP(S)$ has a left invariant mean, then S has a common fixed point on S.

Proof. If $AP(S)$ has a left invariant mean, the fact that the action is weakly separately continuous and weakly quasi-equicontinuous, by using ([6], Lemma 3.1), we get that $h_x(s) = h(s.x), s \in S$ belongs to $WAP(S)$ for every $h \in C(D)$ and $x \in D$. Now, the assumptions of Theorem 2.1 are satisfied with $Z = WAP(S)$ and consequently, we obtain the common fixed point of S in D.

***Remark* 2.3** In the case of nonexpansive mappings, it is easy to observe that for every $n \in \mathbb{N}, T^n$ is nonexpansive which is not the case of Kannan mappings. Indeed, it suffices to take $T : \mathbb{R} \longrightarrow \mathbb{R}$ given by $Tx = 1 - x$, thus T is a Kannan mapping but $T^2 = I$ which is not a Kannan mapping. But if $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by

$$f(x) = \begin{cases} 0 & \text{if } x \leq \ 2 \\ -\frac{1}{2} & \text{if } x > 2. \end{cases}$$

It is easy to observe that f is a Kannan mapping with $f^2 = 0$ and consequently $S = \{0, f\}$ is a Kannan semigroup which acts on $\mathbb{R}$ (it can be equipped with it's discrete topology).

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