Some Results On Generalized Kirk's Process In Banach Spaces and Application

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Work plan

1. Basic definitions and preliminaries
2. Fixed points formulas
3. The case of asymptotically regular mappings
4. Convergence of generalized Kirk's processes
5. Application to a nonlinear system

- In applied sciences, many problems are modeled by equations

$$u - Tu = f \tag{1}$$

where T is nonlinear and $f \in X$ (convenable Banach space).

- u_0 is a solution of (1) if and only if u_0 is a fixed point of T_f

$$T_f u = Tu + f \tag{2}$$

Definition 1.1

Let C be a nonempty subset of a normed space $X. T : C \longrightarrow C$ is said to be **nonexpansive** if

$$\|Tx - Ty\| \leq \|x - y\| \qquad \text{for all } x, y \in C$$

In equations (1) and (2), it is easy to observe that T is nonexpansive if and only if T_f is nonexpansive.

Big Question:

Let X be a Banach space and C a closed bounded convex subset of X. Does every nonexpansive selfmapping T on C has a fixed point?

Some positive answers to the big question

1. If $dim(X) < \infty$ then T has a fixed point.
   (Consequence of Brouwer's Theorem (1912)).
2. If $dim(X) = \infty$ and
   1. C compact then T has a fixed point.
      (Consequence of Schauder's Theorem (1930)).
   2. C weakly compact and has a normal structure then T has a fixed point.
      (W. Kirk, D. Göhde, F. E. Browder (1965-1966)).

A famous negative answer to the big question

- $X = L^1([0,1])$
- $\|f\| = \int_0^1 |f(t)|dt,$
- $C = \{f \in L^1([0,1]), \int_0^1 f(t)dt = 1, 0 \leq f \leq 2\}$
- $T : C \longrightarrow C$ defined by

$$T(f)(t) = \begin{cases} \min\{2f(2t), 2\} & 0 \leq t \leq \frac{1}{2} \\ \max\{2f(2t-1) - 2, 0\} & \frac{1}{2} < t \leq 1. \end{cases}$$

Then T is nonexpansive and fixed point free.
(D. Alpasch (1981)).

Definition 1.2

Let C be a nonempty convex subset of a Banach space X

1. Let $T : C \longrightarrow C$ be a selfmapping. Define a sequence $(x_n)_n \subset C$ by

$$x_{n+1} = \lambda x_n + (1-\lambda)T(x_n) \qquad \lambda \in (0,1)$$

$(x_n)_n$ **is called Krasnoselskii process associated to** T.

2. Let $T_1, T_2, ..., T_k$ be selfmappings on C. Define $(x_n)_n \subset C$ by

$$x_{n+1} = \lambda_0 x_n + ... + \lambda_k T_k(x_n),$$

$$\text{where } \lambda_1 > 0, \text{and } \lambda_i \geq 0, i \neq 1 \text{with } \sum_{i=0}^{k} \lambda_i = 1$$

$(x_n)_n$ **is called generalized Kirk's process associated to the mappings** $T_1, ..., T_k$.

Remark 1.3

1. If $\lambda_2 = \ldots = \lambda_k = 0$ in the case of **generalized Kirk's process**, then it is reduced to **Krasnoselskii process** associated to the mapping T_1.

2. If $T_i = T^i, \forall i \geq 1$ in the case of **generalized Kirk's process**, then it reduced to the **classical Kirk's process** associated to the mapping T.

3. In the following, we denote by $F(T)$ the **set of fixed points** of the mapping T.

We start this section by the following lemma.

Lemma 2.1

Let C be a nonempty convex subset of a Banach space X and let $T_1, ..., T_k$ be a selfmappings on C. For $(\lambda_i)_{i=0}^k \subset [0,1]$ with $\sum_{i=0}^k \lambda_i = 1$, we denote by

$$S = \sum_{i=0}^{k} \lambda_i T_i,$$

with the notation $T_0 = Id_C$, then

$$\bigcap_{i=1}^{k} F(T_i) = F(S) \bigcap \left(\bigcap_{i=1}^{k} F(T_i S) \right).$$

Proof: Let $x_0 \in \bigcap_{i=1}^{k} F(T_i)$ then $x_0 \in F(T_i)$ for all integer $i = 1, ..., k$, which proves that $T_i(x_0) = x_0$ for all $i = 1, ..., k$ and consequently $S(x_0) = \sum_{i=0}^{k} \lambda_i T_i(x_0) = x_0$, this gives that $x_0 \in F(S)$ and consequently $x_0 \in F(S) \bigcap \bigcap_{i=1}^{k} F(T_i S)$.

Conversely, let $x_0 \in F(S) \bigcap \left(\bigcap_{i=1}^{k} F(T_i S) \right)$, then $S(x_0) = x_0$ and $(T_i S)(x_0) = x_0$ for all integer $i = 1, ..., k$ by composition the equality $S(x_0) = x_0$ by $T_i \;\; (i = 1, ..., k)$, we get

$$(T_i S)(x_0) = T_i x_0 = x_0,$$

this implies that $x_0 \in F(T_i), \forall i = 1, ..., k$ and consequently $x_0 \in \bigcap_{i=1}^{k} F(T_i)$, which achieves the proof.

Corollary 2.2

Let C be a **nonempty subset** of a Banach space X and let $T : C \longrightarrow C$ be a selfmapping then for all $k \geq 1$, we have

$$F(T) = F(T^k) \bigcap F(T^{k+1}).$$

Proof: In the proof of Lemma 2.1, it suffices to take that $\lambda_i = 0$, $T_i = T^i$ for all integer $i \neq k$ and $\lambda_k = 1$ together with $T_k = T^k$.

Remark 2.3

It is easy to observe that **the assumption of the convexity** of the subset C can be **dropped** in Corollary 2.2

Theorem 2.4

Let C be a **convex subset** of a Banach space X and let $T_1, T_2, ..., T_k$ be a selfmappings satisfying that $\forall x \in C$, and $\forall i, j = 1, ..., k, (i < j)$ there exists an integer $n(x)$ with $1 \leq i \leq n(x) < j \leq k$ such that

$$\| T_i(x) - T_j(x) \| \leq \| x - T_{n(x)}(x) \| \tag{3}$$

Let $(\lambda_i)_{i=0}^k \subset [0, 1]$ with $\lambda_1 > 0$ and $\sum_{i=0}^{k} \lambda_i = 1$. We denote

$S = \sum_{i=0}^{k} \lambda_i T_i$ (with the notation $T_0 = I_C$). Then

$$\bigcap_{i=1}^{k} F(T_i) = F(S).$$

Proof It is easy to prove that $\bigcap_{i=1}^{k} F(T_i) \subseteq F(S)$. For the converse, let $x_0 \in F(S)$, thus

$$S(x_0) = \left(\sum_{i=0}^{k} \lambda_i T_i\right)(x_0) = x_0,$$

this gives that

$$x_0 = \left(\sum_{i=1}^{k} \left(\frac{\lambda_i}{1-\lambda_0}\right) T_i\right)(x_0) \quad (\lambda_0 \neq 1 \ \ since \ \ \lambda_1 > 0).$$

Let $\delta = \sup\{\|T_i(x_0) - T_j(x_0)\|, i, j = 0, ..., k\}$. Assume that $\delta > 0$, the assumption (3) proves that there exists a smallest integer $p(x_0) \in \{1, ..., k\}$ such that

$$\delta = \|x_0 - T_{p(x_0)}(x_0)\|.$$

Since $\sum_{i=1}^{k} \frac{\lambda_i}{1-\lambda_0} = 1$, it follows that

$$x_0 = \gamma_0 T_1(x_0) + (1-\gamma_0)z,$$

where $z \in conv\{T_2(x_0), ..., T_k(x_0)\}(\gamma_0 \in (0,1])$. Thus

$$\begin{aligned}\delta = \|x_0 - T_{p(x_0)}(x_0)\| =& \|\gamma_0 T_1(x_0) + (1-\gamma_0)z - T_{p(x_0)}(x_0)\| \\ \leq& \gamma_0 \|T_1(x_0) - T_{p(x_0)}(x_0)\| + (1-\gamma_0)\|z - T_{p(x_0)}(x_0)\| \\ \leq& \gamma_0 \delta + (1-\gamma_0)\delta = \delta.\end{aligned}$$

(i) If $p(x_0) = 1$, this is a contradiction, since, we obtain that $\|T_1(x_0) - T_1(x_0)\| = 0 = \delta$.

(ii) If $p(x_0) > 1$, by the assumption (3), we obtain the existence of an integer $m(x_0) < p(x_0)$ such that

$$\delta \leq \|T_1(x_0) - T_{p(x_0)}(x_0)\| \leq \|x_0 - T_{m(x_0)}(x_0)\|$$

which gives that $\|x_0 - T_{m(x_0)}(x_0)\| = \delta$ and contradicts the fact that $p(x_0)$ is the smallest integer such that $\delta = \|x - T_{p(x_0)}(x_0)\|$. Necessarily, we get $\delta = 0$ and $\|x_0 - T_i(x_0)\| = 0$ for all integer $i = 1, ..., k$, consequently $x_0 \in \bigcap_{i=1}^{k} F(T_i)$ which achieves the proof.

Corollary 2.5

Let C be convex subset of a Banach space X and let $T : C \longrightarrow C$ be nonexpansive . Denote by

$$S = \sum_{i=0}^{k} \lambda_i T^i$$

with the notation $T^0 = I_C$ where $(\lambda_i)_{i=0}^{k} \subset [0, 1]$ together with $\lambda_1 > 0$. and $\sum_{i=0}^{n} \lambda_i = 1$.

Then F(S)= F(T).

Proof: The result follows from Theorem 2.4 by taking $T_i = T^i$ for all integer i. In this case, we have $\bigcap_{i=1}^{k} F(T^i) = F(T)$ since $F(T) \subset F(T^i)$ for all integer $i \geq 1$ and $n(x) = j - i$ $(1 \leq i < j \leq k)$ for all $x \in C$.

Definition 3.1

Let C be a nonempty subset of a Banach space X and let $T : C \longrightarrow C$ is said to be **asymptotically regular** if, for all $x \in C$, we have

$$\lim_{n \longrightarrow \infty} \|T^{n+1}(x) - T^n(x)\| = 0.$$

Remark 3.2

1. If T is a **Banach contraction** then T is **asymptotically regular**.
2. If T is a **nonexpansive**, then $\delta_n = \|T^{n+1}(x) - T^n(x)\|$ is **decreasing** but does not converge necessarily to 0.

   Indeed, it suffices to take

   - $C = X = \mathbb{R}$ equipped with it's usual norm.
   - $T : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $T(x) = 1 - x$.

Definition 3.3

A uniformly convex Banach space X is a Banach space such that for every $0 < \epsilon \leq 2$ there is some $0 < \delta$ such that for any two vectors x, y with $\|x\| = \|y\| = 1$, the condition $\|x - y\| \geq \epsilon$ implies $\frac{\|x+y\|}{2} \leq 1 - \delta$.

This concept was firstly introduced by James. A. Clarckson in (1936).

Remark 3.4

Intuitively, X is a uniformly convex Banach space if it's unit ball is sufficiently round.

Examples 3.5

1. **Hilbert spaces** and $L_p([0,1])(1 < p < \infty)$ are **uniformly convex**
2. $L_1([0,1])$ and $L_\infty([0,1])$ are **not uniformly convex.**

Theorem 3.6

Let C be a convex subset of a uniformly convex Banach space X and let $T_1, T_2, ..., T_k$ be nonexpansive selfmappings on C satisfying assumption (3). Denote by

$$S = \sum_{i=0}^{k} \lambda_i T_i \text{ (with the notation } T_0 = Id_C)$$

where $(\lambda_i)_{i=0}^{k} \subset [0,1]$ and $\lambda_1 > 0$ with $\sum_{i=0}^{k} \lambda_i = 1$. If $\bigcap_{i=1}^{k} F(T_i) \neq \phi$.

Then S is asymptotically regular.

Proof: First of all , since T_i is nonexpansive for all integer $i \in \{1, 2, ..., k\}$, then S is nonexpansive. Moreover, Theorem 2.4 implies that $F(S) = \bigcap_{i=1}^{k} F(T_i) \neq \phi$. Assume that $y \in C$ is a fixed point of S and let $x \in C$. Define a sequence $(x_n) \subset C$ by $x_n = S^n x, n \in \mathbb{N}$ with the notation $S^0 = Id_C$. It is easy to show that the sequence $\{\|x_n - y\|\}_n$ is decreasing, then $\lim_{n \longrightarrow \infty} \|x_n - y\| = \alpha \geq 0$.

[($\imath$)] If $\alpha = 0$, then $\lim_{n \longrightarrow +\infty} x_n = y$, since S is continuous (S is nonexpansive), it follows that
$\lim_{n \longrightarrow +\infty} x_{n+1} = \lim_{n \longrightarrow +\infty} S(x_n) = S(\lim_{n \longrightarrow +\infty} x_n) = S(y) = y$
and consequently

$$\lim_{n \longrightarrow +\infty} \|S^{n+1}(x) - S^n(x)\| = \|y - y\| = 0.$$

$[(ii)]$ If $\alpha > 0$, thus

$$x_{n+1} - z_0 = S(x_n) - y = \sum_{i=0}^{k} \lambda_i T_i(x_n) - y = \lambda_0(x_n - y) - (1 - \lambda_0) z_n,$$

where

$$z_n = \frac{1}{1 - \lambda_0} \sum_{i=1}^{k} \lambda_i \left(T_i(x_n) - y \right).$$

Since $y \in \bigcap_{i=1}^{k} F(T_i)$, we get

$$\| T_i(x_n) - y \| = \| T_i(x_n) - T_i(y) \| \leq \| x_n - y \|.$$

The fact that $\sum_{i=0}^{k} \lambda_i = 1$ implies that $\overline{\lim}\|z_n\| \leq \alpha$. Moreover, since $\lim_{n \longrightarrow +\infty} \|x_n - y\| = \alpha$, gives that $\lim_{n \longrightarrow +\infty} \|x_{n+1} - y\| = \alpha$. From the uniform convexity of X, we get that

$$\lim_{n \longrightarrow +\infty} \|x_n - y - z_n\| = 0,$$

and consequently

$$\lim_{n \longrightarrow +\infty} x_{n+1} - x_n = \lim_{n \longrightarrow +\infty} (1 - \lambda_0)(x_n - y - z_n) = 0,$$

which achieves the proof.

Theorem 4.1

Let X be a uniformly convex Banach space and let $T_1, T_2, ..., T_k$ be nonexpansive compact selfmappings on X satisfying the assumption (3). Denote by S the mapping

$$S = \sum_{i=0}^{k} \lambda_i T_i$$

with the notation $T_0 = Id_X$, where $(\lambda_i)_{i=0}^{k} \subset [0, 1], \lambda_1 > 0$ and $\sum_{i=0}^{k} \lambda_i = 1$.

If $\bigcap_{i=1}^{k} F(T_i) \neq \emptyset$, then for each $x_0 \in X$ the Picard sequence $\{S^n(x_0)\}$ converges to a common fixed point of the mappings $T_1, T_2, ..., T_k$.

Proof: It follows from Theorem 3.6 that S is asymptotically regular with $F(S) = \bigcap_{i=1}^{k} F(T_i) \neq \emptyset$. First of all, we prove that the mapping $I - S$ maps bounded closed subsets of X into closed subsets of X. Indeed, let C an arbitrary bounded closed subset of X and assume that $\lim_{n \longrightarrow +\infty} (y_n - Sy_n) = y, y_n \in C$. We will show that $y \in (I - S)(C)$. The fact that each $T_i, 1 \leq i \leq k$ is compact implies the existence of a subsequence $(y_{n^i(l)})_l$ such that $T_i(y_{n^i(l)})_l$ converges to $z_i \in X, 1 \leq i \leq k$ which proves the existence of a subsequence $\left(y_{f(l)}\right)_l$ of $(y_l)_l$ (with $f(1)$ is the smallest integer multiple of $n^1(1), n^2(1), ..., n^k(1)$) such that $T_i(y_{f(l)})$ converges to $z_i \in X$. Thus

$$\begin{aligned}(I - S)(y_{f(l)}) =& y_{f(l)} - \sum_{i=0}^{k} \lambda_i T_i(y_{f(l)}) \\ =& (1 - \lambda_0) y_{f(l)} - \sum_{i=1}^{k} \lambda_i T_i(y_{f(l)}).\end{aligned}$$

Since $y_{f(l)} - S(y_{f(l)})$ converges to $y \ (l \longrightarrow +\infty)$, we get

$$\lim_{l \longrightarrow +\infty} (1 - \lambda_0) y_{f(l)} = y + \sum_{i=1}^{k} \lambda_i z_i$$

which implies $\lim_{l \longrightarrow +\infty} y_{f(l)} = \frac{y}{1 - \lambda_0} + \sum_{i=1}^{k} (\frac{\lambda_i}{1 - \lambda_0}) z_i \in C$ (since C is closed) then $\lim_{l \longrightarrow +\infty} y_{f(l)} = \widetilde{y} \in C$, which gives that

$$\widetilde{y} - S\widetilde{y} = y,$$

it proves that $y \in (I - S)(C)$ which is the desired result. Now the result follows from Theorem 6 in (F. E. Browder and W. V. Petryshin, *The solution by iteration of linear functional equations in Banach spaces*, Bull. Amer. Math. Soc., (72) (1966), 566-570).

Theorem 4.2

Let X be a uniformly convex Banach space, C a closed bounded convex subset of X, and let $T_1, T_2, ..., T_k$ be a nonexpansive mappings satisfying the assumption (3). Define

$$S = \sum_{i=0}^{k} \lambda_i T_i$$

with the notation $T_0 = Id_C$ where $(\lambda_i)_{i=0}^{k} \subset [0,1], \lambda_1 > 0$ and $\sum_{i=0}^{k} \lambda_i = 1$. **Assume that** $\bigcap_{i=1}^{k} F(T_i) = \{z_0\}$. **Then for each** $x_0 \in C$, **the Picard sequence** $\{S^n(x_0)\}$ **converges weakly to** z_0 **in** C.

Proof: Since S is nonexpansive, then the mapping $I - S$ is demiclosed (F. E. Browder, *Nonexpansive nonlinear operators in a Banach space,* Proc. Nat. Acad. Sci. USA., (54) (1965), 1041-1044).

Now let $x_0 \in C$ and let $(x_n)_n$ the Picard sequence $x_n = S^n x_0 (n \in \mathbb{N})$, since X is uniformly convex, then X is reflexive (K. Goebel and W. A. Kirk, *Topics in metric fixed point theory,* Cambridge Studies in Advanced Mathematics. First edition (1990)), this implies the existence of a subsequence $\{x_{n_k}\}_k$ of $\{x_n\}_n$ such that x_{n_k} converges weakly to y_0. Theorem 3.6 gives that S is asymptotically regular, thus

$$\lim_{k \longrightarrow +\infty} (I - S)(x_{n_k}) = \lim_{k \longrightarrow +\infty} \left(S^{n_k}(x_0) - S^{n_k+1}(x_0) \right) = 0.$$

By definition of demiclosedness, it follows that

$$(I - S)(y_0) = 0,$$

which proves that y_0 is a fixed point of S. But $F(S) = \bigcap_{i=1}^{k} F(T_i)$ (see Theorem 2.4), hence $y_0 = z_0$ and y_0 is the unique fixed point of S. Consequently, every weakly convergent subsequence of $\{x_n\}$ converges weakly to z_0. By a standard argument using the reflexivity of X and the fact that the sequence $\{x_n\}_n$ is bounded, we infer that $\{x_n\}_n$ converges weakly to z_0 which is the desired result.

Remark 4.3

Notice that Theorems 4.1 and 4.2 are extensions respectively of Corollary and Theorem 3 in **(W. A. Kirk, On successive approximations for nonexpansive mappings, Glasgow. Math. J., Vol (2) (1), (1971), 6-9)** by taking $T_i = T^i$ for all integer $i \geq 1$.

Lemma 4.4

(see Lemma 1 in C. W. Groetsch, A nonstationary iterative process for nonexpansive mappings, Proc. Math. Soc., 43 (1) (1974), 155-158)

If $\{x_n\}_n$ and $\{y_n\}_n$ are sequences in a uniformly convex space with

$$\|y_n\| \leq \|x_n\| \text{ and } x_{n+1} = (1-\alpha_n)x_n + \alpha_n y_n \ (0 \leq \alpha_n \leq 1)$$

where $\sum_{n=1}^{\infty} \min(\alpha_n, 1-\alpha_n) = \infty$.

Then $0 \in \overline{\{x_n - y_n, n \in \mathbb{N}\}}$ (where $\overline{C}$ denotes the closure of the set C).

Let $(\alpha_{ij})_{i=0}^{\infty}$ $(j = 0, 1, ..., k)$ a set of positive reals such that $0 \leq \alpha_{ij}, 0 < \alpha \leq \alpha_{i1}$ with $\sum_{j=0}^{k} \alpha_{ij} = 1$ for each i and $\sum_{i=0}^{\infty} \min(\alpha_{i0}, 1 - \alpha_{i0}) = \infty.$

Define the mappings S_i by

$$S_i = \alpha_{i0} I + \alpha_{i1} T_1 + ... + \alpha_{ik} T_k \quad (i = 0, 1, 2, ...,)$$

A non-stationary generalized Kirk's process is given by the formula

$$x_{n+1} = S_n x_n \quad (n = 0, 1, 2, ...) \tag{4}$$

It is easy to observe that if

- $T_1, T_2, ..., T_k$ are nonexpansives mappings,
- $z_0 \in \bigcap_{i=1}^{k} F(T_i)$.

Then

$$\|x_{n+1} - z_0\| = \| \sum_{j=0}^{k} \alpha_{nj}(T_j x_n - T_j z_0)\| \leq \|x_n - z_0\|$$

Proposition 4.5

Let C be a convex subset of uniformly convex Banach space and let $T_1, T_2, ..., T_k$ be nonexpansive selfmappings on C with $\bigcap_{i=1}^{k} F(T_i) \neq \phi$ and let (x_n) defined by equation (4), then $0 \in \overline{\{x_{n+1} - x_n, n \in \mathbb{N}\}}$.

Proof: Let $x_0 \in \bigcap_{i=1}^{k} F(T_i)$. Define $y_n = x_n - x_0$ and

$$z_n = \frac{1}{1 - \alpha_{n0}} \sum_{j=1}^{k} \alpha_{nj}(T_j x_n - T_j x_0).$$

It follows that

$$\begin{aligned} y_{n+1} = x_{n+1} - x_0 =& S_n x_n - x_0 = \alpha_{n0} x_n + \ldots + \alpha_{nk} T_k x_n - (\sum_{j=0}^{k} \alpha_{nj}) x_0 \\ =& \alpha_{n0}(x_n - x_0) + \sum_{j=1}^{k} \alpha_{nj}(T_j x_n - T_j x_0) \\ =& \alpha_{n0} y_n + (1 - \alpha_{n0}) z_n. \end{aligned}$$

We have $\|z_n\| \leq \|x_n - x_0\| = \|y_n\|$, because the mappings $T_1, T_2, \ldots, T_k$ are nonexpansive. It follows by Lemma 4.4, that $0 \in \overline{\{y_n - z_n, n \in \mathbb{N}\}}$. On the other hand,

$$
\begin{aligned}
\|y_n - z_n\| =& \|x_n - x_0 - \frac{1}{1-\alpha_{n0}} \sum_{j=1}^{k} \alpha_{nj} T_j x_n + x_0\| \\
=& \|x_n - \frac{1}{1-\alpha_{n0}} \sum_{j=0}^{k} \alpha_{nj} T_j x_n + \frac{\alpha_{n0}}{1-\alpha_{n0}} x_n\| \\
=& \frac{1}{1-\alpha_{n0}} \|x_n - x_{n+1}\| \\
\geq& \|x_n - x_{n+1}\| \text{ since } \frac{1}{1-\alpha_{n0}} \geq 1
\end{aligned}
$$

this proves the existence of a subsequence $\{x_{n_k}\}$ such that $\lim_{k \longrightarrow +\infty} \|x_{n_k} - x_{n_k+1}\| = 0$, which is the desired result.

Theorem 4.6

Assume in addition to the hypotheses of Proposition 4.5, that the mappings $T_1, T_2, ..., T_k$ satisfy the assumption (3) and each T_i $(1 \leq i \leq k)$ is compact. Then for each $x_1 \in C$, the sequence $\{x_n\}_n$ defined by the equation (4) converges to a common fixed point for the mappings $T_1, T_2, ..., T_k$.

Proof: By the previous Proposition, there exists a subsequence $\{x_{x_k}\}$ with $x_{n_{k+1}} - x_{n_k} \longrightarrow 0$. The assumption given on the set $(\alpha_{ij})_{i=0}^{\infty}$ $(j = 0, 1, ..., k)$ shows that, we can extract a subsequences $\alpha_{m_k j}$ of the sequence $\{\alpha_{n_k j}\}$ such that $\lim_{k \longrightarrow +\infty} \alpha_{m_k j} = \alpha_j \in [0, 1]$ with $\alpha_1 > 0$.

Let

$$S = \alpha_0 I + \alpha_1 T_1 + ... + \alpha_k T_k.$$

We get

$$x_{m_k} - Sx_{m_k} = x_{m_k} - S_{m_k} x_{m_k} + S_{m_k} x_{m_k} - Sx_{m_k},$$

Where

$$x_{m_k} - S_{m_k} x_{m_k} = x_{m_k} - x_{m_k+1} \longrightarrow 0.$$

If $x_0 \in \bigcap_{i=1}^{k} F(T_i)$, since the sequence $\{\|x_n - x_0\|\}_n$ is decreasing and the mappings $T_1, T_2, ..., T_k$ are nonexpansive, it follows that

$$\|T_j x_{m_k} - x_0\| = \|T_j x_{m_k} - T_j x_0\| \leq \|x_{m_k} - x_0\| \leq \|x_1 - x_0\|.$$

Since

$$\|T_j x_{m_k} - x_0\| \leq \|x_1 - x_0\|.$$

We obtain that

$$\|T_j x_{m_k}\| \leq \|x_1 - x_0\| + \|x_0\| = \gamma \text{ for all } j = 0, 1, ..., k$$

Thus

$$\|S_{m_k}x_{m_k} - Sx_{m_k}\| = \|\sum_{j=0}^{k}(\alpha_{m_kj} - \alpha_j)T_jx_{m_k}\|$$
$$\leq \gamma \sum_{j=0}^{k}|\alpha_{m_kj} - \alpha_j| \longrightarrow 0 \ (k \longrightarrow +\infty).$$

We infer that $x_{m_k} - Sx_{m_k} \longrightarrow 0 \ (k \longrightarrow +\infty)$. Since each $T_i \ (i \leq 1 \leq k)$ is compact, Theorem 4.1 shows that $I - S$ maps closed bounded subsets into closed subsets. On the other hand, from the decreasness of the sequence $\{\|x_n - x_0\|\}_n$, we deduce that $\{\overline{x_n, n \in \mathbb{N}}\}$ is closed and bounded. Afterwards, Proposition 4.5 implies that $0 \in (I - S)(\{\overline{x_n, n \in \mathbb{N}}\})$. This proves the existence of $y_0 \in \{\overline{x_n, n \in \mathbb{N}}\}$ with $S(y_0) = y_0$ and here y_0 is a fixed point of S. Now, by Theorem 2.4, we get $y_0 \in \bigcap_{i=1}^{k} F(T_i)$. Apply for a second time the decreasness of the sequence $\{\|x_n - y_0\|\}_n$, it follows that $x_n \longrightarrow y_0 \ (n \longrightarrow +\infty)$, which completes the proof.

Let be the nonlinear system

$$\begin{cases} x - T_1 x & = f_1 \\ \ldots\ldots\ldots & = \ldots \\ \ldots\ldots\ldots & = \ldots \\ \ldots\ldots\ldots & = \ldots \\ x - T_k x & = f_k \end{cases} \qquad (\star)$$

in a convex subset C of a Banach space X where $f_i \in C$ for all $i = 1, ..., k$ and $T_1, ..., T_k$ are selfmappings on C.

Denote by $B_i, i = 1, ..., k$ the mapping given by $B_i x = T_i x + f_i$ with the notation $B_0 = Id_X$. For all $(\lambda_i)_{i=0}^{k} \subset [0, 1]$ with $\lambda_1 > 0$ and $\sum_{i=0}^{k} \lambda_i = 1$, if we denote by $\gamma_i = \frac{\lambda_i}{1 - \lambda_0}$ $(i = 1, ..., k)$, then we have

Lemma 5.1

Let $z_0 \in X$. Then z_0 is a solution of the system $(\star)$ if and only if z_0 is at the same time the solution of the nonlinear equation

$$x = \sum_{i=1}^{k} \gamma_i B_i x \tag{5}$$

and the system

$$x = B_i \left(\sum_{j=0}^{k} \lambda_j B_j \right) x, \qquad i = 1, ..., k \tag{$\star\star$}$$

Lemma 5.2

Assume that the mappings $(B_i)_{i=1}^k$ given in $(\star)$ satisfy the assumption (3). Then x is a solution of the system $(\star)$ if and only if x is the solution of the nonlinear equation (5).

Let X be a Banach space and C a convex subset of X. For a finite family of nonexpansive selfmappings $\{T_i\}_{i=1}^k$ of C. For $\alpha \in]0,1[$, **P. Kuhfittig (Common fixed points of nonexpansive mappings by iteration, Pacific. J. Math., Vol (97) (1), (1981), 137-139)** has defined the following iterative process

$$x_{n+1} = U_k(x_n), \qquad n = 0, 1, ...,$$

where

$$\begin{cases} U_0 & = I \\ U_1 & = (1-\alpha)I + \alpha T_1 U_0 \\ ... & = ... \\ ... & = ... \\ U_k & = (1-\alpha)I + \alpha T_k U_{k-1} \end{cases}$$

Theorem 5.3

Let C be a convex compact subset of a strictly convex Banach space X and let $\{T_i\}_{i=1}^{k}$ be a family of nonexpansive selfmappings of C. If the nonlinear equation (5) has at least a solution and the mappings $\{B_i\}_{i=1}^{k}$ satisfy the assumption (3).

Then for an arbitrary $z_0 \in C$, the sequence $\{U_k^n z_0\}$ converges strongly to a solution of the system $(\star)$.

Theorem 5.4

If X is a Hilbert space and C is a closed convex subset of X. Assume that the mappings $\{T_i\}_{i=1}^{k}$ are nonexpansive selfmappings of C. If the nonlinear equation (5) has at least a solution and the mappings $\{B_i\}_{i=1}^{k}$ satisfy the assumption (3).

Then for any $z_0 \in C$, the sequence $\{U_k^n z_0\}$ converges weakly to a solution of the system $(\star)$.

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Thank you for your attention