A new properties of semi-Fredholm and Fredholm perturbations in Banach spaces

Abdelkader Dehici and Ines Walha

Abstract

In this paper, we study some topological properties of semi-Fredholm and Fredholm perturbations in Banach spaces, many problems and questions associated to these classes are established. Moreover, our contribution extend and improve many well known results in the literature.

Mathematics Subject Classification (2010): 47A10 47A53, 47A55

1 Introduction

It is well known that the theory of normaly solvable operators play a crucial role both in pure and applied mathematics. Let X and Y be two Banach spaces and A a bounded linear operator from X to Y, then A is called normally solvable if the equation $Ax = y$ ($y \in Y$) has a solution if and only if $f_{|R(A)} = 0$ for all functional f in the conjugate space $Y^\star$ where $f_{|R(A)}$ is the restriction of f to $R(A)$ given as the range of A. [17] proved that the set of normally solvable operators is nothing but else the set of operators having closed ranges. This theory includes that of semi-Fredholm and Fredholm operators together with that of stability by perturbations associated to them. The first result in this direction is due to J. Dieudonné [15] who has showed that the index of Fredholm operator is unchanged by perturbation with small norms. After, B. Yood [51] and [5, 18]proved independently this result for compact operators. Calkin's result [8] has a capital impact in functional analysis showing that in a separable Hilbert space $\mathcal{H}$, the set of compact operators is the only closed two-sided ideal in the algebra of bounded linear operators $\mathcal{L}(\mathcal{H})$. Afterwards, I. Gohberg, Markus and Feldmann [16] have showed that this result holds also for the case of the Banach spaces $l_p(1 \leq p < \infty)\bigcup c_0$. Since, the finitenes and infiniteness of the closed two sided ideals sparked the curiosity of many mathematicians on which who have worked by exploring one of the complicated fraweworks connected directly to the geometry of Banach spaces. The discovery of strictly singular operators by T. Kato (1957) [33] and that of semi-Fredholm and Fredholm perturbations by I. C. Gohberg,

Keywords: *Closed range, null space, normally solvable operator, upper semi-Fredholm operator, lower semi-Fredholm operator, Fredholm operator, compact operator, strictly singular operator, semi-Fredholm perturbation.*
Corresponding author address: 1. Laboratory of Informatics and Mathematics
University of Souk-Ahras,P.O.Box 1553, Souk-Ahras 41000, Algeria.
E-mails: dehicikader@yahoo.fr

A. Markus and I. A. Feldman (1960) had a considerable impact on this theory by enriching and deepening it. Indeed, the paper of the last authors [16] was a pioneer work which gave an abstract framework to this thematic by extending all results which preceded it in this direction. In this paper, we find the first example of a strictly singular operator which is not compact constructed on $L_p(\mu)$-spaces $(1 \leq p < \infty)$. For $p = 1$, it gives an example of weakly compact (not integral operator) operator which is not compact. In (1977), L. Weis [48] has completed Milman's work [38] by showing the coincidence of all semi-Fredholm and Fredholm perturbation with those of strictly singular and strictly cosingular operators for all $L_p(\mu)$-spaces $(1 \leq p < \infty)$. The same author [49] has established a fairly riche contribution on this theory for closed densely defined semi-Fredholm and Fredholm operators. One of the open problems which has long been open is the coincidence between the classe of upper semi-Fredholm perturbation (resp. lower semi-Fredholm perturbation) with that of strictly singular operators (resp. strictly cosingular operators), it took the discovery of hereditarily indecomposable Banach spaces to give an negative answer to this question by M. Gonzalez (2003) [22]. Some interesting questions on this subject can also be found in [27]. In (1986), the same author gave a characterization of non-semi-Fredholm operators in abstract Banach spaces and separable ones and has extend Bouldin's results [6] established in the case of Hilbert space. For a Banach space X, it is known that the sets of upper and lower semi-Fredholm operators are open in the Banach algebra $\mathcal{L}(X)$ of all bounded linear operators on X, one of the curious question is to determine the boundary of these sets. This problem was solved by H. Skhiri [43] who proved that in separable Hilbert spaces the boundaries sets of upper and lower semi-Fredholm operators coincide with that of Fredholm operators which is not the case of nonseparable one [44]. Also, recall that these classes of perturbations are used to study the stability of various essential spectra of closed densely defined operators on Banach spaces and to understood the phenomena in the case of transport operators involving in kinetic theory of gaz [35].

The organization of this paper is as follows:
In the first section, we give some definitions, notations and preliminaries results which will be used in the rest of the paper. Section 2 is devoted to give some assumptions ensuring the coincidence or not of different classes of semi-Fredholm and Fredholm operators in Banach spaces. In section 3, we extend some results given in section 1 of [34], moreover an answer to the most important question in this paper is given. In section 4, we improve most of the results established by the authors in [28] by using the class of semi-Fredholm perturbations. Finally, the goal of the section 5 is to study the problem of the normally solvable restrictions of semi-Fredholm perturbations to closed subspaces.

2 Preliminaries and Notations

Let X and Y be two complex infinite dimensional Banach spaces, we denote by $\mathcal{L}(X, Y)$ the Banach space of bounded linear operators between X and Y. The subspace of compact operators (resp. finite rank operators) of $\mathcal{L}(X, Y)$ is designated by $\mathcal{K}(X, Y)$ (resp. $\mathcal{F}_R(X, Y)$) and let $\mathcal{N}(X, Y)$ be the space of nuclear operators. A bounded linear operator $A \in \mathcal{L}(X, Y)$ is called normally solvable if its range $R(A)$ is closed in Y, we write $\mathcal{NS}(X,Y)$ for the set of normally solvable operators in $\mathcal{L}(X,Y)$. If $A \in \mathcal{L}(X,Y)$, let $N(A), \alpha(A)$ and $\widetilde{\beta}(A)$ be the null space, the nullity of A defined as the dimension of $N(A)$ and the deficiency of A given as the codimension of $\overline{R(A)}$ (or in other words the dimension of the quotient space $Y/\overline{R(A)}$). The set of upper semi-Fredholm of $\mathcal{L}(X,Y)$ is defined by

$$\Phi_+(X,Y) = \{A \in \mathcal{L}(X,Y) : \alpha(A) = \dim\ (Ker(A)) < \infty \text{ and } R(A) \text{ is closed in } Y\}$$

and the set of lower semi-Fredholm of $\mathcal{L}(X,Y)$ is defined by

$$\Phi_-(X,Y) = \{A \in \mathcal{L}(X,Y) : \widetilde{\beta}(A) < \infty \text{ (thus } R(A) \text{ is closed in } Y)\}$$

Operators in $\Phi_\pm(X,Y) = \Phi_+(X,Y) \bigcup \Phi_-(X,Y)$ are called semi-Fredholm operators while $\Phi(X,Y) = \Phi_+(X,Y) \bigcap \Phi_-(X,Y)$ is the set of Fredholm operators. For $A \in \Phi_\pm(X,Y)$, the integer $i(A) = \alpha(A) - \widetilde{\beta}(A)$ is called the index of A.

***Remark* 2.1** Notice that for Banach spaces X, Y with $X \neq Y$, the sets $\Phi_+(X,Y), \Phi_-(X,Y)$ and consequently $\Phi(X,Y)$ may be empty. Indeed, for example if $X = l_p$ $(1 \leq p \leq \infty)$ and $Y = l_q$ $(1 \leq q \leq \infty)(p \neq q)$, then $\mathcal{L}(X,Y) = \mathcal{F}(X,Y) = \mathcal{S}(X,Y)$ if $p < q$ and $\mathcal{L}(X,Y) = \mathcal{F}(X,Y) = \mathcal{K}(X,Y)$ if $q < p$. But if $X = Y$, we have $\Phi_+(X), \Phi_-(X)$ and $\Phi(X)$ are non empty since the identity operator I belongs to each one of them.

Definition 2.1 Let X, Y be two complex infinite dimensional Banach spaces and let $S \in \mathcal{L}(X,Y)$.

$(\imath)$ S is called an isomorphism if S is one-to-one and $S \in \mathcal{NS}(X,Y)$. Two closed subspaces $M \subset X$ and $M' \subset Y$ are called isomorphic and we write $M \approx M'$ if there exists an isomorphism $J : M \longrightarrow Y$ for which $R(J) = M'$, in this case we denote by $Iso(M, M')$ for the set of isomorphisms between M and M';

$(\imath\imath)$ S is called strictly singular if it's restriction to every infinite dimensional closed subspace of X is not an isomorphism. We denote by $\mathcal{S}(X,Y)$ the set of strictly singular operators between X and Y;

$(\imath\imath\imath)$ S is called strictly cosingular if there is no closed subspace $M \subset Y$ such that $\beta(M) = \infty$ such that the linear operator $Q_M S : X \longrightarrow Y/M$ is onto where $Q_M : Y \longrightarrow Y/M$ is the canonical mapping. We denote by $\mathcal{SC}(X,Y)$ the set of strictly singular operators between X and Y;

$(\imath v)$ S is called upper semi-Fredholm perturbation if $A + S \in \Phi_+(X,Y)$ whenever $A \in \Phi_+(X,Y)$. We denote by $\mathcal{F}_+(X,Y)$ the set of upper semi-Fredholm perturbations between X and Y;

(v) S is called lower semi-Fredholm perturbation if $A + S \in \Phi_-(X,Y)$ whenever $A \in \Phi_-(X,Y)$. We denote by $\mathcal{F}_-(X,Y)$ the set of lower semi-Fredholm perturbations between X and Y;

$(v\imath)$ S is called semi-Fredholm perturbation if $S \in \mathcal{F}_\pm(X,Y) = \mathcal{F}_+(X,Y) \bigcap \mathcal{F}_-(X,Y)$;

$(v\imath\imath)$ S is called Fredholm perturbation if $A + S \in \Phi(X,Y)$ whenever $A \in \Phi(X,Y)$. We denote by $\mathcal{F}(X,Y)$ the set of upper semi-Fredholm perturbations between X and Y.

$(viii)$ Banach spaces X and Y are called totally incomparables (resp. essentially incomparables) if $\mathcal{L}(X,Y) = \mathcal{S}(X,Y)$ (resp. $\mathcal{L}(X,Y) = \mathcal{F}(X,Y)$).

For a good study and properties on these classes, notions and notations, we can quote for example [16] and the references therein. Recall that the sets $\mathcal{S}(X,Y), \mathcal{SC}(X,Y), \mathcal{F}_+(X,Y)$, $\mathcal{F}_-(X,Y)$ and $\mathcal{F}(X,Y)$ are closed in $\mathcal{L}(X,Y)$ and we have

$$\mathcal{F}_R(X,Y) \subseteq \mathcal{N}(X,Y) \subseteq \mathcal{K}(X,Y) \subseteq \mathcal{S}(X,Y) \subseteq \mathcal{F}_+(X,Y) \subseteq \mathcal{F}(X,Y)$$

and

$$\mathcal{F}_R(X,Y) \subseteq \mathcal{N}(X,Y) \subseteq \mathcal{K}(X,Y) \subseteq \mathcal{SC}(X,Y) \subseteq \mathcal{F}_-(X,Y) \subseteq \mathcal{F}(X,Y)$$

The inclusion $\mathcal{S}(X,Y) \subseteq \mathcal{F}_+(X,Y)$ is due to T. Kato [33] while the inclusion $\mathcal{SC}(X,Y) \subseteq \mathcal{F} - (X,Y)$ was proved by Vladimirskii [46]. In general, these classes does not coincide (see [17]).

Proposition 2.1 [17] Let X, Y, Z be three infinite dimensional Banach spaces. Then

(i) If $\Phi(X,Y)$ or $\Phi(Y,Z)$ is nonempty, thus for all $A_1 \in \mathcal{L}(Y,Z), A_2 \in \mathcal{L}(X,Y), F_1 \in \mathcal{F}(X,Y), F_2 \in \mathcal{F}(Y,Z)$, we have $A_1F_1 \in \mathcal{F}(X,Z)$ and $F_2A_2 \in \mathcal{F}(X,Z)$.

(ii) If $\Phi_+(X,Y)$ or $\Phi_+(Y,Z)$ is nonempty, thus for all $A_1 \in \mathcal{L}(Y,Z), A_2 \in \mathcal{L}(X,Y), F_1 \in \mathcal{F}_+(X,Y), F_2 \in \mathcal{F}_+(Y,Z)$, we have $A_1F_1 \in \mathcal{F}_+(X,Z)$ and $F_2A_2 \in \mathcal{F}_+(X,Z)$.

(iii) If $\Phi_-(X,Y)$ or $\Phi_-(Y,Z)$ is nonempty, thus for all $A_1 \in \mathcal{L}(Y,Z), A_2 \in \mathcal{L}(X,Y), F_1 \in \mathcal{F}_-(X,Y), F_2 \in \mathcal{F}_-(Y,Z)$, we have $A_1F_1 \in \mathcal{F}_-(X,Z)$ and $F_2A_2 \in \mathcal{F}_-(X,Z)$.

when $X = Y$, we write $\mathcal{L}(X) = \mathcal{L}(X,X), \mathcal{K}(X) = \mathcal{K}(X,X), \mathcal{F}_R(X) = \mathcal{F}_R(X,X), \mathcal{N}(X) = \mathcal{N}(X,X), \mathcal{K}(X) = \mathcal{K}(X,X), \mathcal{NS}(X) = \mathcal{NS}(X,X), \mathcal{S}(X) = \mathcal{S}(X,X)$, $\mathcal{SC}(X) = \mathcal{SC}(X,X), \mathcal{F}_+(X) = \mathcal{F}_+(X,X), \mathcal{F}_-(X) = \mathcal{F}_-(X,X), \mathcal{F}(X) = \mathcal{F}(X,X)$, $\Phi_+(X) = \Phi_+(X,X), \Phi_-(X) = \Phi_-(X,X), \Phi_\pm(X) = \Phi_\pm(X,X), \Phi(X) = \Phi(X,X)$. It is known that $\mathcal{F}_+(X), \mathcal{F}_-(X)$ and $\mathcal{F}(X)$ are closed two-sided ideals in $\mathcal{L}(X)$ (see [17, 19, 20, 21, 22, 23, 24, 25, 26, 48, 49]).

Proposition 2.2 [40] Let X, Y be two infinite dimensional Banach spaces, we denote by $X^\star$ and $Y^\star$ the adjoint (dual) spaces of X and Y. Then

(i) $A \in \Phi_+(X,Y)$ if and only if $A^\star \in \Phi_-(Y^\star, X^\star)$,

(ii) $A \in \Phi_-(X,Y)$ if and only if $A^\star \in \Phi_+(Y^\star, X^\star)$,

(iii) $A \in \Phi(X,Y)$ if and only if $A^\star \in \Phi(Y^\star, X^\star)$.

Proposition 2.3 [40] Let X, Y, Z be three infinite dimensional Banach spaces. Then

(i) For all $A \in \Phi_+(Y,Z)$ and $B \in \Phi_+(X,Y)$, we have $AB \in \Phi_+(X,Z)$ and $i(AB) = i(A) + i(B)$,

(ii) For all $A \in \Phi_-(Y,Z)$ and $B \in \Phi_-(X,Y)$, we have $AB \in \Phi_-(X,Z)$ and $i(AB) = i(A) + i(B)$,

(iii) For all $A \in \Phi(Y,Z)$ and $B \in \Phi(X,Y)$, we have $AB \in \Phi(X,Z)$ and $i(AB) = i(A) + i(B)$.

Proposition 2.4 [40] Let X, Y, Z be three infinite dimensional Banach spaces and let $A \in \mathcal{L}(X, Y)$ and $B\imath\mathcal{L}(Y, Z)$. Then

$(\imath)$ If $BA \in \Phi_+(X, Z)$ then $A \in \Phi_+(X, Y)$,

$(\imath\imath)$ If $BA \in \Phi_-(X, Z)$ then $B \in \Phi_+(Y, Z)$,

$(\imath\imath\imath)$ If $BA \in \Phi(X, Z)$ then $A \in \Phi_+(X, Y)$ and $B \in \Phi_-(Y, Z)$.

A bounded linear operator $R \in \mathcal{L}(X)$ is said to be Riesz operator if for all $\lambda \in \mathbb{C}\backslash\{0\}$, we have $\lambda I - A \in \Phi(X)$. We denote by $\mathcal{R}(X)$ the set of Riesz operator on X. It is known that $\mathcal{R}(X)$ is not in general an ideal. Moreover, Riesz operators have Riesz-Schauder property of compact operators concerning the spectrum and $\mathcal{R}(X)$ contains all the ideals $\mathcal{F}_-(X), \mathcal{F}_+(X)$ and $\mathcal{F}(X)$. For more details on the set of Riesz operators, we can refer to [10].

Let X be an infinite dimensional Banach space and let $A \in \mathcal{L}(X)$, we denote by $\sigma(A)$ and $\rho(A) = \mathbb{C}\backslash\rho(A)$ respectively the spectrum and the resolvent set of A. Now, we define the following sets

$$\Phi_{+A} = \{\lambda \in \mathbb{C}/\lambda I - A \in \Phi_+(X)\},$$
$$\Phi_{-A} = \{\lambda \in \mathbb{C}/\lambda I - A \in \Phi_-(X)\},$$
$$\Phi_A = \{\lambda \in \mathbb{C}/\lambda I - A \in \Phi(X)\},$$
$$\Phi_A^0 = \{\lambda \in \mathbb{C}/\lambda I - A \in \Phi(X) \text{ and } i(\lambda I - A) = 0\}.$$

It is easy to observe that each one of the sets $\Phi_{+A}, \Phi_{-A}, \Phi_A$ and Φ_A^0 is an open set of the complex plane $\mathbb{C}$ which contains $\rho(A)$. Consequently, if we denote by

$$\sigma_+(A) = \mathbb{C}/\Phi_{+A},$$
$$\sigma_-(A) = \mathbb{C}/\Phi_{-A},$$
$$\sigma_e(A) = \mathbb{C}/\Phi_A,$$
$$\sigma_\omega(A) = \mathbb{C}/\Phi_A^0.$$

Thus, all the sets $\sigma_+(A), \sigma_-(A), \sigma_e(A)$ and $\sigma_\omega(A)$ are nonempty compact sets of $\sigma(A)$ satisfying that

$$\sigma_+(A) \bigcup \sigma_-(A) \subset \sigma_e(A) \subset \sigma_\omega(A) \subset \sigma(A),$$

Moreover, we have (see [2, 31])

$$\partial\sigma_e(A) \subset \sigma_+(A) \bigcap \sigma_-(A).$$

(where $\partial\sigma_e(A)$ is the boundary of the set $\partial\sigma_e(A)$).

Definition 2.2 Let X be an infinite dimensional complex Banach space and M a closed subspace of X. M is said to be complemented in X if there exists a closed subspace $Z \subset X$ such that $X = M \oplus Z$.

***Remark* 2.2** If X is a separable Hilbert space, it is known that every closed subspace M of X is complemented, which is not the case concerning general Banach spaces. For example F. J. Muray (1937) showed that the space $l_p(1 < p < \infty), p \neq 2$ has non-complemented closed subspaces. Fore more details on this fascinate subject, we quote [39].

***Remark* 2.3** It is known that if M is finite dimensional or finite codimensional then M is complemented. Also, Definition 2.2 is equivalent to the existence of a bounded linear projection $P \in \mathcal{L}(X)$ $(P^2 = P)$ such that $R(P) = M$.

Definition 2.3 Let X be an infinite dimensional Banach space. X is said to be decomposable if there exist two closed infinite dimensional subspaces M_1 and M_2 of X such that $X = M_1 \oplus M_2$. X is said to be indecomposable if it is not decomposable.

Definition 2.4 Let X be an infinite dimensional Banach space. X is called hereditarily indecomposable Banach space (and we write H.I) if X and all its closed infinite dimensional subspaces are indecomposable.

***Remark* 2.4** All classical Banach spaces are decomposable, separable Hilbert spaces, L_p-spaces $1 \leq p < \infty$, $C([0,1)$, ... In particular, we prove that if the Banach space X has an unconditional basis $(x_n)_1^\infty$ (see [37] for the definition and other properties), then X is decomposable since $X = [x_{2n}] \oplus [x_{2n+1}]$ where $[x_{2n}]$ and $[x_{2n+1}]$ are respectively the closed subspaces generated by the vectors $(x_{2n})_n$ and $(x_{2n+1})_n$ but the converse is not true, we can find decomposable Banach spaces does not having unconditional basis, for example $L_1(\mu)$ or $C([0,1])$. It is easy to deduce that complex Banach spaces can be divided into the following four categories:

$(\imath)$ H. I Banach spaces (Example: the space of Gowers-Maurey X_{GM} [29]);

$(\imath\imath)$ Indecomposable Banach spaces having closed infinite decomposable subspaces (Example: Schift space X_S [30]);

$(\imath\imath\imath)$ Decomposable Banach spaces having closed infinite dimensional decomposable subspaces (Examples: separable Hilbert spaces, L_p-spaces $(1 \leq p < \infty), C([0,1]))$;

(v) Decomposable Banach spaces having closed infinite dimensional indecomposable subspaces (Examples: $X_{GM} \times Y$ where Y is an infinite dimensional closed subspace of X_{GM} for which $dim(X_{GM}/Y) = \infty$. We have $X_{GM} \times Y = X_{GM} \times \{0\} \oplus \{0\} \times Y$).

2.1 Coincidence and not coincidence between the sets of semi-Fredholm operators

Let X, Y be two complex infinite dimensional Banach spaces. Recall that the sets $\Phi_+(X,Y)\backslash\Phi_-(X,Y)$ and $\Phi_-(X,Y)\backslash\Phi_+(X,Y)$ are closed in $\mathcal{L}(X,Y)$ (see the proof of Corollary 18. 2, page 169 in [40]).

Proposition 2.5 Let X, Y be two complex infinite dimensional Banach spaces. Then

If there exist complemented closed subspaces $M_1 \subset X, M_2 \subset Y$ with codim$(M_1) < \infty$ and codim$(M_2) = \infty$ such that $M_1 \simeq M_2$ then $\Phi_+(X,Y)\backslash\Phi_-(X,Y) \neq \emptyset$ and $\Phi_-(X,Y)\backslash\Phi_+(X,Y) \neq \emptyset$.

Proof. Let $J : M_1 \longrightarrow M_2$ be the isomorphism between M_1 and M_2 and denote by $T : X \longrightarrow Y$ given by $T(x) = J(x)$ if $x \in M_1$ and $T(x) = 0$ if $x \in [v_i]_1^n$ where $[v_i]_1^n$ is the complemented subspace of M_1 is X. Thus we have $T \in \mathcal{L}(X,Y)$ and $\alpha(T) = dim[v_i]_1^n$ and $R(T) = M_2$ which is closed in Y by assumption. It is easy to observe that $T \in \Phi_+(X,Y)\backslash\Phi_-(X,Y)$.

Now, Let $\widetilde{J} : M_2 \longrightarrow M_1$ be the isomorphism between M_2 and M_1 given by $\widetilde{J} = J^{-1}$ and denote by $S : X \longrightarrow Y$ given by $S(x) = \widetilde{J}(x)$ if $x \in M_2$ and $T(x) = 0$ if $x \in M''$ where M'' is the complemented subspace of M_2 in X with $dim(M'') = \infty$. Thus we have $S \in \mathcal{L}(X,Y)$ and $\widetilde{\beta}(S) = codim(M_1) < \infty$ by assumption. It is easy to observe that $S \in \Phi_-(X,Y)\backslash\Phi_+(X,Y)$.

Corollary 2.1 Let X, Y be two complex infinite dimensional Banach spaces. Then

If there exist closed complemented subspaces $M_1 \subset X, M_2 \subset Y$ with $\mathrm{codim}(M_1) < \infty$ and $\mathrm{codim}(M_2) = \infty$ such that $M_1 \simeq M_2$ then $\Phi_-(Y^\star, X^\star)\backslash\Phi_+(Y^\star, X^\star) \neq \emptyset$ and $\Phi_+(Y^\star, X^\star)\backslash\Phi_-(Y^\star, X^\star) \neq \emptyset$.

Proof. The proof follows directly from Proposition 2.2 and Proposition 2.5.

Also, as an immediate consequence of Proposition 2.2, we have

Corollary 2.2 Let X, Y be two reflexive infinite dimensional Banach spaces. Then

($\imath$) $\Phi_+(X,Y)\backslash\Phi_-(X,Y) \neq \emptyset$ if and only if $\Phi_-(Y^\star, X^\star)\backslash\Phi_+(Y^\star, X^\star) \neq \emptyset$;

($\imath\imath$) $\Phi_-(X,Y)\backslash\Phi_+(X,Y) \neq \emptyset$ if and only if $\Phi_+(Y^\star, X^\star)\backslash\Phi_-(Y^\star, X^\star) \neq \emptyset$.

Corollary 2.3 Let X be a separable Hilbert space. Then

$$\Phi_+(X)\backslash\Phi_-(X) \neq \emptyset \text{ and } \Phi_-(X)\backslash\Phi_+(X) \neq \emptyset.$$

Proof. The fact that X is a separable Hilbert space implies that all closed infinite dimensional subspaces of X are isomorphic. Thus the result follows from Proposition 2.2.

***Remark* 2.5** Let X_1, X_2 and X_3 three complex infinite dimensional Banach spaces. If $X_1 \simeq X_2$ and $X_2 \simeq X_3$ then $X_1 \simeq X_3$.

Now, we prove that Corollary 2.2 holds for Banach spaces $l_p(1 \leq p < \infty), p \neq 2$ and c_0. More precisely,

Corollary 2.4 Let X one of the Banach spaces $l_p(1 \leq p < \infty), p \neq 2$ or c_0. Then

$$\Phi_+(X)\backslash\Phi_-(X) \neq \emptyset \text{ and } \Phi_-(X)\backslash\Phi_+(X) \neq \emptyset.$$

Proof. ($\imath$) Let X one of the Banach spaces $l_p(1 \leq p < \infty), p \neq 2$ or c_0, then complemented subspaces of X are isomorphic to X (Theorem 2.2.4 in [3]) and hence they are isomorphic by Remark 2.5. Now, let M, M' be two complemented closed subspaces of X such that $codim(M) < \infty$ and $codim(M') = \infty$, thus $M \simeq M'$ and the result follows from Proposition 2.5.

***Remark* 2.6** Let X one of the Banach spaces $l_p(1 \leq p < \infty), p \neq 2$ or c_0, it is easy to deduce that the existence of infinite dimensional complemented subspaces M of X is always ensured. Indeed, X has an unconditional basic sequence $\{x_n\}_n$, then it suffices to take $M = [x_{2n}]$ or $M = [x_{2n+1}]$.

***Remark* 2.7** Let X be complex infinite dimensional Banach space. Then if $\Phi_+(X) = \Phi_-(X)$ (and consequently $\Phi_+(X) = \Phi_-(X) = \Phi(X)$) then we deduce that $\mathcal{F}(X) = \mathcal{F}_+(X) = \mathcal{F}_-(X)$. But the converse is in general not true as Corollary 2.3 and 2.4 show, since if X is one of the Banach spaces $l_p(1 \leq p < \infty)$ or c_0, we have $\mathcal{F}(X) = \mathcal{F}_+(X) = \mathcal{F}_-(X) = \mathcal{K}(X)$ [17].

Corollary 2.5 Let X one of the Banach spaces $L_p([0,1])(1 \leq p < \infty)$ or $C([0,1])$. Then

$$\Phi_+(X) \backslash \Phi_-(X) \neq \emptyset \text{ and } \Phi_-(X) \backslash \Phi_+(X) \neq \emptyset.$$

Proof. Following Proposition 2.5, if X is one Banach spaces $L_p([0,1])(1 \leq p < \infty)$ or $C([0,1])$, it suffices to find two complemented closed subspaces $M_1, M_2 \subset X$ such that $codim(M_1) < \infty$ and $codim(M_2) < \infty$ such that $M_1 \approx M_2$. Let $M_1 \subset X$ be a closed subspace such that $codim(M_1) = 1$, then $M_1 \approx X$ (see [**?**]). Moreover, there exist a complemented closed subspace $M_2 \subset X$ such that $codim(M_2) = \infty$ (see [7]), then $M_2 \approx X$ since X is a primary Banach space (see for the definition [1, 7] and page 1594 of [32]). Finally, Remark 2.6 implies that $M_1 \approx M_2$ which is the desired result.

Corollary 2.6 Let X be an infinite dimensional Banach space such that $\Phi_+(X) \subset \Phi_-(X)$ (and consequently $\Phi(X) = \Phi_+(X)$) or $\Phi_-(X) \subset \Phi_+(X)$ (and consequently $\Phi(X) = \Phi_-(X)$). Let M_1 and M_2 be two closed subspaces such that $codim(M_1) < \infty$ and $dim(X/M_2) = \infty$, then nor M_2 is not complemented in X or $Iso(M_1, M_2) = \emptyset$ if M_2 is complemented.

Proposition 2.6 Let X be an infinite dimensional Banach space. Then

($\imath$) $\sigma_e(A) = \sigma_\omega(A)$ for all $A \in \mathcal{L}(X)$ if and only if $\Phi(X) = \Phi_0(X)$;

($\imath\imath$) If $\Phi_+(X)$ (resp. $\Phi_-(X), \Phi(X)$) is connected in $\mathcal{L}(X)$ then $\Phi(X) = \Phi_+(X) = \Phi_0(X)$ (resp. $\Phi(X) = \Phi_-(X) = \Phi_0(X), \Phi(X) = \Phi_0(X)$);

($\imath\imath\imath$) $\Phi_+(X) = \Phi_-(X) = \Phi(X) = \Phi_0(X)$ if and only if $\sigma_+(A) = \sigma_-(A) = \sigma_e(A) = \sigma_\omega(A)$;

($\imath v$) If $\sigma_e(A)$ has an empty interior for all $A \in \mathcal{L}(X)$ then $\Phi(X) = \Phi_+(X) = \Phi_-(X)$;

(v) If $\sigma_e(A)$ has an empty interior with Φ_A connected for all $A \in \mathcal{L}(X)$ then $\Phi_+(X) = \Phi_-(X) = \Phi(X) = \Phi_0(X)$.

Proof. ($\imath$) Assume that $\sigma_e(A) = \sigma_\omega(A)$ for all $A \in \mathcal{L}(X)$. To prove that $\Phi(X) = \Phi_0(X)$, it suffices to prove the inclusion $\Phi(X) \subset \Phi_0(X)$. Let $A \in \Phi(X)$, thus $0 \notin \sigma_e(A) = \sigma_\omega(A)$ which proves that $A \in \Phi_0(X)$. Conversely, assume that $\Phi(X) = \Phi_0(X)$ and let $\lambda \notin \sigma_\omega(A)$ hence $\lambda I - A \in \Phi_0(X) = \Phi(X)$, this proves that $\lambda \notin \sigma_e(A)$.

$(\imath\imath)$ Assume that the set $\Phi_+(X)$ is connected in $\mathcal{L}(X)$ and let $A \in \Phi_+(X)$, since $I \in \Phi_+(X)$, the stability of the index on $\Phi_+(X)$ (see Corollary 2, page 169 in [40]) shows that $i(A) = 0$ and consequently $A \in \Phi_0(X)$. The case of $\Phi_-(X)$ or $\Phi(X)$ can be established by the same way.

$(\imath\imath\imath)$ Can be proved by a same argument as in $(\imath)$.

$(\imath v)$ See the proof of Proposition 3.11 in [14].

(v) Following $(\imath v)$, it suffices to prove that if Φ_A is connected for all $A \in \mathcal{L}(X)$ then $\Phi(X) = \Phi_0(X)$. By $(\imath)$, this is equivalent to prove that $\sigma_e(A) = \sigma_\omega(A)$ for all $A \in \mathcal{L}(X)$. Let $\lambda \notin \sigma_e(A)$, then $\lambda I - A \in \Phi(X)$. On the other hand, we have $\rho(A) = \mathbb{C}\backslash\sigma(A) \subset \Phi_A$. The fact that the index is constant on connected components of Φ_A (see Corollary 2, page 169 in [40]) shows that $i(\lambda I - A) = 0$ which implies that $\lambda \notin \sigma_\omega(A)$. Since $\sigma_e(A) \subset \sigma_\omega(A)$ and $\Phi_A \subset \Phi_0(A)$, we conclude that $\sigma_e(A) = \sigma_\omega(A)$ which is the desired result.

Corollary 2.7 Let X be an infinite dimensional Banach space. If for all $A \in \mathcal{L}(X)$, $\sigma_e(A)$ is a finite set then

$$\Phi_+(X) = \Phi_-(X) = \Phi(X) = \Phi_0(X).$$

Proof. This result is an immediate consequence of the assertion (v) in Proposition 2.6.

As an immediate consequence of Corollary 2.2, we have

Corollary 2.8 If X is one of the following Banach spaces:

$(\imath)$ X a H.I Banach space;

$(\imath\imath)$ X a Q.H.I Banach space (quotient hereditarily indecomposable) (see page 223 in [13] for the definition);

$(\imath\imath\imath)$ X a HD_n Banach space (hereditarily finitely indecomposable)(see page 223 in [13] for the definition);

(v) X a QD_n Banach space (quotient hereditarily finitely indecomposable)(see page 223 in [13] for the definition).

Then

$$\Phi_+(X) = \Phi_-(X) = \Phi(X) = \Phi_0(X).$$

Proof. Indeed, if X is one of these spaces, we have for all $A \in \mathcal{L}(X)$, $\sigma_e(A)$ is a finite set. For more details, see Remark 2.2 in [13].

3 Complementation of the Kernel and the Range

This section is inspired essentially by section 2 and 3 of [34], Theorem 2.1 and Lemma 3.1 of this paper are improved and generalized here.

Let X, Y be two complex infinite dimensional Banach spaces and let $\Gamma_R[X,Y], \Gamma_N[X,Y]$ be the following subsets in $\mathcal{L}(X,Y)$.

$$\Gamma_R[X,Y] = \{A \in \mathcal{L}(X,Y) : \overline{R(T)} \text{ is complemented in } Y\}$$

$$\Gamma_N[X,Y] = \{A \in \mathcal{L}(X,Y) : N(T) \text{ is complemented in } X\}$$

We denote by $\Gamma[X,Y] = \Gamma_N[X,Y] \bigcap \Gamma_R[X,Y]$. If $X = Y$, we write $\Gamma_N[X,X] = \Gamma_N[X], \Gamma_R[X,X] = \Gamma_R[X]$ and $\Gamma[X,X] = \Gamma[X]$.

Theorem 3.1 Let X and Y be an infinite dimensional Banach spaces.

($\imath$) If Y is a Hilbert space then $\Gamma_R[X,Y] = \mathcal{L}(X,Y)$ and if X is a Hilbert space then $\Gamma_N[X,Y] = \mathcal{L}(X,Y)$.

($\imath\imath$) $\mathcal{FR}(X,Y) \subset \Gamma_R[X,Y]$; an operator having a finite codimensional kernel is an element of $\Gamma_N[X,Y]$.

($\imath\imath\imath$) If $Isom(X,Y) \neq \emptyset$, then $Isom(X,Y) \subset \Gamma[X,Y] = \Gamma_N[X,Y] \bigcap \Gamma_R[X,Y]$.

(v) If $\Phi_+(X,Y) \bigcap \Gamma_R[X,Y] \neq \emptyset$, then for all $A \in \Phi_+(X,Y) \bigcap \Gamma_R[X,Y]$, we have $A+S \in \Gamma[X,Y]$ for every $S \in \mathcal{S}(X,Y)$.

($\imath v$) If $Isom(X,Y) \neq \emptyset$, then for all $A \in Isom(X,Y)$, we have $R(A+S)$ is closed in Y and for every $S \in \mathcal{S}(X,Y)$ we have $A+S \in \Gamma[X,Y]$.

This theorem can be seen as an extension of Theorem 2.1 in [34] given in the case $X = Y$. The proof of Theorem 3.1 can be established by the same techniques and for assertions (v) and ($\imath v$) on the stability perturbations by strictly singular operators, we can see [11, 12].

Proposition 3.1 Let X an infinite dimensional Banach spaces such that there exists two closed infinite dimensional subspaces M_1 and M_2 such that $M_1 \approx M_2$ with M_1 complemented and M_2 is not complemented in X. Then the set $\Gamma_R[X]$ is not open in $\mathcal{L}(X)$.

Proof. Since $M_1 \approx M_2$ then there exists $J : M_1 \longrightarrow M_2$ an isomorphism. The fact that M_1 is complemented in X implies the existence of a closed subspace $M_1' \subset X$ such that $X = M_1 \oplus M_1'$. Denote by $T \in \mathcal{L}(X)$ defined by $T_{|M_1} = J$ and $T_{|M_1'} = 0$. Let $F \in \mathcal{FR}(X)$ such that $R(F) \bigcap R(T) = R(F) \bigcap M_2 = \{0\}$ (since $R(T) = M_2$). From the assertion ($\imath\imath$) of Theorem 3.1, it follows that $F \in \Gamma_R[X]$. We denote by $S_n = F + \dfrac{1}{n}T$ for all integer $n \geq 1$. It is easy to observe that $S_n \longrightarrow F$ in the topology norm of $\mathcal{L}(X)$. Moreover, $R(S_n) = R(F + \dfrac{1}{n}T) = R(F) \oplus M_2$ which is not complemented in X. Now let $\epsilon > 0$, thus the open ball $B(F,\epsilon)$ contains an infinite elements of S_n which implies that $B(F,\epsilon) \nsubseteq \Gamma_R$ and gives the result.

***Example* 3.1** If X is one of the following Banach spaces, then X has two closed infinite dimensional Banach spaces satisfying the assumptions of Proposition 3.1 (see [49]).

1. $X = L_p(0,\infty) + L_q(0,\infty)(1 < p < q < 2)$ equipped with the norm $\|f\| = \inf\{\|h\|_p + \|g\|_q; f = h + g\}$.

2. $X = L_p(0,\infty) \bigcap L_q(0,\infty)(2 < q < p < \infty)$ equipped with the norm $\|f\| = \max(\|f\|_p, \|f\|_q)$.

3. $X = L(p,q)$ the Lorentz space on $[0,1]$ with the norm $\|f\| = (\frac{q}{p}\int_0^1 [t^{\frac{1}{p}} f^\star(t)]^q \frac{dt}{t})^{\frac{1}{q}}$ where $f^\star$ is a decreasing rearrangement of f.

***Remark* 3.1** We note that the set $\Gamma_R[X]$ is not in general closed. Indeed, in X is one of the Banach spaces indicated above, then there exist an infinite dimensional closed subspaces $M_1, M_2 \subset X$ such that $M_1 \approx M_2 \approx l_p$ $(1 \leq p < \infty)$ or c_0 with M_1 complemented and M_2 not complemented in X. Since M_1 is complemented in X then there exists a projection $P : X \longrightarrow M_1$, hence $B = JP \in \mathcal{F}(X)$ where J is the isomorphism between M_1 and M_2 but $B \notin \Gamma_R[X]$. Thus if we take $A_n = B + \frac{1}{n}I$, thus $A_n \in \Phi(X)$ and consequently $A_n \in \Gamma_R[X]$ but $\lim_{n\longrightarrow\infty} A_n = B \notin \Gamma_R[X]$.

Proposition 3.2 The set $\Gamma_R[X_{GM}]$ is not an open set in $\mathcal{L}(X_{GM})$.

Proof. Let $T = \sum_{i=1}^{\infty} x_i^\star \otimes e_i$ the strictly singular (non compact operator) given in [4] where $(x_i^\star)$ is a seminormalized block sequence in $X_{GM}^\star$ and (e_i) is the unit vector basis of X_{GM} (see for more details [4]). By the construction, $\alpha(T) = \infty$. Thus $N(T)$ is not complemented in X_{GM}. Indeed, if $N(T)$ is complemented in X_{GM} thus necessarily there exists a finite dimensional space $[z_i]_{i=1}^k$ such that $X_{GM} = N(T) \oplus [z_i]_{i=1}^k$. Hence $T_{|[z_i]_{i=1}^k}$ is an isomorphism and T becomes a finite rank operator which is a contradiction. Let $F \in \mathcal{FR}(X_{GM})$ then $F \in \Gamma_N[X_{GM}]$ and let $F_n = F + \frac{1}{n}T$ for all integer $n \geq 1$. It is easy to show that $\alpha(F_n) = \infty$ and $N(F_n)$ is not complemented in X_{GM} which proves that $F_n \notin \Gamma_N$. Now let $\epsilon > 0$, thus the open ball $B(F, \epsilon)$ contains an infinite elements of F_n which implies that $B(F, \epsilon) \nsubseteq \Gamma_N$ and gives the result.

***Remark* 3.2** We note that the set $\Gamma_N[X]$ is not in general closed. Indeed, following the previous example if we take $T_n = T + \frac{1}{n}I$, thus $T_n \in \Phi(X)$ and consequently $T_n \in \Gamma_N[X]$ but $\lim_{n\longrightarrow\infty} T_n = T \notin \Gamma_N[X]$.

***Remark* 3.3** Notice that Propositions 3.1 and 3.2 give an answer to Question 3.1 given in [34].

Theorem 3.2 (see Theorem 14, page 160 in [40]) Let X, Y be two complex infinite dimensional Banach spaces and let $T \in \mathcal{L}(X, Y)$.

$(\imath)$ $T \in \Phi_+(X, Y) \bigcap \Gamma_R[X, Y]$;

$(\imath\imath)$ there exists $S \in \mathcal{L}(X, Y)$ and $K \in \mathcal{K}(X)$ such that $ST = I_X + K$.

Theorem 3.3 (see Theorem 15, page 160 in [40]) Let X, Y be two complex infinite dimensional Banach spaces and let $T \in \mathcal{L}(X, Y)$.

$(\imath)$ $T \in\in \Phi_-(X, Y) \bigcap \Gamma_N[X, Y]$;

$(\imath\imath)$ there exists $S \in \mathcal{L}(X, Y)$ and $K \in \mathcal{K}(Y)$ such that $ST = I_Y + K$.

As in [34], we denote by $\mathcal{F}_l[X,Y] = \Phi_+(X,Y)\bigcap\Gamma_R[X,Y]$ and $\mathcal{F}_r[X,Y] = \Phi_-(X,Y)\bigcap\Gamma_N[X,Y]$. It is easy to observe that Lemma 3.1 of [34] can be extended to the case of Banach spaces X and Y as follows:

Lemma 3.1 Let X, Y be two complex infinite dimensional Banach spaces. Then

($\imath$) $\Phi_+(X,Y) \subseteq \Gamma_N[X,Y]$ and $\Phi_-(X,Y) \subseteq \Gamma_R[X,Y]$ and consequently

$$\Phi(X,Y) \subseteq \Gamma[X,Y] \text{ and } \Phi_+(X,Y)\bigcup\Phi_-(X,Y) \subseteq \Gamma_N[X,Y]\bigcup\Gamma_R[X,Y];$$

($\imath\imath$) $\mathcal{F}_l[X,Y]\backslash\mathcal{F}_r[X,Y] = (\Phi_+(X,Y)\backslash\Phi_-(X,Y))\bigcap\Gamma_R[X,Y] = \Phi_+(X,Y)\backslash\Phi_-(X,Y))\bigcap\Gamma[X,Y]$; and

$\mathcal{F}_r[X,Y]\backslash\mathcal{F}_l[X,Y] = (\Phi_-(X,Y)\backslash\Phi_+(X,Y))\bigcap\Gamma_N[X,Y] = \Phi_-(X,Y)\backslash\Phi_+(X,Y))\bigcap\Gamma[X,Y]$;

($\imath\imath\imath$) $\mathcal{F}_l[X,Y]\bigcap\mathcal{F}_r[X,Y] = \Phi(X,Y)$;

($\imath v$) $\mathcal{F}_l[X,Y]\bigcup\mathcal{F}_r[X,Y] = (\Phi_-(X,Y)\backslash\Phi_+(X,Y))\bigcap\Gamma[X,Y] \subseteq \Gamma[X,Y]$;

(v) If $\Phi_+(X,Y) = \Phi_-(X,Y))$, then $\Phi_+(X,Y) = \mathcal{F}_l[X,Y]$ and $\Phi_-(X,Y) = \mathcal{F}_r[X,Y]$ and consequently $\mathcal{F}_l[X,Y] = \mathcal{F}_r[X,Y]$;

($v\imath$) If $\mathcal{F}_l[X,Y] = \mathcal{F}_r[X,Y]$, then $\Phi_+(X,Y)\bigcap\Gamma_R[X,Y] = \Phi_-(X,Y)\bigcap\Gamma_N[X,Y] = \Phi_+(X,Y)\bigcap\Gamma[X,Y] = \Phi_-(X,Y)\bigcap\Gamma[X,Y] = \Phi(X,Y) = (\Phi_+(X,Y)\bigcup\Phi_-(X,Y))\bigcap\Gamma[X,Y]$ and this case, we have also

$(\Phi_+(X,Y)\backslash\Phi_-(X,Y))\bigcap\Gamma[X,Y] = (\Phi_-(X,Y)\backslash\Phi_+(X,Y))\bigcap\Gamma_N[X,Y] = \emptyset$;

($v\imath\imath$) If $\Gamma_N[X,Y] = \Gamma_R[X,Y]$, then $\Phi_+(X,Y) = \mathcal{F}_l[X,Y]$ and $\Phi_-(X,Y) = \mathcal{F}_r[X,Y]$; moreover,

$\Phi_+(X,Y) = \mathcal{F}_l[X,Y]$ and $\Phi_-(X,Y) = \mathcal{F}_r[X,Y]$ if and only if $\Phi_+(X,Y)\bigcup\Phi_-(X,Y) \subseteq \Gamma[X,Y]$.

Corollary 3.1 Let X, Y two complex infinite dimensional Hilbert spaces, then

$$\Phi_+(X,Y) = \mathcal{F}_l[X,Y] \text{ and } \Phi_-(X,Y) = \mathcal{F}_r[X,Y].$$

Proof. It's an immediate consequence of Lemma 3.1 ($v\imath\imath$).

Corollary 3.2 Let X be a complex infinite dimensional Banach space such that for all $A \in \mathcal{L}(X)$, the set $\sigma_e(A)$ has an empty interior. Then

$$\Phi_+(X) = \mathcal{F}_l[X] \text{ and } \Phi_-(X) = \mathcal{F}_r[X].$$

Proof. It's an immediate consequence of Proposition 2.6 ($\imath v$) and Lemma 3.1 ($v\imath$).

4 The instability of non-semi-Fredholm operators by semi-Fredholm perturbations

The goal of this section is to improve the contributions of M. Gonzalez and V. Onieva in [28]. More precisely, we prove that many of their results hold true by means of the class of semi-Fredholm perturbations.

Theorem 4.1 Let X, Y be two complex infinite dimensional Banach spaces. Then

$(\imath)$ $\overline{\mathcal{L}(X,Y)\backslash\mathcal{NS}(X,Y)} = \mathcal{L}(X,Y)\backslash\Phi_{\pm}(X,Y)$;

$(\imath\imath)$ $T \in \mathcal{L}(X,Y)\backslash\Phi_{\pm}(X,Y)$ if and only if there exists $A \in \mathcal{L}(X,Y)\backslash\mathcal{NS}(X,Y)$ and $F \in \mathcal{F}_{\pm}(X,Y)$ such that $T = A + F$;

$(\imath\imath\imath)$ $T \in \mathcal{L}(X,Y)\backslash\Phi_{\pm}(X,Y)$ if and only if there exists $A \in \mathcal{L}(X,Y)$ with $\alpha(A) = \widetilde{\beta}(A) = \infty$ and $F \in \mathcal{F}_{\pm}(X,Y)$ such that $T = A + F$.

(v) For all $T \in \partial\Phi(X,Y)$ then there exists $A_1 \in \mathcal{L}(X,Y), A_2 \in \mathcal{L}(X,Y)\backslash\mathcal{NS}(X,Y)$ and $F_1, F_2 \in \mathcal{F}_{\pm}(X,Y)$ with $\alpha(A_1) = \widetilde{\beta}(A_1) = \infty$ such that $T = A_1 + F_1 = A_2 + F_2$.

Proof. $(\imath)$ (see Theorem 2.1 (1) in [28]).

$(\imath\imath)$ Assume that $T \in \mathcal{L}(X,Y)\backslash\Phi_{\pm}(X,Y)$, then (Theorem 2.1 (2) in [28]) implies that there exists $A \in \mathcal{NS}(X,Y)$ and $K \in \mathcal{K}(X,Y)$ such that $T = A + K$. Since $\mathcal{K}(X,Y) \subseteq \mathcal{F}_{\pm}(X,Y)$, we get the result for the first implication. Conversely, let $T \in \mathcal{L}(X,Y), A \in \mathcal{L}(X,Y)\backslash\mathcal{NS}(X,Y), F \in \mathcal{F}_{\pm}(X,Y)$ such that $T = A + F$ and assume that $T \in \Phi_{\pm}(X,Y)$, thus $T - F = A \in \mathcal{NS}(X,Y)$ which is a contradiction.

$(\imath\imath\imath)$ the proof of this assertion is based on (Theorem 2.1 (3) in [28]) and the same argument given in $(\imath\imath)$.

(v) is deduced by combining the assertions $(\imath)$ and $(\imath\imath)$ and Lemma 1, page. 169 in [40].

***Remark* 4.1** It is easy to observe that Theorem 4.1 holds true if we replace the class $\mathcal{F}_{\pm}(X,Y)$ by any subclass $I \subseteq \mathcal{F}_{\pm}(X,Y)$.

Definition 4.1 Let X, Y be two complex infinite dimensional Banach spaces and let U_0, U_1, U_2, U_3, U_4 be the sets given in [28]. If $T \in \mathcal{L}(X,Y)$ we denote by $T^\star$ the adjoint operator of T.

$$U_0 = \{T \in \mathcal{L}(X,Y)\backslash\mathcal{NS}(X,Y) : \alpha(T) = \alpha(T^\star) < \infty\}$$
$$U_1 = \{T \in \mathcal{L}(X,Y)\backslash\mathcal{NS}(X,Y) : |\alpha(T) - \alpha(T^\star)| < \infty\}$$
$$U_2 = \{T \in \mathcal{L}(X,Y)\backslash\mathcal{NS}(X,Y) : \alpha(T) - \alpha(T^\star) = -\infty\}$$
$$U_3 = \{T \in \mathcal{L}(X,Y)\backslash\mathcal{NS}(X,Y) : \alpha(T) - \alpha(T^\star) = \infty\}$$
$$U_4 = \{T \in \mathcal{L}(X,Y)\backslash\mathcal{NS}(X,Y) : \alpha(T) = \alpha(T^\star) = \infty\}$$
$$U_5 = \{T \in \mathcal{NS}(X,Y) : \alpha(T) = \alpha(T^\star) = \infty\}$$

The following theorem and it's corollary are an extension of Theorem 3.4 and Corollary 3.5 in [28]. Proofs can be adapted, so they are omitted.

Theorem 4.2 Let X, Y be two infinite dimensional Banach spaces.

$(\imath)$ If X and Y are separable, then $U_j + \mathcal{F}_{\pm}(X,Y) = \mathcal{L}(X,Y)\backslash\Phi_{\pm}(X,Y), j = 0, 1$.

$(\imath\imath)$ If X is separable then $U_2 + \mathcal{F}_{\pm}(X,Y) = \mathcal{L}(X,Y)\backslash\Phi_{\pm}(X,Y)$.

$(\imath\imath\imath)$ If Y is separable then $U_3 + \mathcal{F}_{\pm}(X,Y) = \mathcal{L}(X,Y)\backslash\Phi_{\pm}(X,Y)$.

(v) $U_4 + \mathcal{F}_{\pm}(X,Y) = \mathcal{L}(X,Y)\backslash\Phi_{\pm}(X,Y)$.

Following Theorem 4.2 and Lemma 1 page 169 in [40], we obtain

Corollary 4.1 Let X, Y be two infinite dimensional Banach spaces.

($\imath$) If X and Y are separable, then $\partial(\Phi(X)) \subseteq U_j + \mathcal{F}_\pm(X,Y), j = 0, 1$.

($\imath\imath$) If X is separable then $\partial(\Phi(X)) \subseteq U_2 + \mathcal{F}_\pm(X,Y)$.

($\imath\imath\imath$) If Y is separable then $\partial(\Phi(X)) \subseteq U_3 + \mathcal{F}_\pm(X,Y)$.

(v) $\partial(\Phi(X)) \subseteq U_4 + \mathcal{F}_\pm(X,Y)$.

Corollary 4.2 Let X be a complex infinite dimensional Banach space. Then the following assertions are equivalent

($\imath$) X is separable.

($\imath\imath$) $U_0 + \mathcal{F}_\pm(X) = \mathcal{L}(X)\backslash\Phi_\pm(X)$.

($\imath\imath\imath$) $U_1 + \mathcal{F}_\pm(X) = \mathcal{L}(X)\backslash\Phi_\pm(X)$.

(v) $U_3 + \mathcal{F}_\pm(X) = \mathcal{L}(X)\backslash\Phi_\pm(X)$.

By combining Corollary 4.1, Corollary 3.5 in [28] and Lemma 1, page 169 in [40], we get

Corollary 4.3 Let X be a separable complex infinite dimensional Banach space. Then we have

($\imath$) $\partial(\Phi(X)) \subseteq U_0 + \mathcal{F}_\pm(X) = U_0 + \mathcal{K}(X)$.

($\imath\imath$) $\partial(\Phi(X)) \subseteq U_1 + \mathcal{F}_\pm(X) = U_1 + \mathcal{K}(X)$.

($\imath\imath\imath$) $\partial(\Phi(X)) \subseteq U_3 + \mathcal{F}_\pm(X) = U_3 + \mathcal{K}(X)$.

***Remark* 4.2** Notice that $U_j + \mathcal{F}_\pm(X) = U_j + \mathcal{K}(X)$ ($j = 0, 1, 3$) does not imply necessarily $\mathcal{F}_\pm(X) = \mathcal{K}(X)$.

Proposition 4.1 Let X be an infinite complex dimensional Banach space. Then

$$\mathcal{R}(X) \subseteq \mathcal{L}(X)\backslash\Phi_\pm(X).$$

Proof. It is equivalent to prove that $\mathcal{R}(X)\bigcap\Phi_\pm(X) = \emptyset$. Let $R \in \mathcal{R}(X)\bigcap\Phi_\pm(X)$ hence $\sigma_e(R) = \{0\}$. On the other hand, we have $\sigma_+(R)\bigcup\sigma_-(R) \subseteq \sigma_e(R)$ and $\partial(\sigma_e(R)) \subseteq \sigma_+(R)\bigcap\sigma_-(R)$ we infer that $\sigma_e(R) = \sigma_+(R) = \sigma_-(R) = \{0\}$. By the definition of the sets $\sigma_+(R)$ and $\sigma_-(R)$ we get that $R \notin \Phi_\pm(X)$ which is a contradiction and achieves the proof.

***Remark* 4.3** Notice that the inclusion $\mathcal{R}(X)\bigcap\Phi_+(X)$ was proved firstly in [36] by a different techniques of ours.

As an immediate consequence of Proposition 4.1 and Corollary 4.3, we have

Corollary 4.4 Let X be a separable infinite complex dimensional Banach space. Then for all $R \in \mathcal{R}(X)$ there exist $K_0, K_1, K_2 \in \mathcal{K}(X), A_0 \in U_0, A_1 \in U_1, A_2 \in U_2$ such that

$$R = A_0 + K_0 = A_1 + K_1 = A_2 + K_2.$$

Let X be a complex Banach space. We denote by $\mathcal{Q}(X) = \{B \in \mathcal{L}(X)/\sigma(Q) = \{0\}\}$ the set of quasinilpotent operators on X. One of the complicated and open problems in operator theory is to prove that for all $R \in \mathcal{R}(X)$ then there exist $K \in \mathcal{K}(X), Q \in \mathcal{Q}(X)$ such that $R = K + Q$, this problem is known as the West decomposition of Riesz operators (see [50]). Recently, this problem was reduced just for Banach spaces having Rademacher's type equal to 1 (see [41, 47]).

Corollary 4.5 Let X be a separable infinite complex dimensional Banach space having Rademacher's type 1. Assume that there exists $i \in \{0, 1, 2\}$ such that for all $A_i \in U_i$ there exists $K_i \in \mathcal{K}(X), T_i \in \mathcal{L}(X)$ such that $A_i = K_i + T_i$ and $\sigma(T_i) \subseteq \sigma_\omega(A_i)$. Then for all $R \in \mathcal{R}(X)$, R satisfies the West decomposition.

Proof. Let $R \in \mathcal{R}(X)$, by Proposition 4.1 and Corollary 4.4, it follows that there exists $K_i \in \mathcal{K}(X), S_i \in U_i$ such that $R = K_i + S_i$. Since $R \in \mathcal{R}(X)$, then $R - K_i = S_i \in \mathcal{R}(X)$. The fact that $S_i \in U_i$ and assumptions imply that $\sigma(S_i) \subseteq \sigma_\omega(R) = \sigma_e(R) = \{0\}$. Thus $\sigma(S_i) = \{0\}$ and consequently $S_i \in \mathcal{Q}(X)$ and achieves the proof.

Definition 4.2 Let X be an infinite complex dimensional Banach space and $T \in \mathcal{L}(X)$. T is called relatively regular if there exists $B \in \mathcal{L}(X)$ such that $TBT = T$. We denote by $\mathcal{RG}(X)$ the set of all relatively regular operators.

Proposition 4.2 [45] Let X be a complex infinite dimensional Banach space. Then

$$\mathcal{RG}(X) = \Gamma_N[X] \bigcap \Gamma_R[X].$$

Let X be a complex infinite dimensional Banach space and let π the quotient map from $\mathcal{L}(X)$ onto the Banach space $\mathcal{L}(X)/\mathcal{F}_\pm(X)$ and let us

$$\widetilde{R}(X) = \{T \in \mathcal{L}(X) \backslash \Phi_\pm(X)/\pi(T) \in \mathcal{RG}(\mathcal{L}(X)/\mathcal{F}_\pm(X))\}.$$

Proposition 4.3 [45] Let X be a complex infinite dimensional Banach space. Then

$$(U_5 \bigcap \mathcal{RG}(X)) + \mathcal{F}_\pm(X) \subseteq \widetilde{R}(X) \subseteq \mathcal{L}(X) \backslash \Phi_\pm(X).$$

Moreover, if $dim(\mathcal{L}(X)/\mathcal{F}_\pm(X)) = \infty$ then $\widetilde{R}(X) \neq \mathcal{L}(X) \backslash \Phi_\pm(X)$.

Proof. Let $T \in U_5 \bigcap \mathcal{RG}(X)$ then there exists $B \in \mathcal{L}(X)$ such that $T = TBT$ and let $F \in \mathcal{F}_\pm(X)$, then $\pi(T) = \pi(T+F)\pi(B)\pi(T+F) = \pi(T)\pi(B)\pi(T)$. Since $T \in U_5$ then $T \notin \Phi_\pm(X)$ and consequently $T + F \notin \Phi_\pm(X)$ which implies that $\pi(T) = \pi(T+F) \neq 0$ and shows that $\pi(T + F) \in \mathcal{RG}(\mathcal{L}(X)/\mathcal{F}_\pm(X))$. This gives that $T + F \in \widetilde{R}(X)$.
Now if $\widetilde{R}(X) = \mathcal{L}(X) \backslash \Phi_\pm(X)$, then $dim(\mathcal{L}(X)/\mathcal{F}_\pm(X)) < \infty$ (see [42], page 96) which is a contradiction.

Corollary 4.6 [28] Let $\mathcal{H}$ be an infinite dimensional Hilbert space. Then

$$U_5 + \mathcal{K}(\mathcal{H}) \subseteq \widetilde{R}(\mathcal{H}) \subsetneq \mathcal{L}(\mathcal{H}) \backslash \Phi_\pm(\mathcal{H}).$$

Proof. Follows immediately from the fact that $\mathcal{F}_\pm(\mathcal{H}) = \mathcal{K}(\mathcal{H})$ and $U_5 \subseteq \widetilde{R}(\mathcal{H})$ together with $dim(\mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})) = \infty$.

5 Restrictions of upper-semi-Fredholm operators and upper-semi-Fredholm-perturbations which are normally solvable

In this section, some inherited properties of the upper-semi-Fredholm perturbations to closed subspaces are established. We start by the following Proposition.

Theorem 5.1 [28] Let X, Y be two complex infinite dimensional Banach spaces.

($\imath$) If $K \in \mathcal{K}(X,Y)$ then for all closed infinite dimensional subspace $Z \subseteq X$ if $K(Z)$ is closed in Y then $K_{|Z} : Z \longrightarrow K(Z) \in \mathcal{FR}(Z, K(Z))$;

($\imath\imath$) If $S \in \mathcal{S}(X,Y)$ then for all closed infinite dimensional subspace $Z \subseteq X$ if $S(Z)$ is closed in Y then $S_{|Z} : Z \longrightarrow S(Z) \in \mathcal{S}(Z, S(Z))$;

($\imath\imath\imath$) If $S \in \mathcal{S}(X,Y) \bigcap \mathcal{N}(X,Y)$ with $dim(R(S)) = \infty$ then $\mathcal{S}(X,Y) \neq \mathcal{K}(X,Y)$;

(v) $S \in \mathcal{S}(X,Y)$ if and only if for all closed infinite dimensional subspace $Z \subseteq X$ such that $S(Z)$ is closed in Y then $S_{|Z} : Z \longrightarrow S(Z) \notin \Phi_+(Z, S(Z))$;

($\imath v$) If $S \in \mathcal{S}(X,Y)$ with $\alpha(S) < \infty$ then $R(S) = S(X)$ does not contain any closed infinite dimensional subspace of Y;

($v\imath$) If $T \in \mathcal{F}_+(X,Y)$ then if for all closed infinite dimensional subspace $Z \subseteq X$ such that $T(Z)$ is closed in Y we have $T_{|Z} : Z \longrightarrow T(Z) \in \mathcal{F}_+(Z, T(Z))$ thus $\mathcal{F}_+(X,Y) = \mathcal{S}(X,Y)$;

($v\imath\imath$) If $T \in \mathcal{F}(X,Y)$ then if for all closed infinite dimensional subspace $Z \subseteq X$ such that $T(Z)$ is closed in Y we have $T_{|Z} : Z \longrightarrow T(Z) \in \mathcal{F}(Z, T(Z))$ thus $\mathcal{F}(X,Y) = \mathcal{F}_+(X,Y) = \mathcal{S}(X,Y)$;

Proof. ($\imath$) We have the mapping $K_{|Z} : Z \longrightarrow K(Z)$ is onto then by it is an open mapping, moreover it maps any bounded set in Z to a relatively compact set in $K(Z)$ thus we obtain necessarily that $dim(K(Z)) < \infty$;

($\imath\imath$) Assume that $\mathcal{S}(X,Y) = \mathcal{K}(X,Y)$ then $S \in \mathcal{K}(X,Y)$ with $R(S)$ is a closed subspace for which $dim(R(S)) = \infty$ then $S : X \longrightarrow R(S)$ is a compact onto operator, then by the open mapping theorem $dim(R(S)) < \infty$ which is a contradiction hence $\mathcal{S}(X,Y) \neq \mathcal{K}(X,Y)$.

($\imath\imath\imath$) Assume that $S_{|Z} : Z \longrightarrow S(Z)$ is not strictly singular then there exists an infinite dimensional closed subspace $M \subseteq Z$ such that $S_{|Z} : M \longrightarrow S_{|Z}(M)$ is an isomorphism then $S_{|Z}(M)$ is closed in $S(Z)$ but $S(Z)$ is closed in Y hence $S_{|Z}(M)$ is closed in Y and $M \approx S_{|Y}(M)$, consequently $S : X \longrightarrow Y$ is not strictly singular which is a contradiction.

(v) Assume that there exists $Z \subseteq X$ such that $S(Z)$ is closed in Y and $S_{|Z} : Z \longrightarrow S(Z) \in \Phi_+(Z, S(Z))$ then there exists a closed infinite dimensional subspace M of Z such that $Z = M \oplus H$ where $dim(H) < \infty$ and $M \approx S_{|Z}(M)$, this implies that M is an infinite dimensional closed subspace of X with $S_{|M} : M \longrightarrow Y$ is an isomorphism , consequently $S \notin \mathcal{S}(X,Y)$ which is a contradiction. Conversely, if $S \notin \mathcal{S}(X,Y)$ then there exists a closed infinite dimensional subspace $M' \subseteq X$ such that $S_{|M'} : M' \longrightarrow Y$

is an isomorphism which proves that $S(M')$ is closed in Y and $S_{|M'} : M' \longrightarrow S(M') \in \Phi_+(M', S(M'))$ which is a contradiction.

(iv) Let $S \in \mathcal{S}(X,Y)$ with $\alpha(S) < \infty$ and assume that there exists an infinite dimensional closed subspace M of Y such that $M \subseteq R(S) = S(X)$ then $M' = S^{-1}(M)$ is an infinite closed subspace of X. Consequently, $S_{|M'} : M' \longrightarrow M \in \Phi_+(M', M)$ which is a contradiction by (v).

(vi) Assume that $\mathcal{F}_+(X,Y) \neq \mathcal{S}(X,Y)$ then there exists an infinite dimensional closed subspace $M \subseteq X$ such that $S_{|M} : M \longrightarrow S(M)$ is an isomorphism thus $S(M)$ is closed in Y and $S_{|M} \notin \mathcal{F}_+(M, S(M))$ which is a contradiction (since we have $Iso(M, S(M)) \bigcap \mathcal{F}_+(M, S(M)) = \emptyset$). The converse can be obtained by combining that the restriction of strictly singular operators to closed subspaces such that this restriction has a closed range is strictly singular and the fact that the class of strictly singular operators is included in the class of upper-semi Fredholm perturbations.

(vii) Can be proved as in (vi).

From Theorem 5.1, it is easy to observe that the properties of compactness and strictly singular are inherited by the operators restrictions to infinite dimensional closed subspaces having closed ranges which is not the case of upper semi-Fredholm perturbations and Fredholm perturbations as the following examples show:

***Example* 5.1** Assume that $X = X_{GM} \times X_{GM}$ and $Y = X_{GM}$ then we have $\mathcal{L}(X,Y) = \mathcal{F}(X,Y)$, in particular the projector operator $Pr : X \longrightarrow Y$ defined by $Pr(x,y) = y$ is a Fredholm perturbation but $Pr_{|\{0\}\times Y} : \{0\} \times Y \longrightarrow Y$ is an isomorphism hence it is not a Fredholm perturbation in $\mathcal{L}(\{0\} \times Y, Y)$.

***Example* 5.2** Let X one of the Banach spaces given in Example 3.1 (section 3). With the notations of Remark 3.1, $B \in \mathcal{F}(X)$ but $B_{|M_1} : M_1 \longrightarrow M_2$ is an isomorphism hence it is not a Fredholm perturbation in $\mathcal{L}(M_1, M_2)$.

***Example* 5.3** Assume that $X = X_{GM} \times Y$ where $Y \subset X$ is an infinite dimensional subspace with $dim(X_{GM}/Y) = \infty$. Denote by $J : Y \longrightarrow X_{GM}$ defined by $J(x) = x$ for all $x \in Y$. Then $A = \begin{pmatrix} 0 & J_Y \\ 0 & 0 \end{pmatrix} \in \mathcal{F}(X) = \mathcal{F}_+(X)$ (see [22]) but $A_{|\{0\}\times Y} : \{0\}\times Y \longrightarrow \{0\} \times Y$ defined by $A_{|\{0\}\times Y}(0,y) = (0, J_Y(y))$ is not a Fredholm perturbation or upper-semi-Fredholm perturbation in $\mathcal{L}(\{0\} \times Y)$.

As an application of the assertion (iv), we have the following result:

Corollary 5.1 (Corollary 5.2 in [48]): For $1 < p < 2$, the range of the Fourier transform $\Im : L_p(G,m) \longrightarrow L_q(\Gamma, n)(\frac{1}{p} + \frac{1}{q} = 1)$ does not contain any infinite dimensional closed subspace in L_q (here (G,m) is a locally compact group with its Haar measure m and (Γ, n) is the dual group of (G,m) with its Haar measure n).

Proof. Indeed, the Fourier transform $\Im :: L_p(G,m) \longrightarrow L_q(\Gamma, n)$ is a one-to-one strictly singular operator (see for more details Theorem 5.1 in [48]).

***Remark* 5.1** Notice that the assertion ($\imath\imath\imath$) of Proposition 5.1 shows that in a Banach space X when it's possible to construct a strictly singular operator T having an infinite dimensional range then necessarily $\mathcal{S}(X) \neq \mathcal{K}(X)$ as the example given by [17] to prove that $\mathcal{S}(L_p([-1,1])) \neq \mathcal{K}(L_p([-1,1]))(1 \leq p < \infty)$. Indeed, the example established by the authors has an infinite dimensional closed subspace given by the closed hull of the set of Rademacher's functions (see for more details [17].

Theorem 5.2 Let X, Y be two infinite dimensional Banach spaces such that $\Phi(X, Y) \neq \emptyset$. Let $T \in \mathcal{F}(X, Y)$ and assume that for all infinite dimensional closed subspace M of X with $T(M)$ closed, the subspace $T(M)$ is a complemented subspace of Y for which $\Phi(X, T(M)) \neq \emptyset$. Then

$$\mathcal{F}(X, Y) = \mathcal{F}_+(X, Y) = \mathcal{S}(X, Y).$$

Proof. Assume that there exists $T \in \mathcal{F}(X, Y)\backslash\mathcal{S}(X, Y)$ then there exists an infinite dimensional closed subspace M of X with $T(M)$ closed and $M \approx T(M)$. Now by assumption $T(M)$ is a complemented subspace of Y and thus there exists a projection $P : Y \longrightarrow T(M)$. Hence by the assertion ($\imath$) of Proposition 2.1, the bounded linear operator

$$P \circ T \circ J_M : M \longrightarrow X \longrightarrow Y \longrightarrow T(M) \in \mathcal{F}(M, T(M)).$$

But on the other hand, $P \circ T \circ J_M \in Iso(M, F(M)) \subset \Phi(M, T(M))$ which is a contradiction.

Corollary 5.2 Let X be two infinite dimensional Banach space and let $T \in \mathcal{F}(X)$. Assume that for all infinite dimensional closed subspace M of X with $T(M)$ closed, the subspace $T(M)$ is a complemented subspace of Y for which $\Phi(X, T(M)) \neq \emptyset$. Then

$$\mathcal{F}(X) = \mathcal{F}_+(X) = \mathcal{S}(X).$$

Proof. Follows immediately from Theorem 5.2 by taking $X = Y$ since $\Phi(X) \neq \emptyset$.

***Remark* 5.2** Notice that Corollary 2.2 can be used to prove that in the case of $L_p(\mu)$ spaces $(1 \leq p < 2)$, we have $\mathcal{F}(L_p(\mu)) = \mathcal{F}_+(L_p(\mu)) = \mathcal{S}(L_p(\mu))$ (see for more details [48]).

In the following, we give some conditions ensuring that the restriction of upper semi-Fredholm perturbations and Fredholm perturbations to closed subspaces inherit this property.

Theorem 5.3 Let X be an infinite dimensional Banach space and let $M \subset X$ a closed complemented infinite dimensional subspace of X. Assume that $F \in \mathcal{F}_+(X)$ (resp. $\mathcal{F}(X)$) then if $F(M)$ is closed with $F(M) \subseteq M$, we have $F_{|M} : M \longrightarrow F(M) \in \mathcal{F}_+(M, F(M))$ (resp. $\mathcal{F}(M, F(M))$.

Proof. Since M is complemented in X then there exists a closed subspace Z such that $X = M \oplus Z$. Let us to prove the result for the class of semi-Fredholm perturbation, the same argument can be applied to prove the case for Fredholm perturbations. Assume that $F_{|M} \notin \mathcal{F}_+(M, F(M))$ then there exists $A \in \Phi_+(M, F(M))$ such that $A - F_{|M} \notin \Phi_+(M, F(M))$. Let us define $\widetilde{A} \in \mathcal{L}(X)$ by $\widetilde{A} = \begin{pmatrix} A & 0 \\ 0 & I_Z \end{pmatrix}$, we have $\widetilde{A} \in \Phi_+(X)$ since $\alpha(\widetilde{A}) = \alpha(A) < \infty$ and $R(\widetilde{A}) = R(A) \oplus Z$ which is a closed subspace of X. Moreover, we have $\widetilde{A} - F \notin \Phi_+(X)$. Indeed, since $A - F_{|M} \notin \Phi_+(M, F(M))$ we have two situations:

($\imath$) If $\alpha(A - F_{|M}) = \infty$ then $\alpha(\widetilde{A} - F) = \infty$ hence $A - F \notin \Phi_+(X)$;

($\imath\imath$) If $R(A - F_{|M})$ is not closed in $F(M)$ then $(A - F)(M) = R(A - F_{|M})$ is not closed in X hence $A - F \notin \Phi_+(X)$ (see Theorem 10, page 158 in [40]).

Competing Interests

The author declares that there is no conflict of interest regarding the publication of this paper.

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