

# Fixed points for right reversible semigroups satisfying controllable punctual inequality

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## Abstract

In this paper, we give some fixed point results for right reversible semitopological semigroups acting on convex subsets of Banach spaces and satisfy "controllable punctual inequality". These results extend some ones established by W. A. Kirk "Glasgow. Math. Journal, Vol 2(1), 6-9, (1971)".

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## 1 Introduction

Throughout this paper, if  $X$  is a Banach space and  $A$  is a nonempty subset of  $X$  then  $\overline{A}$  and  $co(A)$  denote respectively the closure and the convex hull of  $A$  in  $X$ .

Let  $S$  be a semitopological semigroup, in other words a semigroup with a Hausdorff topology such that the mappings  $s \in S \rightarrow st$  and  $s \in S \rightarrow ts$  are continuous from  $S$  into  $S$  for each  $t \in S$ .  $S$  is called right reversible if any two closed left ideals of  $S$  has non-void intersection. In this case,  $(S, \leq)$  is a directed system when the binary relation " $\leq$ " on  $S$  is given by  $x_1 \leq x_2$  if and only if  $\{x_1\} \cup \overline{Sx_1} \supseteq \{x_2\} \cup \overline{Sx_2}$ ,  $x_1, x_2 \in S$  and  $x_1 < x_2$  means that  $x_1 \leq x_2$  and  $x_1 \neq x_2$ . The family of right reversible semitopological semigroups contains in particular those of commutative semigroups and right amenable semitopological semigroups as discrete semigroups [7, 8, 9, 10, 11, 12, 13, 14, 18]. By the same way, we can define left reversible semigroups.  $S$  is called reversible if  $S$  is both left and right reversible.

Let  $l^\infty(S)$  be the Banach space of bounded real-valued functions on  $S$  with the supremum norm. For  $s \in S$  and  $g \in l^\infty(S)$ , the left and right translations of  $g$  in  $l^\infty(S)$  are defined by

$$l_s g(t) = g(st) \text{ and } r_s g(t) = g(ts)$$

for all  $t \in S$ .

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Let  $Z$  be a closed linear subspace of  $l^\infty(S)$  containing constants and invariant under translations, i.e.,  $l_s(Z) \subset Z$  and  $r_s(Z) \subset Z$ . A linear functional  $\mu \in Z^*$  is called a left invariant mean on  $Z$  if  $\|\mu\| = \mu(1) = 1$  and  $\mu(l_s g) = \mu(g)$  for each  $s \in S$  and  $g \in Z$ . By a same way, we can define a right invariant mean. Let  $C(S)$  be the closed subalgebra consisting of all continuous functions on  $S$  and let  $LUC(S)$  be the space of left uniformly continuous functions on  $S$ , i.e. all functions  $f \in C(S)$  such that the mappings  $s \in S \rightarrow l_s f$  from  $S$  to  $C(S)$  are continuous if  $C(S)$  is equipped with the sup norm topology. The space of right uniformly continuous functions on  $S$  denoted by  $RUC(S)$  can be defined by the same way. For more details, we quote for example [2, 7, 8, 9, 10, 11, 12, 13, 14, 18]. Additionnaly, in these citations and the references therein, we can find a rich results and contributions on fixed point theory of semigroup actions on convex subsets of Banach spaces and some convergence results are investigated. These results solved many problems in this direction which are related to the Bruck's conjecture [3] asserting that weak fixed point property (w-FPP) is equivalent to the weak fixed point property for left reversible semigroups.

Let  $K$  be a nonempty subset of a Banach space  $X$  with a norm  $\|\cdot\|$ . An action of  $S$  on  $K$  is a mapping of the set  $S \times K$  into  $K$ , denoted by  $(s, x) \rightarrow s.x = T_s x$  for which  $(s_1 s_2).x = s_1.(s_2.x)$  for all  $s_1, s_2 \in S, x \in K$ . The set of mappings  $\mathcal{S} = (T_s, s \in S)$  is called a representation of  $S$ .

Let  $G \subseteq S$ , a point  $x \in K$  is a common fixed point of  $G$  with respect to this action if  $s.x = T_s x = x$  for all  $s \in G$ . The set of common fixed points of  $G$  is denoted by  $F(G)$ . An action of  $S$  on  $K$  is called generalized nonexpansive if it satisfies that for all  $s \in S$ :

$$\|s.x - s.y\| = \|T_s x - T_s y\|$$

$$\leq \alpha_1 \|x - y\| + \alpha_2 \|x - s.x\| + \alpha_3 \|y - s.x\| + \alpha_4 \|y - s.y\| + \alpha_5 \|x - s.y\| \text{ for all } x, y \in K.$$

where  $\alpha_i \geq 0$  for all  $i = 1, \dots, 5$  and  $\sum_{i=1}^5 \alpha_i = 1$ . In the case where  $\alpha_i = 0$  for all  $i = 2, \dots, 5$  (consequently  $\alpha_1 = 1$ ) then the action is called nonexpansive.

If for each  $x \in K$  and  $s \in S$  the maps  $s \rightarrow T_s x$  from  $S$  into  $K$  and  $x \rightarrow T_s x$  from  $K$  into  $K$  are continuous then the action is called jointly continuous.

In [7], A. T. Lau and C. S. Wong gave fixed point results for a generalized nonexpansive action on convex compacts subsets of strictly convex Banach spaces and they assert that these results hold for the case of Hausdorff locally convex topological spaces for which the topology is induced by a family of seminorms. Recently, A. Dehici [4] has established fixed point results for Kannan's actions ( $\alpha_1 = \alpha_3 = \alpha_5 = 0, \alpha_2 = \alpha_4 = \frac{1}{2}$ ) of weakly continuous mappings on weakly compact convex subsets of strictly convex Banach spaces.

## 2 Main Results

First of all, we give the following basic definition.

**Definition 2.1** Let  $S$  be a right reversible semitopological semigroup. Suppose that  $S$  acts on a convex subset  $C$  of a Banach space  $X$ . We say that  $S$  satisfies "controllable

punctual inequality” if  $S$  has a minimal element  $\min(S)$  satisfying that  $T_{\min(S)} = I_C$  and for all  $x \in C$  and all  $\alpha, \beta \in S \setminus \{\min(S)\}$ , there exists  $\gamma(x) \in S \setminus \{\min(S)\}$  with  $\gamma(x) \leq \alpha$  and  $\gamma(x) \leq \beta$  such that

$$\|T_\alpha x - T_\beta x\| \leq \|x - T_{\gamma(x)} x\| \quad (*)$$

Let  $n \geq 2$  be an integer. Under the notations of Definition 2.1, we denote

$$\begin{aligned} \Xi_n = \{ & I \subset S, I \text{ totally directed, } \text{card}(I) = n \text{ and} \\ & \gamma(x) \in I \setminus \{\min(I)\} \text{ for all } x \in C, \alpha, \beta \in I \setminus \{\min(I)\} \} \end{aligned}$$

For  $I \in \Xi_n$ , define

$$\mathfrak{S}_{n,I} = \{ \tilde{T} = \sum_{\alpha \in I} \beta_\alpha T_\alpha, \sum_{\alpha \in I} \beta_\alpha = 1, T_{\min I} = I_C \text{ and } \beta_{\min(I \setminus \{\min I\})} > 0 \}$$

Our first result is given by the following theorem

**Theorem 2.1** Let  $S$  be a right reversible semitopological semigroup which acts on a convex subset  $C$  of a Banach space  $X$ . If  $S$  satisfies a ”controllable punctual inequality”, then for every integer  $n \geq 2$  and for all  $I \in \Xi_n$ , we have

$$F(\mathfrak{S}_{n,I}) = F(I)$$

*Proof.* It is easy to observe that  $F(I) \subseteq F(\mathfrak{S}_{n,I})$ . Now to prove the converse, let  $x_0 \in F(\mathfrak{S}_{n,I})$  thus for every  $\tilde{T} \in \mathfrak{S}_{n,I}$ , we have  $\tilde{T}(x_0) = x_0$ . Hence, there exists a finite set of positive real numbers  $(\beta_\alpha) \subseteq [0, 1]$  with  $\sum_{\alpha \in I} \beta_\alpha = 1$  and  $\beta_{\min(I \setminus \{\min I\})} > 0$  such

that  $T_{\min I} = I_C$  and  $\tilde{T} = \sum_{\alpha \in I} \beta_\alpha T_\alpha$ . Thus

$$\tilde{T}x_0 = \left( \sum_{\alpha \in I} \beta_\alpha T_\alpha \right) x_0 = x_0,$$

which gives that

$$x_0 = \left( \sum_{\alpha \in I \setminus \{\min(I)\}} \left( \frac{\beta_\alpha}{1 - \beta_{\min(I)}} \right) T_\alpha \right) x_0 \quad (\beta_{\min(I)} \neq 1 \text{ since } \beta_{\min(I \setminus \{\min(I)\})} > 0).$$

Let  $\delta = \sup\{\|T_\alpha x_0 - T_\beta x_0\|, \alpha, \beta \in I\}$ . Assume that  $\delta > 0$ , the fact that the action  $S$  satisfies the ”controllable punctual inequality” proves that there exists a minimal element  $\gamma(x_0) \in I \setminus \{\min(I)\}$  such that

$$\delta = \|x_0 - T_{\gamma(x_0)} x_0\|.$$

Since  $\sum_{\alpha \in I \setminus \{\min(I)\}} \frac{\beta_\alpha}{1 - \beta_{\min(I)}} = 1$ , it follows that

$$x_0 = \delta_\mu T_\mu x_0 + (1 - \delta_\mu)z, \quad \delta_\mu \in ]0, 1]$$

where  $z \in \text{co}\{T_\alpha x_0, \alpha \in I \setminus \{\min(I), \min(I \setminus \min\{I\})\}\}$  and  $\mu = \min(I \setminus \{\min(I)\})$ . Thus

$$\begin{aligned} \delta &= \|x_0 - T_{\gamma(x_0)}x_0\| = \|\delta_\mu T_\mu x_0 + (1 - \delta_\mu)z - T_{\gamma(x_0)}x_0\| \\ &\leq \delta_\mu \|T_\mu x_0 - T_{\gamma(x_0)}x_0\| + (1 - \delta_\mu)\|z - T_{\gamma(x_0)}x_0\| \\ &\leq \delta_\mu \delta + (1 - \delta_\mu)\delta = \delta. \end{aligned}$$

(i) If  $\gamma(x_0) = \mu$ , this is a contradiction, since we obtain that  $\|T_{\gamma(x_0)}x_0 - T_{\gamma(x_0)}x_0\| = 0 = \delta$ .

(ii) If  $\gamma(x_0) > \mu$ , by the assumption (\*) and the definition of the family  $\Xi_n$ , we obtain the existence of  $\theta \in I \setminus \{\min(I)\}$  with  $\theta \leq \mu < \gamma(x_0)$  such that

$$\delta \leq \|T_\mu x_0 - T_{\gamma(x_0)}x_0\| \leq \|x_0 - T_\theta x_0\|$$

which gives that  $\|x_0 - T_\theta x_0\| = \delta$  and contradicts the fact that  $\gamma(x_0)$  is the smallest element in  $I \setminus \{\min(I)\}$  such that  $\delta = \|x - T_\alpha x_0\|$ . Necessarily, we get  $\delta = 0$  and  $\|x_0 - T_\alpha x_0\| = 0$  for all  $\alpha \in I$ , consequently  $x_0 \in F(I)$  which achieves the proof.

**Corollary 2.1** (see Theorem 2 in [6]) Let  $C$  be a convex subset of a Banach space  $X$  and let  $T$  be a nonexpansive selfmapping on  $C$ . Let  $n \geq 1$  be an integer. Denote by

$$S_n = \sum_{i=0}^n \lambda_i T^i \quad (\text{with the notation } T^0 = I_C)$$

where  $(\lambda_i)_{i=0}^n \subset [0, 1]$  together with  $\lambda_1 > 0$ . and  $\sum_{i=0}^n \lambda_i = 1$ . Then  $F(S_n) = F(T)$ .

*Proof.* The result follows from Theorem 2.1 by taking  $S = \{0\} \cup \mathbb{N}$  and  $I = \{0, 1, \dots, n\}$ , in this case the action is given by  $r.x = T^r x$  for all  $x \in C$ . The fact that this action satisfies the "controllable punctual inequality" is easy to check. Indeed, it suffices to observe that for all integers  $r, p \geq 1$  ( $p < r$ ) and all  $x \in C$  we have  $\|T^r x - T^p x\| \leq \|x - T^{r-p} x\|$ . Consequently, in this case we can take  $\gamma(x) = r - p$  for each  $x \in C$ .

**Remark 2.1** In the definition of the family  $\mathfrak{S}_{n,I}$ , the assumption that  $\beta_{\min(I \setminus \{\min I\})} > 0$  is crucial to obtain the result of Theorem 2.1. In Corollary 2.1, this assumption is reduced to the fact that  $\lambda_1 > 0$ . To see this, let  $T : [0, 1] \rightarrow [0, 1]$  given by  $T(x) = 1 - x$ , we have  $F(T) = \{\frac{1}{2}\}$  moreover if we take the real polynomial  $P(x) = \frac{1}{2} + \frac{1}{2}x^2$ , thus we have  $F(P(T)) = F_{(I_{[0,1]})} = [0, 1]$  which implies that  $F(T) \subsetneq F(P(T))$ , however it can be observed that  $F(P(T)) = F(T^2)$ .

**Definition 2.2** Let  $X$  be a Banach space and let  $C, D$  be two nonempty subsets of  $X$  such that  $C \subset D$ . A continuous mapping  $P : D \rightarrow C$  is called a retraction from  $C$  to  $D$  if  $Px = x$  for all  $x \in C$ .

The property of sunny retraction and its characterization in the case of smooth Banach spaces plays a crucial role in the theory of nonexpansive mappings, the investigation

of their fixed points and the convergence results related to them (see [17]). In 1975, J. B. Baillon [1] proved the following first ergodic theorem which asserts that if  $C$  is a closed convex subset of a Hilbert space and  $T$  is a nonexpansive selfmapping on  $C$  with  $F(T) \neq \emptyset$ , then for each  $x \in C$ , the Cesaro mean

$$S_n(x) = \frac{1}{n} \sum_{k=1}^n T^k x$$

converges weakly to some  $y \in F(T)$ . As it's indicated in [14], if we put  $Px = y$  for all  $x \in C$ , thus  $P$  is a nonexpansive retraction from  $C$  to  $F(T)$  such that  $PT = TP = P$ , and  $Px \in \overline{\text{co}}\{T^n x, n = 1, 2, \dots\}$  for each  $x \in C$ . By Corollary 2.1, it is easy to conclude that since  $\frac{1}{n} > 0$  for all integer  $n \geq 1$  we deduce that  $F(S_n) = F(T)$  for all integer  $n \geq 1$ . In [5], Hirano and Takahashi obtained the existence of such retraction in the case of Banach spaces and improved many results in the literature.

In the following result, we establish the existence of average mappings and retractions for the family of mappings  $\mathfrak{S}_{n,I}$ .

**Theorem 2.2** Let  $X$  be a reflexive Banach space and let  $C$  be a closed convex subset of  $X$ . Assume that  $S$  is a right reversible semitopological semigroup of nonexpansive mappings which acts on  $C$  and for which  $RUC(S)$  has a right invariant mean. If there exists an element in  $C$  with a bounded orbit, then for all integer  $n \geq 2$ ,  $\tilde{T}_{t_1, t_2, \dots, t_n}$  is a nonexpansive average mapping of  $C$  such that  $\tilde{T}_{t_1, t_2, \dots, t_n} x \in \overline{\text{co}}\{T_\alpha x\}_{\alpha \in I, I \in \Xi_n}$  for all  $x \in C$  and there exists a nonexpansive retraction  $P$  on  $C$  such that  $\tilde{T}_{t_1, t_2, \dots, t_n} P = P$ . Moreover, if  $P$  is affine, we have  $\tilde{T}_{t_1, t_2, \dots, t_n} P = P \tilde{T}_{t_1, t_2, \dots, t_n} = P$ .

*Proof.* Denote  $\tilde{T}_{t_1, t_2, \dots, t_n} = \sum_{t_1, \dots, t_n} \beta_{t_i} T_{t_i}$  where  $\sum_{t_1, \dots, t_n} \beta_{t_i} = 1$ . It is easy to observe that  $\tilde{T}_{t_1, t_2, \dots, t_n}$  is an average nonexpansive mapping of  $\mathcal{S}$ . The existence of a retraction  $P$  on  $C$  such that  $T_s P = P T_s = P$  (see the proof of Lemma 6 in [10]) for each  $s \in S$  implies that  $\tilde{T}_{t_1, t_2, \dots, t_n} P = P$ . Now, if  $P$  is affine, we infer that  $P(\sum_{t_1, \dots, t_n} \beta_{t_i} T_{t_i}) = \sum_{t_1, \dots, t_n} \beta_{t_i} P T_{t_i} = (\sum_{t_1, \dots, t_n} \beta_{t_i}) P = P$  which gives the result.

**Remark 2.2** Observe that if the set  $I \in \Xi_n$  has a bounded orbit in  $C$  then the family  $\mathfrak{S}_{n,I}$  has also a bounded orbit and  $F(I) = F(\mathfrak{S}_{n,I})$  even in the empty case if the action satisfies the controllable punctual inequality.

### 3 Some remarks and questions

In the case of nonexpansive mappings, it is easy to construct much examples  $T$  for which the set  $\{T^n, n \geq 1\}$  is infinite. To see this, it suffices to take  $X = \mathbb{R}$  equipped with its usual norm and  $T : \mathbb{R} \rightarrow \mathbb{R}$  given by  $Tx = \sqrt{x^2 + 1}$ . A sample calculation shows that  $T$  is nonexpansive. Additionally, we have  $T^n x = \sqrt{x^2 + n}$  for all  $n \geq 2$ . Thus,  $T^n \neq T^m$  for all integers  $n, m$  with  $n \neq m$ .

Let  $X$  be a Banach space and let  $C$  be a nonempty subset of  $X$ . Define the families  $\sum_{1,C}$  and  $\sum_{2,C}$  of selfmappings on  $C$  by

$$\sum_{1,C} = \{T : C \longrightarrow C\}$$

satisfying that

$$\|Tx - Ty\| \leq \frac{\|Tx - x\|\|Tx - y\| + \|Ty - y\|\|Ty - x\|}{\|Tx - y\| + \|Ty - x\|}, \quad x, y \in C, x \neq y$$

with the additional assumption that  $\|Tx - y\| + \|Ty - x\| \neq 0$

and

$$\sum_{2,C} = \{T : C \longrightarrow C\}$$

satisfying that

$$\|Tx - Ty\| \leq \alpha_1\|x - y\| + \alpha_2(\|x - Tx\| + \|y - Ty\|) + \alpha_3(\|x - Ty\| + \|y - Tx\|),$$

for which

$$\alpha_1, \alpha_2, \alpha_3 \geq 0 \text{ and } 3\alpha_1 + 2\alpha_2 + 4\alpha_3 \leq 1$$

It is easy to observe that  $\sum_{1,C}$  and  $\sum_{2,C}$  are nonempty families since they contains constant mappings.

**Example 3.1** Let  $X = \mathbb{R}$  and  $C = [0, 1]$ . Define  $T : [0, 1] \longrightarrow [0, 1]$  by  $Tx = 1 - x$ . Thus  $T \in \sum_{1,[0,1]}$ . However,  $T^2 \notin \sum_{1,[0,1]}$  since  $T^2 = I$ .

It is shown that for all  $T \in \sum_{i,C}$  ( $i = 1, 2$ ) and for all  $x \in C$ , if  $n, m \geq 1$  are arbitrary integers with  $n \neq m$ , we have  $\|T^n x - T^m x\| \leq \|x - Tx\|$  (see [15, 16]).

*Question:* Is it possible to find a Banach space  $X$  and a nonempty subset  $C$  of  $X$  such that for all  $T \in \sum_{i,C}$  ( $i = 1, 2$ ) and for all integer  $n \geq 2$ , we have  $T^n \in \sum_{i,C}$  ( $i = 1, 2$ ) and the set  $\{T^n\}_{n \geq 1}$  of mappings is infinite.

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