SARAJEVO JOURNAL OF MATHEMATICS DOI: 10.5644/SJM.11.2.08
Vol.11 (24), No.2, (2015), 219–234

ON SOME PROPERTIES OF SPECTRA AND ESSENTIAL SPECTRA IN BANACH SPACES

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Abstract. In this paper, we study diverse properties satisfied by the spectra, Wolf and Weyl essential spectra of bounded linear operators and their links with the structures of Banach spaces.

1. Introduction and notations

There is no secret that spectral theory is an important part of functional analysis which plays a key role in the development of pure and applied mathematics at the same time but it is related closely to the functional frameworks on which it is defined and their geometrical structures. Leaving from the simple topological framework in which they are used, Banach spaces and their classification (up to isomorphism) turned out to be a very difficult subject and out of reach. The first questions that have attracted the attention of specialists consist whether there exist indecomposable Banach spaces. This question is not a coincidence because in general Banach spaces which we are accustomed are crosslinked (functions spaces, sequences space, ...) whose structures are well understood.

Work in this direction began with the relevant results from 1991 and established by T. Gowers and B. Maurey solving the unconditional bases problem. Indeed, they constructed a Banach space without an unconditional basic sequence such that its norm appears to be a fixed point of a convenable functional. However their space is reflexive (and hence is separable), and it possesses a very strange property; it is a H.I. Banach space (hereditarily indecomposable Banach space), in other words, it does not contain any decomposable closed infinite-dimensional subspace. Moreover, this space is not isomorphic to its closed subspaces, in particular it is not isomorphic to its hyperplanes, this answers negatively a question given by S. Banach that remained open for a longtime. This discovery has allowed us

2010 *Mathematics Subject Classification.* 47A10 47A53.
Key words and phrases. Banach space, Spectrum, Wolf essential spectrum, Weyl essential spectrum, Fredholm operator, index.

to divide the structures of Banach spaces into two categories, those which have subspaces that have an unconditional basis and those which contain hereditarily indecomposable subspaces.

The purpose of this paper is to study and make the rounds on a lot of questions within the scope of bounded linear operators theory and Fredholm, semi-Fredholm perturbations by exploiting the two directions of the geometry of Banach spaces.

The paper is organized as follows: In Section 2, the surjectivity of the spectra, Wolf and Weyl essential spectra maps are discussed. Section 3 is devoted to the description and characterization of lifting sets for various Banach spaces while the Section 4 deals with the problem of Salinas (also called West-Stampfli decomposition) and its extension to the case of semi-Fredholm perturbations and finally, we close this work by some comments and interesting questions.

Let X be a complex infinite-dimensional Banach space and let $\mathcal{L}(X)$ the space of all bounded linear operators on X while $\mathcal{K}(X)$ designates the subspace of all compact operators on X. If $A \in \mathcal{L}(X)$, we write $N(A) \subseteq X$ and $R(A) \subseteq X$ for the null space and range of A. We set $\alpha(A) := \dim N(A)$, $\beta(A) := \operatorname{codim} R(A)$. Let $A \in \mathcal{L}(X)$ have a closed range. Then A is a Φ_+-operator ($A \in \Phi_+(X)$) if $\alpha(A) < \infty$, and A is a Φ_--operator ($A \in \Phi_-(X)$) and if $\beta(A) < \infty$. For $A \in \Phi_\mp(X) = \Phi_-(X) \bigcup \Phi_+(X)$, the index of A is defined by $i(A) = \alpha(A) - \beta(A) \in \mathbb{Z} \bigcup \{\pm\infty\}$. An operator T is semi-Fredholm if T has closed range and $\min .\mathrm{ind}(T) = \min\{\alpha(T), \beta(T)\}$ is finite. The set of semi-Fredholm operators $\Phi_\mp(X)$ is open, and the index is a locally constant function invariant under compact perturbations ([30], Theorems 16 and 17, p. 161). Operators in $\Phi(X) = \Phi_+(X) \bigcap \Phi_-(X)$ are called Fredholm operators. We denote by $\Phi_0(X)$ the set of Fredholm operators of indices 0. The spectrum of A will be denoted dy $\sigma(A)$. The point spectrum of A, $\sigma_p(A)$ is the set of scalars $\lambda \in \mathbb{C}$ such that $\lambda I - A$ is not one-to-one (the set of eigenvalues of A). the residual spectrum of A, $\sigma_r(A)$ is the set of scalars $\lambda \in \mathbb{C}$ such that $\lambda I - A$ is one-to-one and $R(A)$ is not dense in X while the continuous spectrum $\sigma_c(A)$ is the set of scalars $\lambda \in \mathbb{C}$ such that $\lambda I - A$ is one-to-one and $R(A)$ is dense but not closed in X. The resolvent set of A, $\rho(A)$, is the complement of $\sigma(A)$ in the complex plane. A complex number λ is in φ_{+A}, φ_{-A}, $\varphi_{\mp A}$, φ_A or φ_A^0 if $\lambda I - A$ is in $\Phi_+(X), \Phi_-(X), \Phi_\mp(X)$, $\Phi(X)$ or $\Phi_0(X)$.

Let $A \in \mathcal{L}(X)$. A point $\lambda \in \sigma(A)$ is in the Kato essential spectrum, $\sigma_K(A)$ if $\lambda \notin \varphi_{\mp A}$. A point $\lambda \in \sigma_e(A)$ is in the Wolf essential spectrum, if $\lambda \notin \varphi_A$. The Weyl essential spectrum $\sigma_w(A) = \bigcap_{K \in \mathcal{K}(X)} \sigma(A + K)$ is nothing else but $\mathbb{C} \backslash \varphi_A^0$. All these essential spectra are non-empty compact sets in the complex plane with the following inclusions:

$$\partial\sigma_e(A) \subseteq \sigma_K(A) \subseteq \sigma_e(A) \subseteq \sigma_w(A) \subseteq \sigma(A).$$

(where $\partial\sigma_e(A)$ is the boundary of the set $\sigma_e(A)$).
The set of non-smooth points of $\varphi_{\mp A}$ denoted by $\sigma_{ns}(A)$ is given by

$$\sigma_{ns}(A) = \{\lambda \in \varphi_{\mp A} : \min \operatorname{ind}(\lambda I - A) \neq 0\}.$$

Remark 1.1. Noting that the modern names of Kato essential spectrum and Wolf essential spectrum are respectively, semi-Fredholm essential spectrum and Fredholm essential spectrum (see [1, 2]).

Proposition 1.1.

(*i*) *$\varphi_{+A}, \varphi_{-A}, \varphi_{\mp A}, \varphi_A$ and φ_A^0 are open sets in the complex plane,*
(*ii*) *$\alpha(\lambda I - A)$ and $\beta(\lambda I - A)$ are constant on any component of φ_A except at a discrete set of points.*

It is known (see [8]) that the set $\varphi_{\mp A}$ can be written as a disjoint union of components $\bigcup_{k=0}^{\infty} C_k$ where the components $C_k, k \neq 0$ are bounded and only C_0 is the unique unbounded component.

Now, we give some definitions and properties of diverse classes of perturbations and some categories of Banach spaces which will be used latter.

Let $F \in \mathcal{L}(X)$. F is called a Fredholm perturbation if $U + F \in \Phi(X)$ whenever $U \in \Phi(X)$. F is called a upper (resp. lower) semi-Fredholm perturbation if $F + U \in \Phi_+(X)$ (resp. $\Phi_-(X)$) whenever $U \in \Phi_+(X)$ (resp. $\Phi_-(X)$). The sets of Fredholm perturbations, upper (lower) semi-Fredholm perturbations are denoted by $\mathcal{F}(X), \mathcal{F}_+(X), \mathcal{F}_-(X)$. It is shown that all these sets are closed two-sided ideals in $\mathcal{L}(X)$ [14, 32, 33].

Proposition 1.2. ([14], Proposition 3, p. 70) *Let X be a complex infinite-dimensional Banach space and let $F \in \mathcal{F}(X)$, then $ind(A + F) = ind(A)$ for all $A \in \Phi(X)$.*

Remark 1.2. An operator $R \in \mathcal{L}(X)$ is called a Riesz operator if $\sigma_e(R) = \{0\}$. Let $\mathcal{R}(X), \mathcal{S}(X)$ and $\mathcal{CS}(X)(X)$ denote respectively the class of Riesz operators, strictly singular and strictly cosingular operators on X (see [26, 32, 36]). We recall that Riesz operators satisfy the Riesz-Schauder theory of compact operators, $\mathcal{R}(X)$ is not in general an ideal of $\mathcal{L}(X)$ [7] and we have the following inclusions

$$\mathcal{K}(X) \subseteq \mathcal{S}(X) \subseteq \mathcal{F}_+(X) \subseteq \mathcal{F}(X) \subseteq \mathcal{R}(X).$$

and

$$\mathcal{K}(X) \subseteq \mathcal{CS}(X) \subseteq \mathcal{F}_-(X) \subseteq \mathcal{F}(X) \subseteq \mathcal{R}(X).$$

The containment $\mathcal{S}(X) \subseteq \mathcal{F}_+(X)$ is due to Kato [26] while the inclusion $\mathcal{CS}(X) \subseteq \mathcal{F}_-(X)$ was proved by Vladimirskii [36].

Let $\{x_n\}_n$ be a sequence of vectors in a Banach space X. A series $\sum_{n=1}^{\infty} x_n$ such that $\sum_{n=1}^{\infty} x_{\pi(n)}$ converges for every permutation π of $\mathbb{N}$ is said to be unconditionally convergent. A basis $\{x_n\}_n$ of a Banach space X is said to be unconditional if for every $x \in X$, its expansion in terms of the basis $\sum_{n=1}^{\infty} a_n x_n$ converges unconditionally.

The basis constant of $\{x_n\}_{n=1}^{\infty}$ is defined as the smallest K such that for any choice of scalars $\{a_n\}_n$ and any integers $m < n$, we have

$$\|\sum_{i=1}^{m} a_i x_i\| \leq K \|\sum_{i=1}^{n} a_i x_i\|.$$

If $\{x_n\}_n$ is an unconditional basic sequence with an unconditional constant K. Then, for every choice of scalars $\{a_n\}_n$ such that $\sum_{n=1}^{\infty} a_n x_n$ converges and every choice of bounded scalars $\{\lambda_n\}_{n=1}^{\infty}$, we have

$$\|\sum_{n=1}^{\infty} \lambda_n a_n x_n\| \leq 2K \sup_n |\lambda_n| \; \|\sum_{n=1}^{\infty} a_n x_n\|.$$

For more details concerning the unconditional basic sequence and its properties, we can refer to [27].

Definition 1.1. [4] Let X be a complex infinite-dimensional Banach space and let $A \in \mathcal{L}(X)$. We say that A satisfies the problem of Salinas if there exists $K \in \mathcal{K}(X)$ such that $\sigma(A + K) = \sigma_\omega(A)$.

Remark 1.3. Notice that if the problem of Salinas is satisfied for every bounded linear operator on X, then Riesz operators have the classical West decomposition [38], in other words, every Riesz operator R can be written as a sum $R = K + Q$ where K is compact and Q is quasinilpotent ($\sigma(Q) = \{0\}$).

It is well known that Tsirelson space [35] space is the first example of a Banach space in which neither an l_p space nor a c_0 space can be embedded. This Banach space is reflexive. Recently, the construction of this space was the root for the development of several results in Banach space theory (see for example [5, 21]).

Remark 1.4. Recall that the problem of Salinas is valid for all bounded linear operators defined on each one of these Banach spaces (with unconditional bases)

1. Separable Hilbert spaces;
2. $l_p(1 \leq p < \infty) \bigcup c_0$;
3. $L_p([0, 1])(1 < p < \infty)$;
4. Tsirelson Banach space.

(See respectively [8, 38–40]).

Definition 1.2. ([9–11, 16, 18]) Let X be a complex infinite-dimensional Banach space

ı) X is said to be decomposable if it is the topological direct sum of two closed infinite dimensional subspaces.

ıı) X is said to be hereditarily indecomposable (in short H.I.) if it does not contain any decomposable closed subspace.

ııı) X is said to be quotient hereditarily indecomposable (in short Q.H.I.) if no infinite dimensional quotient of a closed subspace of X is decomposable.

ıv) X is hereditarily finitely decomposable if the maximal number of (infinite dimensional) closed subspaces of X forming a direct sum in X is finite. For $n \geq 1$, X is HD_n if this number is equal to n.

v) X is n-quotient decomposable and we write $X \in QD_n$, if n is the maximal number of the integers k such that X has a quotient which is the direct sum of k closed infinite dimensional subspaces.

Proposition 1.3. ([9], Lemma 2.1) *Let X be a complex infinite-dimensional Banach space, then*

(*ı*) *If X is a H.I. Banach space, then $\mathcal{L}(X) = \mathbb{C}I \oplus \mathcal{S}(X)$;*

(*ıı*) *If X is a Q.H.I. Banach space, then $\mathcal{L}(X) = \mathbb{C}I \oplus \mathcal{CS}(X)$.*

The class of hereditarily indecomposable Banach spaces was first introduced and investigated by T. Gowers and B. Maurey [18]. Notice also that the hereditarily indecomposable Banach space X_{GM} constructed by the last authors was the first example of an indecomposable Banach space.

2. Surjectivity of spectra and essential spectra maps in some Banach spaces

Let $\mathbb{C}$ be the complex plane and $\mathcal{P}(\mathbb{C})$ the set of all subsets of $\mathbb{C}$. We denote by $\mathcal{K}(\mathbb{C})$ the set of the collection of nonempty compact sets in $\mathcal{P}(\mathbb{C})$.

Now, we are ready to give our first result in this section.

Theorem 2.1. *Let X be a complex infinite-dimensional Banach space with unconditional bases, then the map $\sigma : \mathcal{L}(X) \longrightarrow \mathcal{K}(\mathbb{C})$ is surjective.*

Proof. Let $K \in \mathcal{K}(\mathbb{C})$ and let $(x_n)_{n=1}^{\infty}$ be the unconditional bases of X. Let $(\lambda_n)_{n=1}^{\infty}$ be a dense sequence in K. Define T as follows: $T(x) = T(\sum_{n=1}^{\infty} a_n x_n) = \sum_{n=1}^{\infty} \lambda_n a_n x_n$, then $\|T(x)\| \leq 2K \sup_n |\lambda_n| \|\sum_{n=1}^{\infty} a_n x_n\| = M\|x\|$ (where $M = 2K \sup_n |\lambda_n|$). It is clear that T is a bounded linear operator on X and $\sigma(T)$ contains K (since $\{\lambda_n\}_n \subseteq \sigma_p(T) \subseteq \sigma(T)$). Now, we prove the opposite inclusion, if $\lambda \notin K$, then $\inf\{|\lambda - \beta|; \beta \in K\} > 0$ and so $S(x) = S(\sum_{n=1}^{\infty} \alpha_n x_n) = \sum_{n=1}^{\infty}(\lambda - \lambda_n)^{-1}\alpha_n x_n$ is a bounded operator on X and $(\lambda I - T)S = S(\lambda I - T) = I$, yielding $\lambda \notin \sigma(T)$ and this completes the proof. □

The following theorem translates in a certain sense the richness of separable Hilbert spaces.

Theorem 2.2. *Let $\mathcal{H}$ be a separable Hilbert space, then the maps $\sigma_e : \mathcal{L}(\mathcal{H}) \longrightarrow \mathcal{K}(\mathbb{C})$, $\sigma_\omega : \mathcal{L}(\mathcal{H}) \longrightarrow \mathcal{K}(\mathbb{C})$ are surjective.*

Proof. This result was announced without proof in [31]. First of all, we give proofs for the interesting cases of the unit disc and the unit circle in $\mathbb{C}$. Taking $K = \overline{D}(0,1)$ equipped with the Lebesgue measure ν on $\mathbb{R}^2$ and $\mathcal{H} = L^2(\overline{D}(0,1))$. Define A on $\mathcal{H}$ by the following:

$$(Af)(\lambda) = \lambda f(\lambda); \lambda \in \overline{D}(0,1), f \in \mathcal{H}.$$

Thus $\sigma(A) = K$ (see [22], Problem 52). To show that $\sigma(A) = \sigma_e(A)$, it suffices to prove the inclusion $\sigma(A) \subseteq \sigma_e(A)$. Assume that there exists $\lambda_0 \in \sigma(A)$ and $\lambda_0 \notin \sigma_e(A)$, hence $\lambda_0 \in \varphi_A \subseteq \varphi_{\mp A}$. Since A is normal, the operators A and $A^\star$ have SVEP at λ_0 (see [2], p. 115), this gives that λ_0 is an isolated point of $\sigma(A)$ (see [2], Corollary 3.21) which is a contradiction. Consequently, we conclude that $\sigma(A) = K = \sigma_e(A) = \sigma_\omega(A)$.

For $K = \{z \in \mathbb{C}; |z| = 1\}$, taking H a separable infinite dimensional Hilbert space with an orthonormal basis $(\xi_n)_{n\in\mathbb{Z}}$, we define $U(\xi_n) = \xi_{n+1}$. Then U is isometric and surjective, so it is a unitary and it is easy to check that $\sigma(U) = \sigma_e(U) = \sigma_w(U) = K$ (for more details, see ([22], Problem 68) and ([25], Lemma 3.2.13)).

Now, if K is a non-empty compact set in $\mathbb{C}$, let $\{\lambda_n\}_1^\infty$ be a dense set in K, then the operator A can be chosen a diagonal operator such that each λ_i is repeated infinitely many times. □

The reasoning given above concerning the general case of a non empty compact set K can be extended to the case of Banach space with unconditional basis. More precisely, we have.

Theorem 2.3. *Let X be a complex infinite-dimensional Banach space with unconditional basis, then the maps $\sigma_e : \mathcal{L}(X) \longrightarrow \mathcal{K}(\mathbb{C})$, $\sigma_\omega : \mathcal{L}(X) \longrightarrow \mathcal{K}(\mathbb{C})$ are surjective.*

Proof. Let $K \in \mathcal{K}(\mathbb{C})$ and let $(\lambda_n)_{n=1}^\infty$ be a dense set in K. If $T \in \mathcal{L}(X)$ given as in the proof of Theorem 2.1 with the additional condition that for every integer $n \geq 1$ there exists an infinite number of integers k_n such that $\lambda_n = \lambda_{k_n}$. Thus, in this case each λ_n is an eigenvalue of T with infinite algebraic multiplicity and consequently $\{\lambda_n\}_{n=1}^\infty \subseteq \sigma_e(T)$. Indeed, assume that there exists an integer $n_0 \geq 1$ such that $\lambda_{n_0} \notin \sigma_e(T)$, hence $\lambda_{n_0} \in \varphi_T$, this implies that $\alpha(\lambda_{n_0}I - T)$ is finite which is a contradiction. By passage to the closure and taking account that the set $\sigma_e(T)$ is closed, we obtain that $K \subseteq \sigma_e(T) \subseteq \sigma_\omega(T)$. Now, we prove the opposite inclusion, if $\lambda \notin K$, we argue as in the second part of the proof of Theorem 2.1 by defining the bounded linear operator S on X by the same expression. Since $\lambda \notin \sigma(T)$, we have $\lambda \notin \sigma_e(T)$ and $\lambda \notin \sigma_\omega(T)$, hence $\mathbb{C}\backslash K \subset \mathbb{C}\backslash\sigma(T) \subseteq \mathbb{C}\backslash\sigma_e(T)$ and also $\mathbb{C}\backslash K \subset \mathbb{C}\backslash\sigma_\omega(T)$ (since $\sigma_\omega(T) \subseteq \sigma(T)$). By passage to the complements, these last inclusions give that $\sigma_e(T) \subseteq K$ and $\sigma_\omega(T) \subseteq K$ and consequently, we obtain that $K = \sigma_e(T) = \sigma_\omega(T)$ which is the desired result. □

Remark 2.1. Let X be a complex infinite-dimensional Banach space. It is easy to observe that if the map $\sigma_\omega : \mathcal{L}(X) \longrightarrow \mathcal{K}(\mathbb{C})$ is surjective and the problem of Salinas is satisfied for each bounded linear operator on X, yielding that the map $\sigma : \mathcal{L}(X) \longrightarrow \mathcal{K}(\mathbb{C})$ is surjective.

Let us note that there exist Banach spaces such that all the spectrum and essential spectra maps given above are not surjective. This fact is illustrated by the following two propositions

Proposition 2.1. *Let X be a complex infinite-dimensional Banach space such that the Wolf essential spectrum of every bounded operator on X does not contain any boundary of a nonempty open set in the complex plane, then the maps $\sigma_e, \sigma_\omega, \sigma : \mathcal{L}(X) \longrightarrow \mathcal{K}(\mathbb{C})$ are not surjective.*

Proof. First of all, it is easy to observe that the range of the map $\sigma_e : \mathcal{L}(X) \longrightarrow \mathcal{K}(\mathbb{C})$ is included in the family of non-empty compact sets with empty interiors of the complex plane $\mathbb{C}$. Now, we prove that the ranges of the other maps satisfy this same property, and to do this, it suffices to consider the spectra map. Let us consider $A \in \mathcal{L}(X)$. First of all, we prove that the set $Int(\sigma(A))\backslash\sigma_e(A)$ is empty. Assume that $Int(\sigma(A))\backslash\sigma_e(A) \neq \emptyset$, let Ω be a connected component of this set and let $\lambda \in \overline{\Omega}$ such that $\lambda \notin \Omega$; this gives that $\lambda \in \partial\sigma(A)$. Next, we put $S_\lambda = A - \lambda I$; since $\lambda \notin \sigma_e(A)$, the operator S_λ is Fredholm with index 0. Indeed, the fact that $0 \in \partial\sigma(S_\lambda)$ asserts the existence of invertible operators arbitrary close to S_λ, the assertion follows from the continuity of the index. Hence, 0 is an isolated point of $\sigma(S_\lambda)$ (see [1], Lemma 7.43), this shows that λ is isolated in $\sigma(A)$ and contradicts the fact that $\lambda \in \overline{\Omega}\backslash\Omega$. The set $Int(\sigma(A))\backslash\sigma_e(A)$ must be empty and consequently $Int(\sigma(A)) \subseteq \sigma_e(A)$ which gives that $Int(\sigma(A)) = \emptyset$. □

Remark 2.2. $H.I., Q.H.I., HD_n$ and QD_n are Banach spaces that are a good examples of Banach spaces that satisfy the assumption given in the Proposition 2.1. Indeed, their Wolf essential spectra are finite sets in the complex plane (see [16]).

Proposition 2.2. *Let X be the Shift Banach space constructed in [19], then the map $\sigma_e : \mathcal{L}(X) \longrightarrow \mathcal{K}(\mathbb{C})$ is not surjective.*

Proof. In fact, the range of this map is included in the family of non-empty compact connected sets of the complex plane $\mathbb{C}$ (fore more details, see [15], p. 9). □

3. Lifting sets in Banach spaces

We start this section by giving the definition of lifting sets.

Definition 3.1. [31] Let X be a complex infinite-dimensional Banach space and $\Omega \in \mathcal{K}(\mathbb{C})$. We say that Ω is a lifting set for X if for every $A \in \mathcal{L}(X)$ with $\sigma_e(A) = \Omega$, there exists $K \in \mathcal{K}(X)$ satisfying that $\sigma(A+K) = \Omega$.

Theorem 3.1. *Let X be a complex infinite-dimensional Banach space such that for all $A \in \mathcal{L}(X)$, A satisfies the problem of Salinas, then Ω is a lifting set if and only $\sigma_e(A) = \sigma_\omega(A)$ for every $A \in \mathcal{L}(X)$ such that $\sigma_e(A) = \Omega$.*

Proof. Assume that Ω is a lifting set and $A \in \mathcal{L}(X)$ for which $\sigma_e(A) = \Omega$, then there exists $K \in \mathcal{K}(X)$ such that $\sigma_e(A) = \sigma(A+K) = \Omega$. Since $\sigma_e(A) \subseteq \sigma_\omega(A) = \sigma_\omega(A+K) \subseteq \sigma(A+K) = \Omega = \sigma_e(A)$, we obtain that $\sigma_e(A) = \sigma_\omega(A)$. Conversely, let $\Omega \in \mathcal{K}(\mathbb{C})$ such that $\sigma_e(A) = \sigma_\omega(A)$ for each $A \in \mathcal{L}(X)$ satisfying $\sigma_e(A) = \Omega$. The fact that A satisfies the problem of Salinas implies the existence of $K \in \mathcal{K}(X)$ with $\sigma(A+K) = \sigma_\omega(A) = \sigma_e(A) = \Omega$ and hence Ω is a lifting set. □

Definition 3.2. Let $K \subset \mathbb{C}$ be a compact subset of the complex plane. The polynomially convex hull of K denoted by $\widehat{K}$, is defined by

$$\widehat{K} = \{z \in \mathbb{C} : |P(z)| \leq \max_{\xi \in K} |P(\xi)| \text{ for all polynomials}\}.$$

A compact set K is said to be polynomially convex if $K = \widehat{K}$. If $K \subsetneq \widehat{K}$, a connected component of $\widehat{K} \backslash K$ (considered as a topological space) is called a hole of K.

Let $\Omega \in \mathcal{K}(\mathbb{C})$ and denote by $\widehat{\Omega}$ the polynomially convex hull of Ω in the complex plane $\mathbb{C}$. Let Γ_Ω runs through the set of holes of Ω (if Ω is polynomially convex, we take $\Gamma_\Omega = \emptyset$) and let Ψ the subset of $\mathcal{K}(\mathbb{C}) \times \mathcal{P}(\mathbb{C})$ given as follows:

$$\Psi = \{(\Omega, \Gamma_\Omega), \Omega \in \mathcal{K}(\mathbb{C}) \text{ and } \Gamma_\Omega \text{ is a hole of } \Omega\}$$

The following theorem gives a characterization of lifting sets for some classes of Banach spaces having a rich structure in a certain sense.

Theorem 3.2. *Let X be a complex infinite-dimensional Banach space such that the following conditions are satisfied:*

1) *The map* $\sigma_e : \mathcal{L}(X) \longrightarrow \mathcal{K}(\mathbb{C})$ *is surjective;*
2) *Every* $T \in \mathcal{L}(X)$ *satisfies the problem of Salinas;*
3) *The function* Λ *from* $\mathcal{L}(X)$ *to* Ψ *given by* $\Lambda : \mathcal{L}(X) \longrightarrow (\sigma_e, \sigma_\omega \backslash \sigma_e)$ *is surjective.*

If $\Omega \neq \emptyset$, *then* Ω *is a lifting set if and only if* Ω *is polynomially convex.*

Proof. Assume that Ω is not polynomially convex, it follows that the set $\widehat{\Omega}\backslash\Omega$ is not empty. Let Γ_Ω be any hole in Ω, by the assumption 3, there exists $A \in \mathcal{L}(X)$ such that $\sigma_e(A) = \Omega$ and $\sigma_\omega(A)\backslash\sigma_e(A) = \Gamma_\Omega$ which is a contradiction by taking account Theorem 3.1. Conversely, if Ω is a polynomially convex set in the complex plane, then $\Omega = \widehat{\Omega}$. Then the use of the assumption 1 implies the existence of $B \in \mathcal{L}(X)$ such that $\sigma_e(B) = \Omega$. On the other hand, the second assumption gives the existence of $K \in \mathcal{K}(X)$ with $\sigma_\omega(B) = \sigma(B+K)$. Moreover, we know that $\sigma_\omega(B) \subseteq \widehat{\sigma_e(B)}$ (see for example [23]), this shows that $\sigma_\omega(B) \subseteq \widehat{\sigma_e(B)} = \widehat{\Omega} = \Omega = \sigma_e(B)$ and consequently $\sigma_\omega(B) = \sigma_e(B)$ which completes the proof. $\square$

As an application of this theorem, we have

Proposition 3.1. *Let* X *one of these Banach spaces:*

1) *Separable Hilbert spaces;*
2) $l_p(1 \leq p < \infty) \bigcup c_0$ *Banach spaces;*
3) $L^p([0,1]), (1 < p < \infty)$.

Then Ω *is a lifting set for* X *if and only if* Ω *is polynomially convex.*

Proof. It suffices to prove that the three assumptions of the above theorem hold.

1. In the case of separable Hilbert spaces, the assumption 1 of Theorem 3.2 is given by Theorem 2.2, the second one is established by C. Apostol [4] while the third one is an immediate consequence of the results obtained by Berger and Shaw [6].

2. The first assumption is an immediate consequence of Theorem 2.3, for the assumption 2 of Theorem 3.2 (see [8]), while the third one, it can be established by some interpolation techniques (see for example [28]).

3. It suffices to take $p > 2$, the case $1 < p < 2$ is deduced by duality since $\sigma_e(A) = \sigma_e(A^*)$ and $\sigma_\omega(A) = \sigma_\omega(A^*)$. If $p > 2$, then the space $L_p([0,1])$ contains a complemented subspace denoted by M which is isomorphic to l_2 [24], thus $L_p([0,1]) = M \oplus N$ where N is an infinite dimensional closed subspace of $L_p([0,1])$. The fact that $\sigma_e \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} = \sigma_e(A)$ and $\sigma_\omega \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} = \sigma_\omega(A)$

where $A \in \mathcal{L}(M)$ proves that assumptions 1 and 3 are satisfied. For the second assertion (see [39]). $\square$

In the next result, we give a description of lifting sets for the case of hereditarily indecomposable Banach spaces.

Proposition 3.2. *Let X be a H.I. Banach space, then*

1) *If X is a Argyros-Haydon space* [5], *then the lifting sets are the subsets of the form $\{\lambda\}$ $(\lambda \in \mathbb{C})$;*
2) *The lifting sets are the sets $\{\lambda\}$ $(\lambda \in \mathbb{C})$ if and only if strictly singular operators on this space have West decompositions.*

Proof. 1. If X is a H.I. Banach space of Argyros-Haydon, then $\mathcal{L}(X) = \mathbb{C}I \oplus \mathcal{K}(X)$, this implies that for each $A \in \mathcal{L}(X)$, there exists $\lambda \in \mathbb{C}$ and $K \in \mathcal{K}(X)$ such that $A = \lambda I + K$, hence $\sigma_e(A) = \{\lambda\} = \sigma(\lambda I) = \sigma(\lambda I + K - K) = \sigma(A - K)$, which gives the result.

2. In this case, we have $\mathcal{L}(X) = \mathbb{C}I \oplus \mathcal{S}(X)$, thus for each $A \in \mathcal{L}(X)$, there exist $\lambda \in \mathbb{C}$ and $S \in \mathcal{S}(X)$ such that $A = \lambda I + S$, this gives that $\sigma_e(A) = \{\lambda\}$. If there exists $K \in \mathcal{K}(X)$ such that $\sigma_e(A) = \{\lambda\} = \sigma(\lambda I + S + K) = \lambda + \sigma(S + K)$, we infer that $\sigma(S + K) = \{0\}$, which implies that $S = K' + Q$ where $K' \in \mathcal{K}(X)$ and Q is quasinilpotent. Conversely, let $A = \lambda I + S$ where $S \in \mathcal{S}(X)$, then there exists $K' \in \mathcal{K}(X)$ such that $S = K' + Q'$ where Q' is quasinilpotent, we obtain that $\sigma(A - K') = \sigma(\lambda I + S - K') = \sigma(\lambda I + K' + Q' - K') = \sigma(\lambda I + Q') = \lambda + \sigma(Q') = \lambda + \{0\} = \{\lambda\} = \sigma_e(A)$. $\square$

4. Semi-Fredholm perturbation decomposition

To prove the fundamental result of this section, we recall some definitions and preparatory results.

Definition 4.1. Let X be a complex infinite-dimensional Banach space and let $A \in \mathcal{L}(X), \lambda \in \mathbb{C}$, the point λ is called a Riesz point of $\sigma(A)$ if $\lambda \in \varphi_A^0$ and λ is an isolated point in $\sigma(A)$. The set of all Riesz points of $\sigma(A)$ is denoted by R_A.

Definition 4.2. Let X be a complex infinite-dimensional Banach space and Let $A \in \mathcal{L}(X)$. We denote by $\sigma_s(A)$ the set $\sigma_K(A) \bigcup \sigma_{ns}(A)$.

ı) A is called a generalized Riesz operator, if $\sigma_{ns}(A) = R_A$;

ıı) If there exists $S \in \mathcal{F}_+(X) \bigcap \mathcal{F}_-(X)$ such that $\sigma(A + S) = \sigma(A) \backslash R_A$, A is said to have the semi-Fredholm perturbation decomposition;

ııı) If there exists $S \in \mathcal{F}_+(X) \bigcap \mathcal{F}_-(X)$ such that $\sigma(A + S) = \sigma_\omega(A)$, then A is said to make semi-Fredholm perturbation theorem true;

iv) If there exists $S \in \mathcal{F}_+(X) \bigcap \mathcal{F}_-(X)$ such that $\sigma(A+S) = \sigma_\omega(A)$ and the set $\sigma_{ns}(A+S)$ has no inner points, then A is said to make the strong semi-Fredholm perturbation theorem true;

v) If there exists $S \in \mathcal{F}_+(X) \bigcap \mathcal{F}_-(X)$ such that $\sigma_s(A+S) = \sigma_K(A)$, then A is said to have semi-Fredholm perturbation correction.

Lemma 4.1. ([34], Corollary 3.1) *Let X be a complex infinite-dimensional Banach space and let $B \in \mathcal{L}(X)$. If each cluster point of $\sigma_s(B)\backslash\sigma_K(B)$ is on the boundary of $\sigma_K(B)$, then for any $\epsilon > 0$, there is $K \in \mathcal{K}(X)$, $\|K\| < \epsilon$, such that $B + K$ is a generalized Riesz operator and $\rho(B) \subseteq \rho(B + K)$.*

Now, we extend the fundamental result given in ([34], Theorem 3.2) to the case of semi-Fredholm perturbations.

Theorem 4.1. *Let X be a complex infinite-dimensional Banach space. Then, the following statements are equivalent:*

($\imath$) Every generalized Riesz operator on X has the semi-Fredholm perturbation decomposition;

($\imath\imath$) Every bounded linear operator on X makes the strong semi-Fredholm perturbation theorem true;

($\imath\imath\imath$) Every bounded linear operator on X has the semi-Fredholm perturbation correction.

Proof. ($\imath$) $\Longrightarrow$ ($\imath\imath$) Let $A \in \mathcal{L}(X)$, then there exists $K \in \mathcal{K}(X)$ such that $A+K$ is a generalized Riesz operator (see [34], Theorem 3.1). Since ($\imath$) is satisfied, $A+K$ has the semi-Fredholm perturbation decomposition, then there exists $S \in \mathcal{F}_+(X) \bigcap \mathcal{F}_-(X)$ such that $\sigma(A+K+S) = \sigma(A+K)\backslash R_{A+K}$. Let $S' = S + K$, then $S' \in \mathcal{F}_+(X) \bigcap \mathcal{F}_-(X)$ because $\mathcal{K}(X) \subseteq \mathcal{F}_+(X) \bigcap \mathcal{F}_-(X)$. We show that $\sigma(A + S') = \sigma_\omega(A)$ which is equivalent to proving that $\sigma(A+S') \bigcap \varphi^0_A = \sigma(A+S') \bigcap \varphi^0_{A+S'} = \emptyset$. Indeed, assume that $\sigma(A+S') = \sigma_\omega(A)$ and $\sigma(A+S') \bigcap \varphi^0_A \neq \emptyset$, then there exists $\lambda_0 \in \sigma(A+S') \bigcap \varphi^0_A$. Hence, $\lambda_0 \in \sigma(A+S')$ and $\lambda_0 \in \varphi^0_A$, then $\lambda_0 I - A \in \Phi_0(X)$, this shows that $\lambda_0 \notin \sigma_\omega(A) = \sigma(A+S')$ which is a contradiction. Consequently the first implication is proved. On the other hand, if $\sigma(A+S') \bigcap \varphi^0_A = \emptyset$, we will prove that $\sigma(A+S') = \sigma_\omega(A) = \sigma_\omega(A+S')$ (since the Weyl essential spectrum is invariant by perturbation by any element in $\mathcal{F}_+(X) \bigcap \mathcal{F}_-(X)$). It suffices then to prove that $\sigma(A+S') \subseteq \sigma_\omega(A+S')$. Assume that there exists $\lambda_1 \in \sigma(A+S')$ and $\lambda_1 \notin \sigma_\omega(A+S')$, hence $\lambda_1 I - A - S' \in \Phi_0(X)$, this gives that $(\lambda_1 I - A - S') + S' = \lambda_1 I - A \in \Phi_0(X)$ (since the set $\Phi_0(X)$ is invariant by perturbation by elements in $\mathcal{F}_+(X) \bigcap \mathcal{F}_-(X)$), we obtain that $\lambda_1 \in \varphi^0_A$ which implies that $\lambda_1 \in \sigma(A+S') \bigcap \varphi^0_A$ which is a contradiction. Hence the second implication is proved.

Now, assume that there exists $\lambda \in \sigma(A+S') \bigcap \varphi^0_{A+S'} = \sigma(A+S') \bigcap \varphi^0_{A+K} = [\sigma(A+K)\backslash R_{A+K}] \bigcap \varphi^0_{A+K}$. Since $A+K$ is a generalized Riesz operator, then $\lambda \in \sigma(A+K)\backslash R_{A+K}$ implies that $\min.\mathrm{ind}(A+K-\lambda I) = 0$ and $\mathrm{ind}(A+K-\lambda I) = \alpha(A+K-\lambda I) - \beta(A+K-\lambda I) = 0$, thus $\alpha(A+K-\lambda I) = \beta(A+K-\lambda I) = 0$ which gives that $\lambda \in \rho(A+K)$. This is a contradiction since $\lambda \in \sigma(A+K)$. Now, we prove that the set $\sigma_{ns}(A+S')$ is empty. Indeed, if $\lambda \in \sigma_{ns}(A+S')$, then $\lambda \in \Phi_{\pm}(A+K)$. Moreover, since $A+K$ is a generalized Riesz operator, we have $\sigma_{ns}(A+K) = R_{A+K}$. On the other hand, if $\lambda \notin \sigma_{ns}(A+K) = R_{A+K}$, then $\lambda I - A - K \in \Phi_0(X)$ and λ is not isolated in $\sigma(A+K)$ which gives that $\lambda \in \rho(A+K)$ which is a contradiction. Thus $\sigma_{ns}(A+S') \subseteq \sigma_{ns}(A+K) = R_{A+K}$ which also is a contradiction, thus $\sigma_{ns}(A+S')$ is empty.

$(ii) \Longrightarrow (iii)$ Let $A \in \mathcal{L}(X)$, then there exists $S \in \mathcal{F}_+(X) \bigcap \mathcal{F}_-(X)$ such that $\sigma(A+S) = \sigma_\omega(A)$ and $\sigma_{ns}(A+S)$ has no inner points. We prove that all possible cluster points of $\sigma_{ns}(A+S)$ are only in $\partial\sigma_K(A+S)$ (the boundary of $\sigma_K(A+S)$). Denote by $acc(\sigma_{ns}(A+S))$ the set of cluster points of $\sigma_{ns}(A+S)$. Let $\lambda \in acc(\sigma_{ns}(A+S))\backslash\sigma_K(A+S)$, then $\lambda \in \Phi_{\mp}(A+S)$, the use of Proposition 1.1 implies the existence of a small neighborhood Λ_λ of λ such that for any $\zeta \in \Lambda_\lambda\backslash\{\lambda\}$, we have $\min.\mathrm{ind}(\zeta I - A - S)$ is constant. We have two possibilities, if the constant is zero, then $\lambda \notin acc(\sigma_{ns}(A+S))$ which is a contradiction; also, if the constant is non-zero, then λ is in the interior of the set $\sigma_{ns}(A+S)$ which is a contradiction.

Now, according to the Lemma 4.1, there exists $K \in \mathcal{K}(X)$ such that $A+S+K$ is a generalized Riesz operator and $\rho(A+S) \subseteq \rho(A+S+K)$. Let $S' = S+K$. We will prove that the set $\sigma_{ns}(A+S')$ is empty. Since $A+S' = A+S+K$ is a generalized Riesz operator, it suffices to show that $\sigma_{ns}(A+S') = R_{A+S'} = \emptyset$. If $R_{A+S'}$ is not empty, then, there exists $\lambda \in R_{A+S'}$. Since $\rho(A+S) \subseteq \rho(A+S')$, we infer that $\sigma(A+S') \subseteq \sigma(A+S)$, this gives that $\lambda \in \sigma(A+S) \bigcap \varphi^0_{A+S'} = \sigma(A+S) \bigcap \varphi^0_{A+S}$. Moreover, we have $\sigma(A+S) = \sigma_\omega(A) = \sigma_\omega(A+S)$, this implies that $\lambda \in \sigma_\omega(A+S) \bigcap \varphi^0_{A+S} = \emptyset$ which is a contradiction.

$(iii) \Longrightarrow (i)$ Let A be a generalized Riesz operator, then by $iii)$, there exists $S \in \mathcal{F}_+(X) \bigcap \mathcal{F}_-(X)$ such that $\sigma_s(A+S) = \sigma_K(A)$, we will prove that $\sigma(A+S) = \sigma(A)\backslash R_A$.

1) Let $\lambda_0 \in \sigma(A+S)$.

a) If $\lambda_0 \in \sigma_s(A+S) = \sigma_K(A)$, then $\lambda_0 \in \sigma(A)$ (since $\sigma_K(A) \subseteq \sigma(A)$). Now assume that $\lambda_0 \in R_A = \sigma_{ns}(A)$, thus $\lambda_0 \in \varphi_{\mp A}$ which is a contradiction since $\sigma_K(A) \bigcap \varphi_{\mp A} = \emptyset$. This gives that $\lambda_0 \notin R_A$ and consequently $\lambda_0 \in \sigma(A)\backslash R_A$.

b) If $\lambda_0 \notin \sigma_s(A+S)$, then $\lambda_0 \in \varphi_{\mp A+S}$ and $\min.\mathrm{ind}(\lambda_0 I - A - S) = 0$. Assume that $\lambda_0 \in \rho(A)$, then $\lambda_0 I - A$ is invertible, hence $i(\lambda_0 I - A) =$

$i(\lambda_0 I - A - S) = 0$, this gives that $\lambda_0 \in \rho(A+S)$ which is a contradiction. It follows that $\lambda_0 \in \sigma(A)$. Now if $\lambda_0 \in R_A$, hence $\lambda_0 \in \varphi_A^0$, thus $i(\lambda_0 I - A) = i(\lambda_0 I - A - S) = 0$, which implies also that $\lambda_0 \in \rho(A + S)$. This is a contradiction, consequently $\lambda_0 \notin R_A$, this leads that $\lambda_0 \in \sigma(A)\backslash R_A$.

From **1)**, we conclude that $\sigma(A + S) \subseteq \sigma(A)\backslash R_A$.

2) Now, assume that $\lambda_0 \in \sigma(A)\backslash R_A$.

c) If $\lambda_0 \in \sigma_K(A) = \sigma_s(A + S)$, then $\lambda_0 \in \sigma(A + S)$ (since $\sigma_s(A + S) \subseteq \sigma(A + S)$).

d) If $\lambda_0 \notin \sigma_K(A) = \sigma_s(A+S)$. The fact that $\lambda_0 \notin R_A$ gives the following two situations:

Or $\lambda_0 \in \sigma_\omega(A) = \sigma_\omega(A + S) \subseteq \sigma(A + S)$ and hence $\lambda_0 \in \sigma(A + S)$ or $\lambda_0 \in \varphi_A^0$ and λ_0 is not an isolated point in $\sigma(A)$. Since $\min.\mathrm{ind}(\lambda_0 I - A) = 0$ and $i(\lambda_0 I - A) = 0$, we get that $\lambda_0 \in \rho(A)$ which is a contradiction. Thus $\lambda_0 \in \sigma_\omega(A) = \sigma_\omega(A + S) \subseteq \sigma(A + S)$.

From **2)**, we conclude that $\sigma(A)\backslash R_A \subseteq \sigma(A + S)$.

Finally, from **1)** and **2)**, we obtain that $\sigma(A + S) = \sigma(A)\backslash R_A$ which is the desired result. $\square$

Remark 4.1. Notice that in general $\mathcal{K}(X) \subsetneq \mathcal{F}_+(X) \bigcap \mathcal{F}_-(X)$. Indeed if X if one of the following Banach spaces:

1. $X = L_p([0, 1])(1 \leq p < \infty)$;
2. X is the hereditarily indecomposable Banach space of Gowers-Maurey;

then $\mathcal{K}(X) \subsetneq \mathcal{F}_+(X) \bigcap \mathcal{F}_-(X) = \mathcal{S}(X) \bigcap \mathcal{CS}(X) = \mathcal{S}(X) = \mathcal{CS}(X)$ (see [3, 14, 29, 37]).

5. Comments and some questions

The dichotomy theorem of T. Gowers (1996) [17] was the source of the classification of Banach spaces. In (2002), the same author [20] refined this result by using the Ramsey theory which enabled him to establish the four "inevitable" list of Banach spaces. A few years later, classification programs for separable Banach spaces by means of the descriptive theory were constructed, giving birth to subdivisions to these classes [12, 13]. We do not know yet if these four list are nothing else but a complicated interpretation of category of Banach spaces for which the maps spectrum and Wolf essential spectrum are surjective or not, since the origin of the difficulty is the same, which lies in the complexity to give a good comprehension in general to the structure of Calkin algebra $\mathcal{L}(X)/\mathcal{F}(X)$ or $\mathcal{L}(X)/\mathcal{K}(X)$.

In finalizing this study, the first question that may come to our mind is the following:

Question 1: Does there exist Banach spaces X such that some of the maps $\sigma, \sigma_e, \sigma_\omega : \mathcal{L}(X) \longrightarrow \mathcal{K}(\mathbb{C})$ are surjective but the other do not possess this property?

On the other hand, it is easy to observe that all known Banach spaces for which the problem of Salinas is satisfied for all bounded linear operators have unconditional basis, this pushes us to ask the following questions

Question 2: Does the problem of Salinas hold for all bounded linear operators on Banach spaces with unconditional bases noting that the problem in abstract Banach spaces is open?

Question 3: Is the result of Proposition 3.1 true for the case of Tsirelson space, or more general for Banach spaces with unconditional bases?

Acknowledgements. The author would like to thank Professor Maria Teresa Alvarez Seco (University of Oviedo, Spain) for her relevant remarks and helpful councils and the anonymous referee for his/her valuable comments which helped to improve this paper. Without forgetting the interesting correspondences with the professors T. Gowers (Cambridge University, England) and V. Ferenczi (University of Sao Paolo, Brazil).

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(Received: December 30, 2014)
(Revised: July 9, 2015)

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