Available online at http://scik.org
Adv. Fixed Point Theory, 5 (2015), No. 1, 56-70
ISSN: 1927-6303

ON SOME FIXED POINTS OF GENERALIZED CONTRACTIONS WITH RATIONAL EXPRESSIONS AND APPLICATIONS

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Abstract. In this paper, we study some results of existence and uniqueness of fixed points for a class of operators satisfying an inequality of rational expressions. We prove that they are Picard mappings. Under certain conditions imposed on the parameters of the inequality, the Φ-quasi nonexpansive framework of this class is established. Mann and Ishikawa iterative methods are investigated in the framework of convex metric spaces.

Keywords: metric space; fixed point; comparison function; c-comparision function; Φ-quasinonexpansive operator.

2010 AMS Subject Classification: 47H10, 54H25.

1. Introduction

The Banach contraction principle [1] has been the starting point of the development of a very interesting field which is the fixed point theory and its applications. Banach's work provided an abstraction of the classical method of successive approximations introduced by Liouville, used by Cauchy and developed in a first time by Picard in the proof of the existence and uniqueness

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Received August 28, 2014

of solutions of differential equations in the late 19th century. We note that Banach's work was established in the case of normed spaces and extended in metric spaces by Caccioppoli. After, almost a century, this area has become a thriving field, for more details; see [2, 3, 5, 6, 12, 15, 28, 30, 32]. Note that the works of Kirk [14] and Browder [4] have developed the ideas by introducing the geometry properties of spaces in the subject for the case of nonexpansive self mappings.

In the mid-sixties ten, other fixed points results dealing with general contractive conditions with rational expressions were appeared. The early works in this direction were established by Dass and Gupta [7], Khan [13] and Jaggi [15]. For these contributions, the authors exploited the continuous and not necessarily continuous cases of selfmappings depending on the nature of the rational expression. For more details, see Rhoades [20] and the references therein.

In this work, we establish some results of the existence and uniqueness of fixed points concerning a class of selfmappings involving general rational expressions by treating the continuous and not necessarily continuous cases, this enable us to extend Khan's [13] theorem. On the other hand, based on our principal result given by Theorem 2.1, we check the Φ-quasinonexpansive framework of our context. Moreover, by using Ruiz's [26] results, we establish the convergence of Mann and Ishikawa processes and the almost stability of Picard process.

2. Preliminaries

Definition 2.1. Let (X,d) be a metric space and f a self mapping on X. We say that

(1) f is lipschitzian with constant of Lipschitz $k \in \mathbb{R}_+$ (or k-lipschitzian) if, for all $(x,y) \in X^2$, we have

$$d(f(x), f(y)) \leq kd(x,y);$$

(2) f is nonexpansive if, for all $(x,y) \in X^2$, we have

$$d(f(x), f(y)) \leq d(x,y);$$

(3) f is a contraction if f is k-lipschitzian with $0 \leq k < 1$.

Definition 2.2. A function $\Phi : [0,+\infty[\longrightarrow [0,+\infty[$ is called a comparison function if it satisfies the following conditions;

($\imath$): Φ is monotone increasing;

($\imath\imath$): $\lim_{n\longrightarrow+\infty} \Phi^n(t) = 0$; for all $t > 0$ (Φ^n stands for the nth iterate of Φ).

Definition 2.3. A function $\Phi : [0,+\infty[\longrightarrow [0,+\infty[$ is called a c-comparison function if it satisfies;

($\imath$): Φ is monotone increasing;

($\imath\imath$): $\sum_{n=0}^{\infty} \Phi^n(t) < \infty$ for all $t > 0$.

Remark 2.1. Every comparison function satisfies $\Phi(0) = 0$ and $\Phi(t) < t, \forall t > 0$.

In the following we denote by C_1 (resp. C_2) the classes of comparison (resp. c-comparison) functions on $[0,+\infty[$. It is easy to observe that $C_2 \subset C_1$ and the inclusion is strict as the following example shows.

Example 2.1. Let $\Phi : [0,+\infty[\longrightarrow [0,+\infty[$ defined by $\Phi(t) = \dfrac{t}{t+1}$. Then Φ is a comparison function but not a c-comparison function since $\Phi^n(t) = \dfrac{t}{nt+1}$ for $t \geq 0$.

Definition 2.4. Let (X,d) be a metric space and $T : X \longrightarrow X$ a self mapping. Let $x_0 \in X$ be fixed, we define the sequence $\{x_n\}_n$ recursively by

$$x_{n+1} = T(x_n) = T^{n+1}(x_0), \quad \forall n \in \mathbb{N}. \tag{2.1}$$

The sequence defined by (2.1) is known as the sequence of successive approximations or Picard iteration. The set of all fixed points of T will be denoted by $F(T)$.

Definition 2.5. Let (X,d) be a metric space. A mapping $T : X \longrightarrow X$ is called a (strict) Picard mapping if there exists $x^\star \in X$ such that $F(T) = \{x^\star\}$ and

$$T^n(x_0) \longrightarrow x^\star \quad \text{for all } x_0 \in X;$$

In other words, the Picard iteration converges to the unique fixed point for any guess $x_0 \in X$.

Example 2.2. Let (X,d) be a complete metric space, the following examples are Picard self mappings.

(1) (Banach 1922) [1]: Any contraction self mapping T on X;

(2) (Edelstein 1962) [9]: If (X,d) is a compact metric space and $f : X \longrightarrow X$ satisfying that

$$d(f(x), f(y)) < d(x,y) \quad \text{for all } (x,y) \in X^2;$$

(3) (Kannan 1968) [12]: $T : X \longrightarrow X$ is a mapping for which there exists $a \in [0, \frac{1}{2}[$ such that

$$d(T(x), T(y)) \leq a[d(x, T(x)) + d(y, T(y))] \quad \text{for all } (x,y) \in X^2;$$

(4) (Boyd-Wong 1969) [3]: T a Φ-contraction, i.e.,

$$d(T(x), T(y)) \leq \Phi(d(x,y)) \quad \text{for all } (x,y) \in X^2;$$

where Φ is a comparison function.

(5) (Meir-Keeler 1969) [17]: $T : X \longrightarrow X$ satisfying the following condition: given $\varepsilon > 0$, there exists $\eta > 0$ such that

$$\varepsilon \leq d(x,y) < \varepsilon + \eta \Longrightarrow d(T(x), T(y)) < \varepsilon.$$

(6) (Zamfirescu 1972) [33]: $T : X \longrightarrow X$ is a mapping for which there exist real numbers α, β and γ satisfying $0 \leq \alpha < 1, 0 \leq \beta < \frac{1}{2}$ and $0 \leq \gamma < \frac{1}{2}$ such that, for each $x, y \in X$, at least one of the following is true

$(\imath)$**:** $d(T(x), T(y)) \leq \alpha d(x,y)$;

$(\imath\imath)$**:** $d(T(x), T(y)) \leq \beta[d(x, T(x) + d(y, T(y)])$;

$(\imath\imath\imath)$**:** $d(T(x), T(y)) \leq \gamma[d(x, T(y) + d(y, T(x)])$.

(7) (Ciric 1981): $T : X \longrightarrow X$ satisfying the following condition:

$$d(T(x), T(y)) < d(x,y) \quad \text{for all } (x,y) \in X^2, x \neq y;$$

and given $\varepsilon > 0$, there exists $\eta > 0$ such that

$$\varepsilon < d(x,y) < \varepsilon + \eta \Longrightarrow d(T(x), T(y)) < \varepsilon$$

Remark 2.2. We note that the conditions indicated in 1, 2 and 4 of the above example ensures that the self mapping T is continuous at any point of X which is not the case for the other self mappings.

Example 2.4. Let $(X,d) = (\{-1,0,1,2\},|.|)$ and T a self mapping on X defined by

$$T(x) = \begin{cases} 0 & \text{if } x \neq 2, \\ -1 & \text{if } x = 2. \end{cases}$$

Thus, T is a Kannan mapping. Indeed, we have

$$d(T(x),T(y)) \leq 1 = \frac{1}{3}d(2,T(2)) \leq \frac{1}{3}d(x,T(x)) + \frac{1}{3}d(2,T(2)).$$

But it is easy to observe that T is not continuous at the point 2.

3. Main results

We start our results by the following theorem

Theorem 3.1. *Let T be a continuous selfmapping defined on a complete metric space (X,d) satisfying the following condition:*

$$d(T(x),T(y)) \leq \frac{\Phi_1[d(x,T(x))]\Phi_2[d(x,T(y))] + \Phi_4[d(y,T(y)),d(y,T(x))]}{\Phi_2[d(x,T(y))] + \Phi_3[d(y,T(x))]} \tag{3.1}$$

for all $x,y \in X$. Here, without loss of generality, we assume that

$$\Phi_2[d(x,T(y))] + \Phi_3[d(y,T(x))] \neq 0, \tag{$\star$}$$

where:

H_1): $\Phi_1,\Phi_2,\Phi_3 : [0,+\infty[\longrightarrow [0,+\infty[$ *such that* $\sum_{n=1}^{+\infty} \Phi_1^n(t) < +\infty$ *together with* Φ_1 *nondecreasing and* $\Phi_2(t) = \Phi_3(t) = 0$ *if and only if* $t = 0$.

H_2): $\Phi_4 : [0,+\infty[\times [0,+\infty[\longrightarrow [0,+\infty[$ *and* $\Phi_4(t_1,t_2) = 0$ *if* $t_1 = 0$ *or* $t_2 = 0$.

Then T is a Picard mapping on X.

Proof. First, we show the uniqueness. Suppose there exist $u,v \in X$ with $u = T(u)$ and $v = T(v)$ satisfying $(\star)$ with $u \neq v$. Then:

$$\begin{aligned} d(u,v) = d(T(u),T(v)) &\leq \frac{\Phi_1[d(u,T(u))]\Phi_2[d(u,T(v))] + \Phi_4[d(v,T(v)),d(v,T(u))]}{\Phi_2[d(u,T(v))] + \Phi_3[d(v,T(u))]} \\ &= \frac{\Phi_1[d(u,u)]\Phi_2[d(u,v)] + \Phi_4[d(v,v),d(v,u)]}{\Phi_2[d(u,v)] + \Phi_3[d(v,u)]} \\ &= 0. \end{aligned}$$

The fact that $d(u,v) \neq 0$ implies that $d(u,v) < 0$ which is a contradiction, consequently $u = v$.

To show the existence, Let us select $x_0 \in X$ and define the sequence $x_n = T(x_{n-1}) = T^n(x_0)$. For $n \geq 1$, we have

$$d(x_n,x_{n+1}) \leq \frac{\Phi_1[d(x_{n-1},x_n)]\Phi_2[d(x_{n-1},x_{n+1})] + \Phi_4[d(x_n,x_{n+1}),d(x_n,x_n)]}{\Phi_2[d(x_{n-1},x_{n+1})] + \Phi_3[d(x_n,x_n)]},$$

which implies that

$$\begin{aligned} d(x_n,x_{n+1}) &\leq \Phi_1[d(x_{n-1},x_n)] \\ &\leq \Phi_1^{(2)}[d(x_{n-2},x_{n-1})] \\ &\vdots \\ &\leq \Phi_1^{(n)}[d(x_0,x_1]. \end{aligned}$$

For $m > n \geq 0$, we obtain

$$\begin{aligned} d(x_n,x_m) &\leq d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + \ldots + d(x_{m-1},x_m) \\ &\leq \Phi_1^{(n)}[d(x_0,x_1)] + \Phi_1^{(n+1)}[d(x_0,x_1)] + \ldots + \Phi_1^{(m-1)}[d(x_0,x_1)] \\ &= \sum_{k=n}^{m-1} \Phi_1^{(k)}[d(x_0,x_1)]. \end{aligned}$$

The fact that $\Phi_1 \in C_2$ implies that

$$d(x_n,x_m) \longrightarrow 0 \text{ as } m,n \longrightarrow +\infty.$$

This gives that $\{x_n\}_{n=0}^{+\infty}$ is a Cauchy sequence and since X is complete, there exists $x^* \in X$ with $x_n \longrightarrow x^*$ if $n \longrightarrow +\infty$. Moreover, the continuity of T yields that

$$x^* = \lim_{n \longrightarrow +\infty} x_{n+1} = \lim_{n \longrightarrow +\infty} T(T^n(x_0)) = T(x^*).$$

Therefore x^* is a fixed point of T.

Remark 3.1. In the case where there exist $x,y \in X$ for which $\Phi_2(d(x,T(y))) + \Phi_3(d(y,T(x))) = 0$, we add the assumption $d(T(x),T(y)) = 0$. For this situation, the existence of the fixed point is obvious. Indeed, we obtain that $d(x,T(y)) = d(y,T(x)) = 0$, which gives that $y = T(x) = T(y)$ and hence y is a fixed point of T.

In the next theorem, we study the existence of unique fixed points of T without the continuous hypothesis.

Theorem 3.2. *Let T be a selfmapping defined on a complete metric space (X,d). Under assumptions $H_1)$ and $H_2)$ of Theorem 3.1 and, if one of the following cases hold:*

(1) $\frac{\Phi_4(t_3,t_4)}{\Phi_3(t_4)} \leq \widetilde{\Phi}(t_3), \forall t_3,t_4 \in]0,+\infty[$ *where* $\widetilde{\Phi} : [0,+\infty[\longrightarrow [0,+\infty[$ *satisfying that* $I-\widetilde{\Phi} : [0,+\infty[\longrightarrow [0,+\infty[$ *is bijective and strictly nondecreasing.*

(2) Φ_1, Φ_3 *and* $t \longrightarrow \Phi_4(.,t)$ *are continuous at 0 together with the continuity of* Φ_2 *on* $]0,+\infty[$.

Then T is a Picard mapping on X.

Proof.

(1) The uniqueness part is obvious. To prove the existence, let $x_0 \in X$, and $x_n = T^n(x_0), n \geq 1$. Assume that $x_n \neq x_{n+1}$. By (3.1) and $(\star)$, we have

$$\begin{aligned} d(x_n,x_{n+1}) &= d(T(x_{n-1}),T(x_n)) \\ &\leq \frac{\Phi_1[d(x_{n-1},T(x_{n-1}))]\Phi_2[d(x_{n-1},T(x_n))] + \Phi_4[d(x_n,T(x_n)),d(x_n,T(x_{n-1}))]}{\Phi_2[d(x_{n-1},T(x_n))] + \Phi_3[d(x_n,T(x_{n-1}))]}, \\ &= \Phi_1[d(x_{n-1},T(x_{n-1}))], \end{aligned}$$

which implies $d(x_n,x_{n+1}) \leq \Phi_1^{(n)}[d(x_0,x_1)]$ Thus, for $m > n$, we deduce that

$$d(x_n,x_m) \leq \sum_{k=n}^{m-1} \Phi_1^{(k)}[d(x_0,x_1)].$$

Since $\Phi_1 \in C_2$, it follows that $d(x_n,x_m) \longrightarrow 0, (n,m \longrightarrow +\infty)$, which shows that $\{x_n\}_{n=0}^{+\infty}$ is a Cauchy sequence. But (X,d) is a complete metric space, therefore $\{x_n\}_{n=0}^{+\infty}$ converges to some $x^* \in X$.

On the other hand, if $y \neq T(x)$, we have

$$\begin{aligned} d(T(x),T(y)) &\leq \frac{\Phi_1[d(x,T(x))]\Phi_2[d(x,T(y))] + \Phi_4[d(y,T(y)),d(y,T(x))]}{\Phi_2[d(x,T(y))] + \Phi_3[d(y,T(x))]} \\ &\leq \Phi_1[d(x,T(x))] + \frac{\Phi_4[d(y,T(y)),d(y,T(x))]}{\Phi_3[d(y,T(x))]} \\ &\leq \Phi_1[d(x,T(x))] + \widetilde{\Phi}[d(y,T(y))]. \end{aligned}$$

In the case $y = T(x)$, we obtain that $d(T(x),T(y)) \leq \Phi_1[d(x,T(x))]$. By combining the two cases, we get

$$d(T(x),T(y)) \leq \Phi_1[d(x,T(x))] + \widetilde{\Phi}[d(y,T(y))]$$

for all x,y. It follows that

$$\begin{aligned} d(x^*,T(x^*)) &\leq d(x^*,x_n)+d(x_n,T(x^*)) \\ &= d(x^*,x_n)+d(T(x_{n-1}),T(x^*)) \\ &\leq d(x^*,x_n)+\Phi_1(d(x_{n-1},T(x_{n-1})))+\widetilde{\Phi}(d(x^*,T(x^*))). \end{aligned}$$

Thus,

$$\begin{aligned} d(x^*,T(x^*)) &\leq (I-\widetilde{\Phi})^{-1}[\Phi_1(d(x_{n-1},x_n))+d(x^*,x_n)] \\ &\leq (I-\widetilde{\Phi})^{-1}[\Phi_1(\Phi_1^{(n-1)}d(x_0,x_1))+d(x^*,x_n)] \\ &= (I-\widetilde{\Phi})^{-1}[\Phi_1^{(n)}(d(x_0,x_1))+d(x^*,x_n)]. \end{aligned}$$

The fact that $I-\widetilde{\Phi}$ is bijective and strictly nondecreasing implies that $I-\widetilde{\Phi}$ is continuous with $\widetilde{\Phi}(0)=0$, hence $(I-\widetilde{\Phi})^{-1}$ is bijective, strictly nondecreasing and continuous mapping. Using this with the fact that $C_2 \subseteq C_1$ and letting $n \longrightarrow +\infty$, we get $d(x^*,T(x^*))=0$. This gives that $T(x^*)=x^*$. This gives the result for the first case.

(2) By the same analysis given above, assume that $T(x^\star) \neq x^\star$. Then

$$d(x_n,T(x^*)) \leq \frac{\Phi_1[d(x_{n-1},x_n)]\Phi_2[d(x_{n-1},T(x^*))]+\Phi_4[d(x^*,T(x^*)),d((x^*,x_n)]}{\Phi_2[d(x_{n-1},T(x^*))]+\Phi_3[d(x^*,x_n)]}.$$

Taking the limit as $n \longrightarrow \infty$ and following our assumptions, yields $d(x^\star,T(x^\star)) \leq 0$, which is contradiction. Thus $x^\star$ is a fixed point of T. This completes the proof for the case 2.

Example 3.1. In the case $\Phi_1(t)=kt, \Phi_2(t)=t, \Phi_4(t_1,t_2)=kt_1t_2$ and $\Phi_3(t)=t$ with $0 \leq k < 1$, we find Khan's fixed point theorem [13].

Example 3.2. Let $X=\{z \in \mathbb{C}/z=e^{i\theta}\ (0 \leq \theta \leq \pi)\}$ equipped with the metric $d(x,y)=d(e^{i\theta_1},e^{i\theta_2})=|\theta_1-\theta_2|$. Let $(0 \leq \alpha < 1)$ and let T be a self mapping on X satisfying that $d(T(x),T(y)) \leq \dfrac{\alpha d(x,T(x))(e^{d(x,T(y))}-1)+(1-cos(d(y,T(y))))(1-cos(d(y,T(x))))}{(e^{d(x,T(y))}-1)+ln(1+d(y,T(x)))}$ together with $(\star)$. Then T has a unique fixed point in X.

4. Applications

We start this section by to give the concept of φ-quasinonexpansive mappings.

Definition 4.1. Let T be a selfmapping defined on a metric space (X,d). We say that $T : X \longrightarrow X$ is a φ-quasinonexpansive mapping if $F(T) \neq \emptyset$ and there exists a function $\varphi : [0,+\infty[\longrightarrow [0,+\infty[$ such that

$$d(T(x),z) \leq \varphi(d(x,z)),$$

for all $x \in X, z \in F(T)$.

Example 4.1. Let (X,d) be a complete metric space, the following mappings are φ-quasinonexpansives:

(1) k-lipschitzian mappings, by taking $\varphi(t) = kt, t \in \mathbb{R}^+$.
(2) Kannan mappings, by taking $\varphi(t) = \dfrac{a}{1-a}t, t \in \mathbb{R}^+$.
(3) Zamfirescu mappings, by taking $\varphi(t) = \xi t, t \in \mathbb{R}^+$, here $\xi = \max\{\alpha, \dfrac{\beta}{1-\beta}, \dfrac{\gamma}{1-\gamma}\}$.

Fore more details on the class of φ-quasinonexpansive mappings, we refer to [26].

Definition 4.2. A function $\Phi : [0,+\infty[\longrightarrow [0,+\infty[$ is called subadditive if for all $t_1,t_2 \in [0,+\infty[$, we have $\Phi(t_1+t_2) \leq \Phi(t_1) + \Phi(t_2)$.

Theorem 4.1. *Assume that the assumptions of Theorem 3.1 (resp. Theorem 3.2) are satisfied. If in addition Φ_1 is a subadditive strictly nondecreasing function with $\Phi_1 \leq \min\{\Phi_2,\Phi_3\}$, then T is Φ_2-quasinonexpansive selfmapping.*

Proof. We have proven that T has a unique fixed point x^*. Let $x \in X$ with $x \neq x^*$, then

$$\begin{aligned} d(T(x),x^*) = d(T(x),T(x^*)) &\leq \frac{\Phi_1[d(x,T(x))]\Phi_2[d(x,x^*)] + \Phi_4[d(x^*,x^*),d(x^*,T(x))]}{\Phi_2[d(x,x^*)] + \Phi_3[d(x^*,T(x))]} \\ &= \frac{\Phi_1[d(x,T(x))]\Phi_2[d(x,x^*)]}{\Phi_2[d(x,x^*)] + \Phi_3[d(x^*,T(x))]}. \end{aligned}$$

The triangle inequality gives that

$$d(x,T(x)) \leq d(x,x^*) + d(x^*,T(x)).$$

Since Φ_1 is nondecreasing, we find that

$$\Phi_1(d(x,T(x))) \leq \Phi_1(d(x,x^*) + d(x^*,T(x))).$$

Following the subadditivity of Φ_1, we get

$$\Phi_1(d(x,T(x))) \leq \Phi_1(d(x,x^*)) + \Phi_1(d(x^*,T(x))).$$

The fact that $\Phi_1 \leq \min\{\Phi_2, \Phi_3\}$ gives that

$$\Phi_1(d(x,T(x))) \leq \Phi_2(d(x,x^*)) + \Phi_3(d(x^*,T(x))).$$

Consequently, we have

$$d(T(x),x^*) \leq \frac{\Phi_1[d(x,T(x))]\Phi_2[d(x,x^*)]}{\Phi_1[d(x,T(x))]} = \Phi_2[d(x,x^*)],$$

which gives the result.

Now, we give the definition of convex metric spaces introduced by Takahashi [31] which play an important role in the development of the fixed point theory as an extension of Banach spaces.

Definition 4.3. A convex metric space $(X,d,\oplus)$ is a metric space (X,d) together with a convexity mapping $\oplus : X \times X \times [0,1] \longrightarrow X$ satisfying

$$d(z,(1-\lambda)x \oplus \lambda y) \leq (1-\lambda)d(z,x) + \lambda d(z,y),$$

for all $x,y,z \in X, \lambda \in [0,1]$.

Example 4.2. Normed spaces, Hilbert ball and $\mathbb{R}$-trees are good examples of convex metric spaces.

Mann iteration ([16]*):* If $(X,d,\oplus)$ is a convex metric space. The normal Mann iteration procedure or Mann iteration, starting from $x_0 \in X$ is the sequence defined by

$$x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n T(x_n), \text{for all } n \in \mathbb{N}, \tag{4.1}$$

where $\{\alpha_n\}_n \subset [0,1]$.

Originally, the Mann iteration was defined in a matrix formulation (see Chapter 4 of [2]). This iterative process was introduced in 1953 by Mann, its convergence was established in the framework of Banach spaces and extended to the locally convex Haussdorff linear topological spaces setting by Dotson [8].

Ishikawa iteration ([11]*):* The Ishikawa scheme is given by

$$\begin{cases} y_n = (1-\beta_n)x_n \oplus \beta_n T(x_n), \\ x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n T(y_n), \text{ for all } n \in \mathbb{N}, \end{cases} \tag{4.2}$$

where $x_0 \in X$ and $\{\alpha_n\}_n$ and $\{\beta_n\}_n$ are sequences in $[0,1]$. It can be seen as a sort of two-step Mann iteration with two different parameters sequences. This iterative process was first introduced by Ishikawa in 1974, in order to approximate fixed points of some classes of operators for which Mann iteration does not converge as the following examples shows.

Hicks and Kubicek (1977): Let H be the complex plane, $K = \{z \in H : |z| \leq 1\}$ and $T : K \longrightarrow K$ given by

$$T(re^{i\theta}) = \begin{cases} 2re^{i(\theta+\frac{\pi}{3})}, & \text{if } 0 \leq r \leq \frac{1}{2}; \\ e^{i(\theta+\frac{2\pi}{3})}, & \text{if } \frac{1}{2} < r \leq 1. \end{cases}$$

The above example shows that T is not continuous, its unique fixed point is the point $(0,0)$, but the Mann iteration with $\alpha_n = \dfrac{1}{n+1}$ does not converge to this fixed point.

In the case where $b_n = 0$, Ishikawa iteration reduces to the Mann iteration. There is not a general dependence between convergence results for Picard, Mann and Ishikawa iterations. However, some partial results on the equivalence of these processes have been given by Rhoades and Soltuz (see [21, 22, 23, 24, 25, 29]).

By using Theorem 4.1 together with ([26], Theorem 3.7), we obtain the following result for the convergence of the iterative schemes of Mann and Ishikawa.

Proposition 4.1 *Let $(X,d,\oplus)$ be a convex complete metric space. Let $\{\alpha_n\}_n$ and $\{\beta_n\}_n$ be two real sequences in $[0,1]$ such that $\{\alpha_n\beta_n\}_n$ converges to some positive real number, let $x_0 \in X$. Under the assumptions of Theorem 4.1 with Φ_2 a continuous comparison function. Then, the Ishikawa sequence given by (4.2) converges to the unique fixed point of T. Moreover, if $\{\beta_n\}_n$ is the constant sequence equal to 0, the Mann iteration given by (4.1) converges to the same unique fixed point of T.*

Remark 4.1. Notice that for Picard, Mann and Ishikawa iterations, each of them has its peculiar advantage. The merit of the Picard iteration is that is simple. Also, if we make a mistake during computation of fixed points when using this process, the particular point (at which error is introduced) will be converted to another initial point there by needing more time to reach the solution but this is not true for other techniques.

Recall that for the case of numerical stability, we say that a fixed point iteration process is numerically stable if small perturbation in the initial data induces a small influence of the computed value of the fixed point. For the remainder of our study, we need the following two definitions about stability of a general iterative processes.

Definition 4.4. Let (X,d) be a metric space, $T : X \longrightarrow X$ a self mapping of X. Let $\{x_n\}_n \subset X$ be the sequence generated by an iteration involving T and defined by

$$x_{n+1} = f(T,x_n), \text{for all } n \in \mathbb{N} \tag{4.3}$$

where $x_0 \in X$ and f is some function. Assume that $\{x_n\}_n$ converges to a fixed point z_0 of T. Let $\{y_n\}_n \subset X$ and we define

$$\varepsilon_n := d(y_{n+1}, f(T,y_n)) \text{ for all } n \in \mathbb{N}$$

Then

$(\imath)$**:** the iteration process (4.3) is said to be T-stable if $\lim\limits_{n\longrightarrow\infty} \varepsilon_n = 0$ implies $\lim\limits_{n\longrightarrow\infty} y_n = z_0$.

$(\imath\imath)$**:** the iteration process (4.3) is said to be almost T-stable if $\sum\limits_{n\in\mathbb{N}} \varepsilon_n < \infty$ implies $\lim\limits_{n\longrightarrow\infty} y_n = z_0$.

For more informations and interesting comments on these notions of stability, we can see [18]. On the other hand, it is easy to observe that an iterative process (4.3) which is T-stable is almost T-stable but the converse is not true in general (see the counter example given in [19]). Furthermore, the iterative processes can converge without being stable. Indeed, the following example given in [10, 19] confirms this.

Example 4.3. Let $(X,d) = ([0,1],|.|)$ and $T : [0,1] \longrightarrow [0,1], T(x) = x$. It is easy to observe that $F(T) = [0,1]$.

The case of Picard iteration: Let $z_0 \in]0,1]$ and $x_{n+1} = T(x_n) = T^{n+1}(x_0)$ is a stationary sequence which equal to x_0, this implies that its limit is x_0. On the other hand, if we take $y_0 = 0$

and $y_n = \frac{1}{n}$ for $n \geq 1$, we obtain that

$$|y_{n+1} - T(y_n)| = \frac{1}{n(n+1)} \longrightarrow 0.$$

but $\lim_{n \longrightarrow +\infty} y_n = 0 \neq z_0$. This shows that Picard iteration converges but not T-stable.

The case of Mann iteration: Let $z_0 \in]0,1]$, $\{\alpha_n\}_n \subseteq [0,1]$ and $x_{n+1} = (1-\alpha_n)x_n + \alpha_n x_n = x_n = x_0$ for all $n \geq 1$. Let $x_0 = y_0$ and $y_n = \frac{1}{n+1}$. Thus

$$\varepsilon_n = |y_{n+1} - (1-\alpha_n)y_n - \alpha_n T(y_n)| = \frac{1}{(n+1)(n+2)} \longrightarrow 0.$$

but $\lim_{n \longrightarrow +\infty} y_n = 0 \neq z_0$. This shows that Mann iteration converges but not T-stable.

In the following result, we establish the almost stability of Picard's iterative process for our context of self mappings.

Corollary 4.1. *Let (X,d) be a complete metric space. Assume that $T : X \longrightarrow X$ is a self mapping of X satisfying the assumptions of Theorem 3.1 with Φ_2 a continuous comparison function. If z_0 is the unique fixed point of T. Let $x_0 \in X$ and $x_{n+1} := T(x_n), n \in \mathbb{N}$ be the Picard process. Let $\{y_n\}_n \subset X$ and define $\{\varepsilon_n\}_n$ by*

$$\varepsilon_n := d(y_{n+1}, T(y_n)) \text{ for all } n \in \mathbb{N}$$

If $\sum_{n \in \mathbb{N}} \varepsilon_n < \infty$, then $\lim_{n \longrightarrow \infty} y_n = z_0$. In other words, the Picard process is almost T- stable.

Proof. The result is established by combining the fact that T is Φ_2-quasinonexpansive together with Theorem 4.5 in [26].

Conflict of Interests

The authors declare that there is no conflict of interests.

Acknowledgement

the authors are indebted to Professors M. O. Olatinwo (Obafemi Awolowo University, Nigeria) and E. Karapinar (Atilim University, Turkey) for valuable discussions and comments.

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