Hindawi Publishing Corporation
Journal of Function Spaces
Volume 2015, Article ID 370294, 8 pages
http://dx.doi.org/10.1155/2015/370294

Research Article

Some Compactness and Interpolation Results for Linear Boltzmann Equation

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Received 23 November 2014; Accepted 30 July 2015

Academic Editor: Giuseppe Marino

We discuss some compactness results in L_p $(1 \leq p < \infty)$ spaces related to the spectral theory of neutron transport equations for general classes of collision operators and Radon measures having velocity spaces as supports covering most physical models. We show in particular that the asymptotic spectrum of the transport operator is independent of p.

1. Introduction and Notations

The Boltzmann equation (1872) is an integrodifferential equation of the kinetic theory which is devoted to the study of evolutionary behavior of the gas in the one particle phase space of position and velocity. The time evolution of the state of a gas which is contained in a vessel D bounded by solid walls is determined on one hand by the behavior of the gas molecules at collisions with each other and on the other hand by the influence of the walls as well as by external forces; in the case where there are no external forces, this state is described by a scalar function $f(x, v, t)$ which models the density function of gas particles having position $x \in D$ and velocity $v \in \mathbb{R}^3$ at time $t \in \mathbb{R}$. The integral of this function $\int\int_{D\times\mathbb{R}^3} f(x, v, t)dx\,dv$ gives the expectation value (statistical average) of the total mass of gas contained in the phase space $D\times\mathbb{R}^3$. Under some assumptions, function f must satisfy the Boltzmann equation

$$\frac{\partial f}{\partial t}(x, v, t) = -v \cdot \nabla_x f(x, v, t) + J(f(x, \cdot, t))(v) \qquad (\star)$$

completed by boundary and initial conditions. The first term in $(\star)$ is called streaming operator which is responsible for the motion of the particles between collisions, while the second one $J(f(x, \cdot, t))$, which is bilinear, describes the mechanism of collisions. A solution to the initial boundary value problem for $(\star)$ and a proof of H-theorem are given by treating it under its abstract form (for more details, see [1]).

This equation is applied also to the transport of photons involved in studies of nuclear reactors, including calculations on the protection against radiation and calculations of warm-up of materials. The quantum behavior of neutrons occurs in collisions with nuclei, but for physicists these events of collisions can be considered as one-time events and instantaneous, which only the consequences are interested in. According to the energy of the incident neutron and the nucleus with which it interacts, different types of reactions can occur. The neutron can be absorbed or broadcasted or it causes the fission of the nucleus. Each reaction is characterized by the microscopic cross section. Between collisions, neutrons behave as classical particles, described by their position and speed. Uncharged (neutral particles), they move in a straight line at least for short distances for which we neglect the effect of the gravitation. The neutronic equations are naturally linear. Indeed, the neutron-neutron interactions can be neglected vis-a-vis neutron-matter interactions. The relationship between the neutron density and the density of the propagation medium (water, uranium oxyde,...) is of the order 10^{-15}, which justifies this approximation. This assumption leads to simplifying the nonlinear version of the Boltzmann equation used in the kinetic theory of gases.

Without delayed neutrons, these equations can be written under the form

$$\frac{\partial \psi}{\partial t}(x,v,t) + v\cdot\nabla_x\psi(x,v,t) - \sigma(v)\psi(x,v,t) + \int_V \kappa(x,v,v')\psi(x,v',t)\,d\mu(v') = 0 \quad (1)$$

with initial data $\psi(x,v,0) = \psi_0(x,v)$, where $(x,v) \in D\times V$. D is a smooth open subset of $\mathbb{R}^n$ and $\mu(\cdot)$ is a positive Radon measure on $\mathbb{R}^n$ such that $\mu(\{0\}) = 0$ and V (admissible velocity space) denotes the support of μ. The function $\psi(x;v;t)$ describes the distribution of the neutrons in a nuclear reactor occupying the region D. The functions $\sigma(\cdot)$ and $\kappa(\cdot,\cdot,\cdot)$ are called, respectively, the collision frequency and the scattering kernel.

Here, the boundary conditions which represent the interaction between the particles and ambient medium are given by a boundary bounded operator H satisfying

$$\psi_- = H(\psi_+), \quad (2)$$

where ψ_- (resp., ψ_+) is the restriction of ψ to Γ_- (resp., Γ_+) with Γ_- (resp., Γ_+) being the incoming (resp., outcoming) part of the phase space boundary and H is a linear bounded operator from a suitable function space on Γ_+ to a similar one on Γ_-. The classical boundary conditions (vacuum boundary, specular reflections, diffuse reflections, and periodic and mixed type boundary conditions) are special examples of our framework.

Let $(x,v) \in \overline{D}\times V$. We define the positive real numbers $t^{\pm}(x;v)$ by

$$t^{\pm}(x,v) = \sup\{t>0;\ x\pm sv\in D,\ \forall 0<s<t\}. \quad (3)$$

Physically, $t^{\pm}(x,v)$ is the time taken by a neutron initially in $x\in D$ with animated speed $\pm v$ to achieve (for the first time) the boundary of D.

We denote by $\Gamma_{\pm}$ the set

$$\Gamma_{\pm} = \{(x,v)\in\partial D\times V;\ \pm v\cdot n_x \geq 0\}, \quad (4)$$

where n_x is the outer unit normal vector at $x\in\partial D$.

Let $1\leq p<\infty$; we introduce the functional spaces

$$W_p = \{\psi\in X_p \text{ such that } v\cdot\nabla_x\psi\in X_p\}, \quad (5)$$

where

$$X_p := L_p(D\times V; dx\,d\mu(v)). \quad (6)$$

The spaces of traces are $L_p^{\pm} := L_p(\Gamma_{\pm}; |v\cdot n_x|d\gamma(x)d\mu(v))$. Here $d\gamma(\cdot)$ is the Lebesgue measure on ∂D.

Recall that, for every $\psi\in W_p$, we can define the traces $\psi_{\pm}$ on $\Gamma_{\pm}$; unfortunately, these traces do not belong to $L_p^{\pm}$. The traces lie only in $L_{p,\mathrm{loc}}^{\pm}$ or precisely in a certain weighted L_p space (see [2–4], for details).

Define

$$\widetilde{W_p} = \{\psi\in W_p;\ \psi_{\pm}\in L_p^{\pm}\}. \quad (7)$$

In this case $H\in\mathscr{L}(L_p^+, L_p^-)$ $(1\leq p<\infty)$ and the associated advection operator T_H is given as follows:

$$\begin{aligned} T_H : D(T_H)\subseteq X_p &\longrightarrow X_p,\\ \varphi &\longrightarrow (T_H\varphi)\\ &= -v\cdot\nabla_x\varphi(x,v) - \sigma(v)\varphi(x,v), \end{aligned} \quad (8)$$

with domain

$$D(T_H) = \{\psi\in\widetilde{W_p} \text{ such that } \psi_- = H(\psi_+)\}, \quad (9)$$

where the collision frequency $\sigma(\cdot)\in L_+^{\infty}(V)$ (in other words, a positive bounded function).

Let $\lambda\in\mathbb{C}$; consider the boundary value problem

$$\begin{aligned} \lambda\psi(x,v) + v\cdot\nabla_x\psi(x,v) + \sigma(v)\psi(x,v) &= \varphi(x,v),\\ \psi_- &= H(\psi_+), \end{aligned} \quad (10)$$

where $\varphi\in X_p$ and the unknown ψ must belong to $D(T_H)$. Let

$$\lambda^* := \mu - \operatorname*{ess\,inf}_{v\in V}\sigma(v). \quad (11)$$

For $\mathrm{Re}\lambda + \lambda^* > 0$, (10) can be solved formally by

$$\psi(x,v) = \psi(x - t^-(x,v)v, v)\,e^{-(\lambda+\sigma(v))t^-(x,v)} + \int_0^{t^-(x,v)} e^{-(\lambda+\sigma(v))s}\varphi(x-sv,v)\,ds. \quad (12)$$

Moreover, if $(x,v)\in\Gamma_+$, (10) becomes

$$\psi_+(x,v) = \psi_- e^{-(\lambda+\sigma(v))\tau(x,v)} + \int_0^{\tau(x,v)} e^{-(\lambda+\sigma(v))s}\varphi(x-sv,v)\,ds, \quad (13)$$

where $\tau(x,v) = t^+(x,v) + t^-(x,v)$. On the other hand, for every $(x,v)\in\overline{D}\times V$, we have $(x - t^-(x,v)v, v)\in\Gamma_-$ (for more details on the time numbers t^+, t^-, and τ, see [1]).

For the abstract formulation of (12) and (13), we define the following operators depending on the parameter λ:

$$\begin{aligned} M_\lambda : L_p^- &\longrightarrow L_p^+,\\ u &\longrightarrow M_\lambda u := u e^{-(\lambda+\sigma(v))\tau(x,v)};\\ B_\lambda : L_p^- &\longrightarrow X_p,\\ u &\longrightarrow B_\lambda u := u e^{-(\lambda+\sigma(v))t^-(x,v)};\\ G_\lambda : X_p &\longrightarrow L_p^+,\\ \varphi &\longrightarrow \int_0^{\tau(x,v)} e^{-(\lambda+\sigma(v))s}\varphi(x-sv,v)\,ds;\\ C_\lambda : X_p &\longrightarrow X_p,\\ \varphi &\longrightarrow \int_0^{t^-(x,v)} e^{-(\lambda+\sigma(v))s}\varphi(x-sv,v)\,ds. \end{aligned} \quad (14)$$

Straightforward calculations using Hölder's inequality show that all these operators are bounded on their respective spaces. More precisely, we have, for $\mathrm{Re}\lambda > -\lambda^*$,

$$\begin{aligned} \|M_\lambda\| &\leq 1, \\ \|B_\lambda\| &\leq \left(p\left(\mathrm{Re}\lambda + \lambda^*\right)\right)^{-1/p}, \\ \|G_\lambda\| &\leq \left(q\left(\mathrm{Re}\lambda + \lambda^*\right)\right)^{-1/q}, \\ \|C_\lambda\| &\leq \frac{1}{\mathrm{Re}\lambda + \lambda^*}\left(\frac{1}{p} + \frac{1}{q} = 1\right). \end{aligned} \tag{15}$$

1.1. Collision Operators. The collision operator K given as a perturbation of the advection transport operator T_H is defined on X_p by

$$\begin{aligned} K : X_p &\longrightarrow X_p \\ \psi &\longrightarrow \int_V \kappa\left(x, v, v'\right)\psi\left(x, v'\right) d\mu\left(v'\right). \end{aligned} \tag{16}$$

Note that the operator K is local in x; it describes the physics scattering and production of particles (fission), so it can be viewed as mapping:

$$K(\cdot) : x \in D \longrightarrow K(x) \in \mathscr{L}\left(L_p(V)\right). \tag{17}$$

We assume that $K(\cdot)$ is strongly measurable,

$$\begin{aligned} &x \in D \longrightarrow K(x)\varphi \in L_p(V) \text{ is measurable for any } \varphi \\ &\in L_p(V), \end{aligned} \tag{18}$$

and bounded,

$$\operatorname*{ess\,sup}_{x\in D} \|K(x)\|_{\mathscr{L}(L_p(V))} < \infty. \tag{19}$$

It follows that K defines a bounded operator on the space $L_p(D \times V)$ according to the formula

$$\varphi \in L_p(D \times V) \tag{20}$$

($L_p(D \times V) \simeq L_p(D; L_p(V))$) and

$$\|K(x)\|_{\mathscr{L}(L_p(D\times V))} \leq \operatorname*{ess\,sup}_{x\in D} \|K(x)\|_{\mathscr{L}(L_p(V))}. \tag{21}$$

The final assumption on K is

$$K(x) \in \mathscr{K}\left(L_p(V)\right) \text{ almost everywhere,} \tag{22}$$

where $\mathscr{K}(L_p(V))$ denotes the set of compact linear operators on the space $L_p(V)$.

We give now the concept of regular collision operators introduced by Mokhtar-Kharroubi [5].

Definition 1. A collision operator,

$$K(\cdot) : x \in D \longrightarrow K(x) \in \mathscr{L}\left(L_p(V)\right), \tag{23}$$

is said to be regular if $K(x)$ is compact on $L_p(V)$ almost everywhere on D and

$$K(\cdot) : x \in D \longrightarrow \mathscr{L}\left(L_p(V)\right) \tag{24}$$

is a "Bochner measurable function".

The interest of the class of regular collision operators lies in the following lemma.

Lemma 2 (see [5, Proposition 4.1]). *A regular collision operator K can be approximated, in the uniform topology, by a sequence $\{K_n\}$ of collision operators with kernels of the form*

$$\sum_{i\in I} f_i(x)\, g_i(\xi)\, h_i\left(\xi'\right), \tag{25}$$

where $f_i \in L^\infty(D)$, $g_i \in L_p(V)$ and $h_i \in L_q(V)$ $(1/p + 1/q = 1)$ (I is finite).

It is easy to observe that (1) can be written under the following abstract Cauchy problem:

$$\begin{aligned} \frac{\partial\psi}{\partial t} &= \left(T_H + K\right)\psi(t), \quad (t > 0), \\ \psi(0) &= \psi_0. \end{aligned} \tag{26}$$

Spectral theory of transport operators has known a major development since the pioneering papers of Lehner and Wing and Jörgens in the late 1950s [6–8]. A considerable literature has been devoted to the spectral analysis of the transport operator. This one is studied by means of the nature of the parameters of the equation (nature of boundary conditions, nature of the domain of positions or velocity space, and nature of the collision operator). Let us quote, for example, [1, 4–6, 9–48].

In general, the time asymptotic behavior of solutions of (1) is analyzed under two angles: resolvent approach and the semigroup approach.

(1) Resolvent Approach. For $1 < p < \infty$, this approach is based essentially on the compactness (or the compactness of one iterate) of the bounded linear operator $(\lambda - T_H)^{-1}K$. Indeed, Vidav [44] observed that if this condition is satisfied, it leads via an analytic Fredholm alternative to the fact that the set $\sigma(T_H + K) \cap \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > s_H\}$ (σ is the spectrum, while s_H is the spectral bound of the operator T_H) composed (at most) a set of isolated eigenvalues with finite algebraic multiplicities $\{\lambda_i\}_{i\in J}$, where $\{\lambda_i, \mathrm{Re}\lambda_i \geq \alpha\}$ is a finite set for each $\alpha > s_H$. If $p = 1$, it suffices to treat the weak compactness by taking into account the fact that the square of weakly compact operator on this space is compact [49, Corollary 13, p. 510]. Recall that, among relevant results in this direction, we can cite the works of Mokhtar-Kharroubi [5, 36], Latrach [24–27], and Song [43].

Thus, if T_H generates a c_0-semigroup $(U(t);\ t \geq 0)$, by Dyson-Phillips theorem of perturbation, $T_H + K$ generates a c_0-semigroup $(V(t);\ t \geq 0)$ given by the following formula (see [50, Corollary 7.5, p. 29]):

$$V(t)\psi_0 = \frac{1}{2i\pi}\lim_{\gamma\to\infty}\int_{\nu-i\gamma}^{\nu+i\gamma} e^{\lambda t}(\lambda - T - K)^{-1}\psi_0 d\lambda, \quad (t > 0), \tag{27}$$

where ν is sufficiently large by deforming the contour of integration in Dunford's formula. Recover the residues corresponding to the poles (eigenvalues of $T_H + K$); we can obtain a good comprehension of the asymptotic behavior of solution when the initial data ψ_0 belongs to $D(T_H+K)^2$ (unfortunately this regular condition is not natural).

(2) Semigroup Approach. Even if $\sigma(T_H + K) \cap \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > s_H\}$ is reduced to isolated eigenvalues of finite algebraic multiplicities, the set $\sigma(V(t)) \cap \{\eta \in \mathbb{C} : |\eta| > e^{s_H t}\}$ can contain the continuous spectrum due to the absence of a spectral mapping theorem for the mapping $\lambda \to e^{t\lambda}$. Vidav [45] has shown that the time asymptotic behavior of $V(t)_{t\geq 0}$ is connected to the analysis of its spectrum and the compactness of remainder terms of the Dyson-Phillips expansion $R_n(t) = \sum_{j=n}^{\infty} U_j(t)$ (where $U_0(t) = U(t)$ and $U_n(t) = \int_0^t U(t-s)KU_{n-1}(s)ds$ for all $n \geq 1$) is an appropriate tool to exclude the eventual presence of the continuous spectrum and to restore the following spectral mapping theorem:

$$\begin{aligned}\sigma(V(t)) &\cap \left\{\eta \in \mathbb{C} : |\eta| > e^{s_H t}\right\} \\ &= e^{t\sigma(T_H+K)} \cap \left\{e^{t\lambda} : \lambda > s_H\right\}.\end{aligned} \tag{28}$$

This technique has the advantage of not imposing any condition on the initial data; it has been used by [36, 44, 45, 51] and other authors to study the time asymptotic behavior of solutions of transport equations for absorbing boundary conditions ($H = 0$) or $\psi_{|\Gamma_-} = 0$; in other words, it has been used in the case where each neutron which arrives at a point of ∂D and coming from the interior of D disappears, and no neutron arrives from outside and where D is bounded. Many contributions have been made in this direction, showing in particular the compactness of the second-order remainder of the Dyson-Phillips expansion, sometimes through heavy calculations in the case of non absorbing boundary conditions. Recently and always for absorbing boundary conditions, dealing with regular collision operators by assuming that the domain of positions has a finite volume (not necessarily bounded), Mokhtar-Kharroubi [40] has established the compactness of the first remainder term of the Dyson-Phillips expansion on $L_p(D\times V)$ $(1 < p < \infty)$. This analysis simplifies considerably the spectral analysis of transport equations and extends all known results made in the framework of the study of the compactness of the second-order remainder term; this is due to the fact that if $R_n(t)$ is compact, thus $R_{n+1}(t)$ is also compact, and it implies that $(U(t))_{t\geq 0}$ and $(V(t))_{t\geq 0}$ have the same essential spectra and consequently the same essential types. Unfortunately, this argument cannot be applied to the case where $p = 1$ since its proof was obtained in the framework of $L_2(D \times V)$ (and extended to $L_p(D \times V)$ space $(1 < p < \infty)$ via some interpolation techniques) using some properties of Fourier transform and Hilbert-Schmidt operators. Better than that, Mokhtar-Kharroubi conjectured that the first remainder term of the Dyson-Phillips expansion $R_1(t)$, $t > 0$ is not compact on $L_1(D \times V)$; additionally, its weakly compactness is an open problem (see [5, Problem 7, p. 94]).

In this work, we study the impact of compactness results on p-independence of the asymptotic spectrum of the transport operator A_H. These results are established by means of some geometrical properties of the space of positions D and the Radon measure μ having the velocity space V as a support and the natures of the collision operator K and the boundary linear operator H.

2. Main Results

2.1. Compactness Results and p-Independence of $\sigma_s(A_H)$. We assume that the measure μ satisfies the following specific geometrical property:

$$\int_{c_1\leq\|x\|\leq c_2} d\mu(x)\int_0^{c_3}\chi_A(tx)\,dt \quad \text{as } |A| \longrightarrow 0, \tag{29}$$

for every $0 < c_1 < c_2 < \infty$ and $c_3 < \infty$, where $|A|$ is the Lebesgue measure of the set A and χ_A is the indicator function of A.

Remark 3. As indicated in [5, Remark 4.3], the above condition is satisfied by the Lebesgue measure on $\mathbb{R}^n$ or on spheres (multigroup model).

We start our analysis by the following fundamental compactness result which will be used in the rest of this section.

Theorem 4. *If $p = 1$, let $H \in \mathscr{L}(L_1^+, L_1^-)$, $\|H\| < 1$ be a weakly compact boundary linear operator and let μ be a Radon measure satisfying the condition (29). Assume that the collision operator K is regular. Thus,*

(i) *if D is bounded, then $K(\lambda - T_H)^{-1}K$ is weakly compact on $L_1(D \times V, dxd\mu)$;*

(ii) *if D is a bounded convex set in $\mathbb{R}^n$ and H is let to be compact, then $K(\lambda - T_H)^{-1}K$ is compact on $L_1(D \times V, dxd\mu)$.*

Proof. If $\|H\| < 1$, T_H generates a c_0-semigroup $(U_H(t);\ t \geq 0)$ on X_1, and then its resolvent exists as bounded linear operator satisfying

$$(\lambda - T_H)^{-1} = \Gamma_\lambda^H + C_\lambda, \tag{30}$$

where $\Gamma_\lambda^H = \sum_{n\geq 0} B_\lambda H(M_\lambda H)^n G_\lambda$, and

$$\left\|(\lambda - T_H)^{-1}\right\| \leq \frac{1}{\mathrm{Re}\lambda + \lambda^\star}, \quad (\mathrm{Re}\lambda > -\lambda^\star). \tag{31}$$

Thus,

$$K(\lambda - T_H)^{-1}K = K\Gamma_\lambda^H K + KC_\lambda K. \tag{32}$$

Obviously, if H is weakly compact, then Γ_λ^H is weakly compact. On the other hand, it is easy to observe that C_λ is nothing but the resolvent of the streaming operator with vacuum boundary conditions T_0. Now, under condition (29) and by applying [5, Theorem 4.4($\imath$)], we obtain that $K(\lambda - T_0)^{-1}K$ is weakly compact on X_1 if D is bounded. Moreover, if D is convex, then, by applying [5, Theorem 4.4($\imath\imath$)], we get the compactness of the operator $K(\lambda - T_0)^{-1}K$. Since the compactness and weak compactness property concerning bounded linear operators is stable under summation, we obtain the desired result. □

Question 1. It is known that for $1 < p < \infty$ and if the boundary linear operator H is compact, then T_H generates a c_0-semigroup $(U_H(t);\ t \geq 0)$ on X_p (see [31, Theorem 6.8]); is the result still true for $p = 1$ under the condition that H is weakly compact multiplicative boundary operator ($\|H\| \geq 1$)?

Let (D_i, μ_i), $i = 0, 1$, be measure spaces with σ-finite positive measures μ_i.

Theorem 5 (Riesz-Thorin theorem). *Assume that* $1 \leq p_i$, $q_i \leq \infty$, *for* $i = 0, 1$, *and let* T *be a linear operator which maps* $L_{p_i}(D_0, \mu_0)$ *continuously into* $L_{q_i}(D_1, \mu_1)$ *with norm* M_i. *If* $0 < \theta < 1$ *and* $1/p = (1-\theta)/p_0 + \theta/p_1$, $1/q = (1-\theta)/q_0 + \theta/q_1$, *then* T *maps* $L_p(D_0, \mu_0)$ *continuously into* $L_q(D_1, \mu_1)$ *with norm* $M \leq M_0^{1-\theta} M_1^{\theta}$.

This theorem shows that the boundedness of linear operators can be interpolated between L_p-spaces. In 1960, Krasnoselskii [52] showed that compactness can be also interpolated. Thus, we can announce the following result.

Theorem 6. *Assume that* $1 \leq p_i$, $q_i \leq \infty$, *for* $i = 0, 1$, *and let* $T : L_{p_i}(D_0, \mu_0) \rightarrow L_{q_i}(D_1, \mu_1)$ *be compact. If* $0 < \theta < 1$ *and* $1/p = (1-\theta)/p_0 + \theta/p_1$, $1/q = (1-\theta)/q_0 + \theta/q_1$, *then* $T : L_p(D_0, \mu_0) \rightarrow L_q(D_1, \mu_1)$ *is also compact.*

By combining Theorem 6 and [53, Corollary 1.6.2], the following lemma can be derived.

Lemma 7. *Let* $1 \leq p_0$, $p_1 < \infty$. *Assume that a linear operator* $T : L_{p_0}(D) \cap L_{p_1}(D) \rightarrow L_{p_0}(D) \cap L_{p_1}(D)$ *can be extended to bounded linear operators on* $L_{p_i}(D)$, $(i = 0, 1)$ *such that at least one of them is power compact. Then*

(i) *T can be extended to a power compact operator on $L_s(D)$ for each $s \in (p_0, p_1)$;*

(ii) *denote the extension of T to $L_s(D)$ by T_s. If T_{p_i} is power compact, then $\sigma(T_s) = \sigma(T_{p_i})$ for all $s \in (p_0, p_1)$ and the spectral projections corresponding to nonzero eigenvalues are independent of p.*

Let T_H^p (resp., A_H^p) be the closed densely defined operator T_H (resp., A_H) on X_p, $(1 \leq p < \infty)$. We denote by K_p and $(U_H^p(t);\ t \geq 0)$ the bounded linear operators K and $(U_H(t);\ t \geq 0)$ defined on X_p. Let $\sigma_s^p(A_H) = \sigma(A_H^p) \cap \{\lambda \in \mathbb{C} / \mathrm{Re}\lambda > -\lambda^\star\}$ (the asymptotic spectrum of the operator A_H^p).

Now, we establish the fundamental result of this work which describes p-independence of $\sigma_s^p(A_H)$.

Theorem 8. *Under assumptions of Theorem 4, we have*

(i) *$\sigma_s^p(A_H) = \sigma_s^1(A_H)$ for all $p > 1$;*

(ii) *if $\lambda \in \sigma_s^p(A_H) = \sigma_s^1(A_H)$, we have $\mathcal{N}((\lambda - A_H^p)^m) = \mathcal{N}((\lambda - A_H^1)^m)$ for every positive integer m and every $p > 1$, where $\mathcal{N}(T)$ designates the null space of the linear operator T. As a consequence, both the geometrical multiplicity and algebraic multiplicity of λ are p-independent.*

Proof. Let $p > 1$; we have

$$\begin{aligned} K_{p|L_p \cap L_1} &= K_{1|L_p \cap L_1}, \\ U_H^p(t)_{|L_p \cap L_1} &= U_H^1(t)_{|L_p \cap L_1}, \quad (t \geq 0). \end{aligned} \tag{33}$$

The resolvent of T_H^p can be written as the Laplace transform of $U_H^p(t)$ as follows:

$$\left(\lambda - T_H^p\right)^{-1} \varphi = \int_0^\infty e^{-\lambda t} U_H^p(t) \varphi \, dt. \tag{34}$$

Thus for $\lambda \in \mathbb{C}$ such that $\mathrm{Re}\lambda > -\lambda^\star$, we get

$$\left(\lambda - T_H^p\right)^{-1}_{|L_p \cap L_1} = \left(\lambda - T_H^1\right)^{-1}_{|L_p \cap L_1}. \tag{35}$$

By applying Theorem 4, we obtain the compactness of $((\lambda - T_H^1)^{-1}K_1)^2$ on $L_1(D \times V)$. On the other hand, Lemma 7 implies the compactness of $((\lambda - T_H^p)^{-1}K_p)^2$ on $L_p(D \times V)$ and consequently $\sigma(((\lambda - T_H^p)^{-1}K_p)^2) = \sigma(((\lambda - T_H^1)^{-1}K_1)^2)$ for every $p > 1$. Hence, Gohberg-Schmul'yan theorem [22, Theorem 11.4] shows that $\sigma_s^p(A_H)$ consists of discrete eigenvalues with finite algebraic multiplicity. Moreover, it is easy to observe that $1 \in \sigma(((\lambda - T_H^p)^{-1}K_p)^2)$ if and only if $\lambda \in \sigma_s^p(A_H)$; therefore, taking into account assertion (ii) in Lemma 7, we get that $\sigma_s^p(A_H) = \sigma_s^1(A_H)$ for all $p \geq 1$.

Next, following estimation (31), we obtain that $\lim_{\mathrm{Re}\lambda \rightarrow +\infty} \|(\lambda - T_H^p)^{-1}K_p\| = 0$. This gives that, for $\mathrm{Re}\lambda$ sufficiently large, we have

$$\left(I - \left(\lambda - T_H^p\right)^{-1} K_p\right)^{-1} = \sum_{j=0}^{\infty} \left(\left(\lambda - T_H^p\right)^{-1} K_p\right)^j. \tag{36}$$

Thus, for $\mathrm{Re}\lambda$ sufficiently large, it follows that

$$\begin{aligned} &\left(I - \left(\lambda - T_H^p\right)^{-1} K_p\right)^{-1}_{|L_p \cap L_1} \\ &\quad = \left(I - \left(\lambda - T_H^1\right)^{-1} K_1\right)^{-1}_{|L_p \cap L_1}. \end{aligned} \tag{37}$$

Using the analyticity of the operator valuation function $\lambda \to (I-(\lambda-T_H^p)^{-1}K_p)^{-1}$ on the set $\{\lambda \in \mathbb{C}/\mathrm{Re}\lambda > -\lambda^*\} \setminus \sigma_s^p(A_H)$ for each $p \geq 1$, we get that for

$$\lambda \in \{\mu \in \mathbb{C}/\mathrm{Re}\mu > -\lambda^*\} \setminus \sigma_s^p(A_H) = \{\mu \in \mathbb{C}/\mathrm{Re}\mu > -\lambda^*\} \setminus \sigma_s^1(A_H) \tag{38}$$

we have

$$\left(I-\left(\lambda-T_H^p\right)^{-1}K_p\right)^{-1}_{|L_p\cap L_1} = \left(I-\left(\lambda-T_H^1\right)^{-1}K_1\right)^{-1}_{|L_p\cap L_1}. \tag{39}$$

Using the formula $(\lambda-A_H^p)^{-1} = (I-(\lambda-T_H^p)^{-1}K_p)^{-1}(\lambda-T_H^p)^{-1}$ for each $p \geq 1$ and $\lambda \in \{\mu \in \mathbb{C}/\mathrm{Re}\mu > -\lambda^*\} \setminus \sigma_s^p(A_H)$, one sees that

$$\left(\lambda-A_H^p\right)^{-1}_{|L_p\cap L_1} = \left(\lambda-A_H^1\right)^{-1}_{|L_p\cap L_1} \tag{40}$$

for $\lambda \in \{\mu \in \mathbb{C}/\mathrm{Re}\mu > -\lambda^*\} \setminus \sigma_s^p(A_H)$.

Now, if we denote by $\mathscr{P}_\lambda(A_H^p)$ the spectral projection corresponding to an eigenvalue ζ of A_H^p, then for $\beta > 0$ sufficiently small

$$\mathscr{P}_\lambda\left(A_H^p\right) = \frac{1}{2i\pi}\int_{|z-\zeta|=\beta}\left(z-A_H^p\right)^{-1}dz. \tag{41}$$

According to (40) and (41), it follows that for each $\lambda \in \sigma_s^p(A_H) = \sigma_s^1(A_H)$

$$\mathscr{P}_\lambda\left(A_H^p\right)_{|L_p\cap L_1} = \mathscr{P}_\lambda\left(A_H^1\right)_{|L_p\cap L_1}. \tag{42}$$

Since the space of infinitely differentiable functions with compact supports $\mathscr{C}_0^\infty(D\times V)$ is dense in X_p, then $\mathscr{P}_\lambda(A_H^p)(\mathscr{C}_0^\infty(D\times V))$ is dense in $\mathscr{P}_\lambda(A_H^p)(X_p)$, but these two vector spaces are finite-dimensional; hence

$$\mathscr{P}_\lambda\left(A_H^p\right)\left(\mathscr{C}_0^\infty(D\times V)\right) = \mathscr{P}_\lambda\left(A_H^p\right)\left(X_p\right), \quad (p \geq 1). \tag{43}$$

According to (42) and (43), we obtain that $\mathscr{P}_\lambda(A_H^p)(X_p) = \mathscr{P}_\lambda(A_H^1)(X_1)$ for all $p \geq 1$ and $\lambda \in \sigma_s^p(A_H)$. Afterwards, since, for every $k \geq 1$, we have $\mathscr{N}((\lambda - A_H^p)^k) \subset \mathscr{P}_\lambda(A_H^p)(X_p) = \mathscr{P}_\lambda(A_H^1)(X_1)$ and $\mathscr{N}((\lambda - A_H^1)^k) \subset \mathscr{P}_\lambda(A_H^1)(X_1) = \mathscr{P}_\lambda(A_H^p)(X_p)$, it follows that $\mathscr{N}((\lambda-A_H^1)^k) = \mathscr{N}((\lambda - A_H^p)^k) \subset X_p \cap X_1$.

In the spirit of the above theorem, we can prove the following result without weak compactness hypothesis on the boundary linear operator H and the geometrical property (29) with boundedness of D but based on the weak compactness of one remainder of the Dyson-Phillips expansion.

Let $\varrho_s^p(A_H) = \sigma(A_H^p)\bigcap\{\lambda \in \mathbb{C}/\mathrm{Re}\lambda > s(T_H)\}$, where $s(T_H)$ is the spectral bound of the operator T_H in X_p for all $p \geq 1$. □

Proposition 9. *Let K be a regular collision operator and let $H \in \mathscr{L}(L_p^+, L_p^-)$ $(1 \leq p < \infty)$ such that T_H generates a c_0-semigroup $(U_H(t);\ t \geq 0)$ on X_p. If one of the remainder terms of the Dyson-Phillips series $R_n^H(t)$ is weakly compact on X_1, then*

(i) *$\varrho_s^p(A_H) = \varrho_s^1(A_H)$ for all $p > 1$;*

(ii) *if $\lambda \in \varrho_s^p(A_H) = \varrho_s^1(A_H)$, then $\mathscr{N}((\lambda - A_H^p)^m) = \mathscr{N}((\lambda - A_H^1)^m)$ for every positive integer m and every $p > 1$. As a consequence, both the geometrical multiplicity and algebraic multiplicity of λ are p-independent.*

Proof. Following the proof of Theorem 8, it suffices to show that there exists $m \geq 1$ such that $((\lambda - T_H)^{-1}K)^m$ is compact for $\lambda \in \mathbb{C}$ such that $\mathrm{Re}\lambda > s(T_H)$. Indeed, assume that there exists $n \geq 1$ such that $R_n^H(t)$ is weakly compact on X_1. Then, according to [5, Theorem 2.6], $U_n(t) = ([UK]^n * U)(t)$ is weakly compact and consequently by [5, Theorem 2.3] the strong integral

$$\int_0^N e^{-\lambda t}U_n(t)\,dt \text{ is weakly compact on } X_1. \tag{44}$$

On the other hand, we have

$$\int_0^N e^{-\lambda t}U_n(t)\,dt \longrightarrow \int_0^\infty e^{-\lambda t}U_n(t)\,dt \text{ in } \mathscr{L}(X_1). \tag{45}$$

Hence, $\int_0^\infty e^{-\lambda t}U_n(t)dt$ is weakly compact on X_1. Since the Laplace transform of $([UK]^n * U)(t)$ is nothing but $((\lambda - T_H)^{-1}K)^n(\lambda - T_H)^{-1}$, this gives the weak compactness of $((\lambda - T_H)^{-1}K)^n(\lambda - T_H)^{-1}$ for $\lambda \in \mathbb{C}$ such that $\mathrm{Re}\lambda > \omega$, where ω is the type of $(U_H(t);\ t \geq 0)$; by analytic arguments, we obtain that $((\lambda - T_H)^{-1}K)^n(\lambda - T_H)^{-1}$ is weakly compact for $\lambda \in \mathbb{C}$ such that $\mathrm{Re}\lambda > s(T_H)$ and implies compactness of $((\lambda - T_H)^{-1}K)^{2n+2}$ which gives the needed result. □

Now, we focus our study on the case of slab geometry.

Theorem 10. *If $D =]-a, a[$, $V = [-1, 1]$, $\mu = \nu$ (the Lebesgue measure on $\mathbb{R}$) and H is a bounded boundary linear operator from L_p^+ to L_p^-, then*

(i) *$\sigma_s^p(A_H) = \sigma_s^1(A_H)$ for all $p > 1$;*

(ii) *if $\lambda \in \sigma_s^p(A_H) = \sigma_s^1(A_H)$, then $\mathscr{N}((\lambda - A_H^p)^m) = \mathscr{N}((\lambda - A_H^1)^m)$ for every positive integer m and every $p > 1$. As a consequence, both the geometrical multiplicity and algebraic multiplicity of λ are p-independent.*

Proof. Here, the time of sojourn of particles in D is bounded from below by $2a$; indeed, in this case we have $\inf\{\tau(x, v);\ (x, v) \in \Gamma_+\} = 2a > 0$. As an immediate consequence, T_H generates a c_0-semigroup for any boundary linear operator H [29, Remark 6]. Moreover, we have

$$\left\|\left(\lambda - T_H\right)^{-1}\right\| \leq \frac{\alpha}{\mathrm{Re}\lambda + \lambda^*}, \quad \left(\forall\lambda \in \Lambda_0\right), \tag{46}$$

where α is a positive constant depending on $\|H\|$ and $\Lambda_0 = \{\lambda \in \mathbb{C}/\mathrm{Re}\lambda > \lambda_0\}$ with

$$\lambda_0 = \begin{cases} -\lambda^*, & \text{if } \|H\| \le 1, \\ -\lambda^* + \dfrac{1}{2a}\ln(\|H\|), & \text{if } \|H\| > 1. \end{cases} \quad (47)$$

On the other hand, we have $((\lambda - T_H)^{-1}K)^2$ is compact for all $1 \le p < \infty$ [25, Theorem 2.1]. By adopting the same techniques given in the proof of Theorem 8, we obtain the desired result. □

Remark 11. We note that $K(\lambda - T_H)^{-1}$ is not weakly compact in general on $L_1(D \times V)$ (see [18]).

Question 2. In the case of slab geometry, is $(\lambda - T_H)^{-1}K$ weakly compact on $L_1(]-a, a[\times[-1, 1])$ under the assumption that μ is a diffuse (nonatomic) measure on $\mathbb{R}$?

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] J. Voigt, *Functional Analytic Treatment of the Initial Boundary Value Problem for Collisioness Gases*, Habilitationsschrift, München, Germany, 1981.

[2] M. Cessenat, "Théorèmes de trace pour des espaces de fonctions de la neutronique," *Comptes Rendus de l Académie des Sciences, Series I*, no. 299, p. 16, 1984.

[3] M. Cessenat, "Théorèmes de trace pour des espaces de fonctions de la neutronique," *Comptes Rendus de l'Académie des Sciences, Paris. Série 1*, vol. 300, no. 3, pp. 89–92, 1985.

[4] R. Dautray and J. L. Lions, *Analyse mathématique et calcul numérique*, vol. 9, Masson, Paris, France, 1988.

[5] M. Mokhtar-Kharroubi, *Mathematical Topics in Neutron Transport Theory: New Aspects*, vol. 46 of *Series on Advances in Mathematics for Applied Sciences*, World Scientific, 1997.

[6] K. Jörgens, "An asymptotic expansion in the theory of neutron transport," *Communications on Pure and Applied Mathematics*, vol. 11, pp. 219–242, 1958.

[7] J. Lehner and G. M. Wing, "On the spectrum of an unsymmetric operator arising in the transport theory of neutrons," *Communications on Pure and Applied Mathematics*, vol. 8, no. 2, pp. 217–234, 1955.

[8] J. Lehner and M. Wing, "Solution of the linearized Boltzmann transport equation for the slab geometry," *Duke Mathematical Journal*, no. 23, pp. 125–142, 1956.

[9] S. Albertoni and B. Montagnini, "On the spectrum of neutron transport equation in finite bodies," *Journal of Mathematical Analysis and Applications*, vol. 13, no. 1, pp. 19–48, 1966.

[10] N. Angelescu, N. Marinescu, and V. Protopopescu, "Linear monoenergetic transport with reflecting boundary conditions," *Revue Roumaine de Physique*, vol. 19, pp. 17–26, 1974.

[11] N. Angelescu, N. Marinescu, and V. Protopopescu, "Neutron transport with periodic boundary conditions," *Transport Theory and Statistical Physics*, vol. 5, no. 2-3, pp. 115–125, 1976.

[12] J. Banasiac and L. Arlotti, *Perturbations of Positive Semigroups with Applications*, Springer Monographs in Mathematics, Springer, 2006.

[13] R. Beals and V. Protopopescu, "Abstract time-dependent transport equations," *Journal of Mathematical Analysis and Applications*, vol. 121, no. 2, pp. 370–405, 1987.

[14] A. Belleni-Morante, "Neutron transport in a nonuniform slab with generalized boundary conditions," *Journal of Mathematical Physics*, vol. 11, no. 5, pp. 1553–1558, 1970.

[15] M. Borysiewicz and J. Mika, "Time behaviour of thermal neutrons in moderating media," *Journal of Mathematical Analysis and Applications*, vol. 26, no. 3, pp. 461–478, 1969.

[16] C. Cercignani, *The Boltzmann Equation and Its Applications*, vol. 67 of *Applied Mathematical Sciences*, Springer, 1988.

[17] J. J. Duderstadt and W. R. Martin, *Transport Theory*, John Wiley & Sons, New York, NY, USA, 1979.

[18] F. Golse, P.-L. Lions, B. Perthame, and R. Sentis, "Regularity of the moments of the solution of a transport equation," *Journal of Functional Analysis*, vol. 76, no. 1, pp. 110–125, 1988.

[19] W. Greenberg, C. Van der Mee, and V. Protopopescu, *Boundary Value Problems in Abstract Kinetic Theory*, Birkhäuser, Basel, Switzerland, 1987.

[20] G. Greiner, "Spectral properties and asymptotic behavior of the linear transport equation," *Mathematische Zeitschrift*, vol. 185, no. 2, pp. 167–177, 1984.

[21] H. G. Kaper, "The initial-value transport problem for monoenergetic neutrons in an infinite slab with delayed neutron production," *Journal of Mathematical Analysis and Applications*, vol. 19, no. 2, pp. 207–230, 1967.

[22] H. G. Kaper, C. G. Lekkerkerker, and J. Hejtmanek, *Spectral Methods in Linear Transport Theory*, vol. 5 of *Operator Theory: Advances and Applications*, Birkhäuser, 1982.

[23] E. W. Larsen and P. F. Zweifel, "On the spectrum of the linear transport operator," *Journal of Mathematical Physics*, vol. 15, no. 11, pp. 1987–1997, 1973.

[24] K. Latrach, "Compactness properties for perturbed semigroups and application to transport equation," *Journal of the Australian Mathematical Society: Series A*, vol. 69, no. 1, pp. 25–40, 2000.

[25] K. Latrach, "Compactness properties for linear transport operators with abstract boundary conditions in slab geometry," *Transport Theory and Statistical Physics*, vol. 22, no. 1, pp. 39–65, 1993.

[26] K. Latrach, "Compactness results for transport equations and applications," *Mathematical Models and Methods in Applied Sciences*, vol. 11, no. 7, pp. 1181–1202, 2001.

[27] K. Latrach, *Théorie spectrale d'équations cinétiques [Ph.D. thesis]*, Université de Franche-Comté, Besancon, France, 1992.

[28] K. Latrach and A. Dehici, "Spectral properties and time asymptotic behaviour of linear transport equations in slab geometry," *Mathematical Methods in the Applied Sciences*, vol. 24, no. 10, pp. 689–711, 2001.

[29] B. Lods, "A generation theorem for kinetic equations with noncontractive boundary operators," *Comptes Rendus Mathematique*, vol. 335, no. 7, pp. 655–660, 2002.

[30] B. Lods, *Théorie spectrale des équations cinétiques [Ph.D. thesis]*, Université de Franche-Comté, Besancon, France, 2002.

[31] B. Lods, "Semigroup generation properties of streaming operators with non-contractive boundary conditions," *Mathematical and Computer Modelling*, vol. 42, pp. 1141–1162, 2005.

[32] J. T. Marti, "Mathematical foundations of kinetics in neutron transport theory," *Nucleonik*, vol. 8, no. 3, pp. 159–163, 1966.

[33] J. Mika, "Time dependent neutron transport in plane geometry," *Nucleonik*, vol. 9, no. 4, pp. 200–205, 1967.

[34] J. Mika, "The effects of delayed neutrons on the spectrum of the transport operator," *Nucleonik*, vol. 9, no. 1, pp. 46–51, 1967.

[35] M. Mokhtar Kharroubi, *Propriétés spectrales de l'opérateur de transport dans les cas anisotrope [Thèse 3eme cycle]*, Université de Paris VI, 1982.

[36] M. Mokhtar Kharroubi, "Time asymptotic behavior and compactness in neutron transport theory," *European Journal of Mechanics—B/Fluids*, no. 11, pp. 39–68, 1992.

[37] M. Mokhtar Kharroubi, *Les équations de la neutronique [Thèse de doctorat d'état]*, Université Paris 13, Villetaneuse, France, 1987.

[38] M. Mokhtar Kharroubi, "Effets régularisants en théorie neutronique," *Comptes Rendus de l'Académie des Sciences, Series I*, no. 309, pp. 545–548, 1990.

[39] M. Mokhtar and M. Mokhtar Kharroubi, *A Unified Treatment of the Compactness in Neutron Transport Theory with Applications to Spectral Theory*, Publications Mathématiques de Besançon, 1995-1996.

[40] M. Mokhtar-Kharroubi, "Optimal spectral theory of the linear Boltzmann equation," *Journal of Functional Analysis*, vol. 226, no. 1, pp. 21–47, 2005.

[41] A. Palczewski, "Spectral properties of the space inhomogeneous linearized Boltzmann operator," *Transport Theory and Statistical Physics*, vol. 13, no. 3-4, pp. 409–430, 1984.

[42] M. Sbihi, *Analyse spectrale de modèles neutroniques [Ph.D. thesis]*, Université de Franche-Comté, Besancon, France, 2005.

[43] D. Song, *On the spectrum of neutron transport equation with reecting boundary conditions [Ph.D. thesis]*, Virginia Polytechnic Institut and State University, 2000.

[44] I. Vidav, "Existence and uniqueness of nonnegative eigenfunctions of the Boltzmann operator," *Journal of Mathematical Analysis and Applications*, vol. 22, no. 1, pp. 144–155, 1968.

[45] I. Vidav, "Spectra of perturbed semigroups with applications to transport theory," *Journal of Mathematical Analysis and Applications*, vol. 30, no. 2, pp. 264–279, 1970.

[46] J. Voigt, "On the perturbation theory for strongly continuous semigroups," *Mathematische Annalen*, vol. 229, no. 2, pp. 163–171, 1977.

[47] X. Zhang, "Spectral interpolation of the linear Boltzmann operator in L_p spaces," *Transport Theory and Statistical Physics*, vol. 32, no. 1, pp. 63–71, 2003.

[48] L. W. Weis, "A generalization of the Vidav-Jorgens perturbation theorem for semigroups and its application to transport theory," *Journal of Mathematical Analysis and Applications*, vol. 129, no. 1, pp. 6–23, 1988.

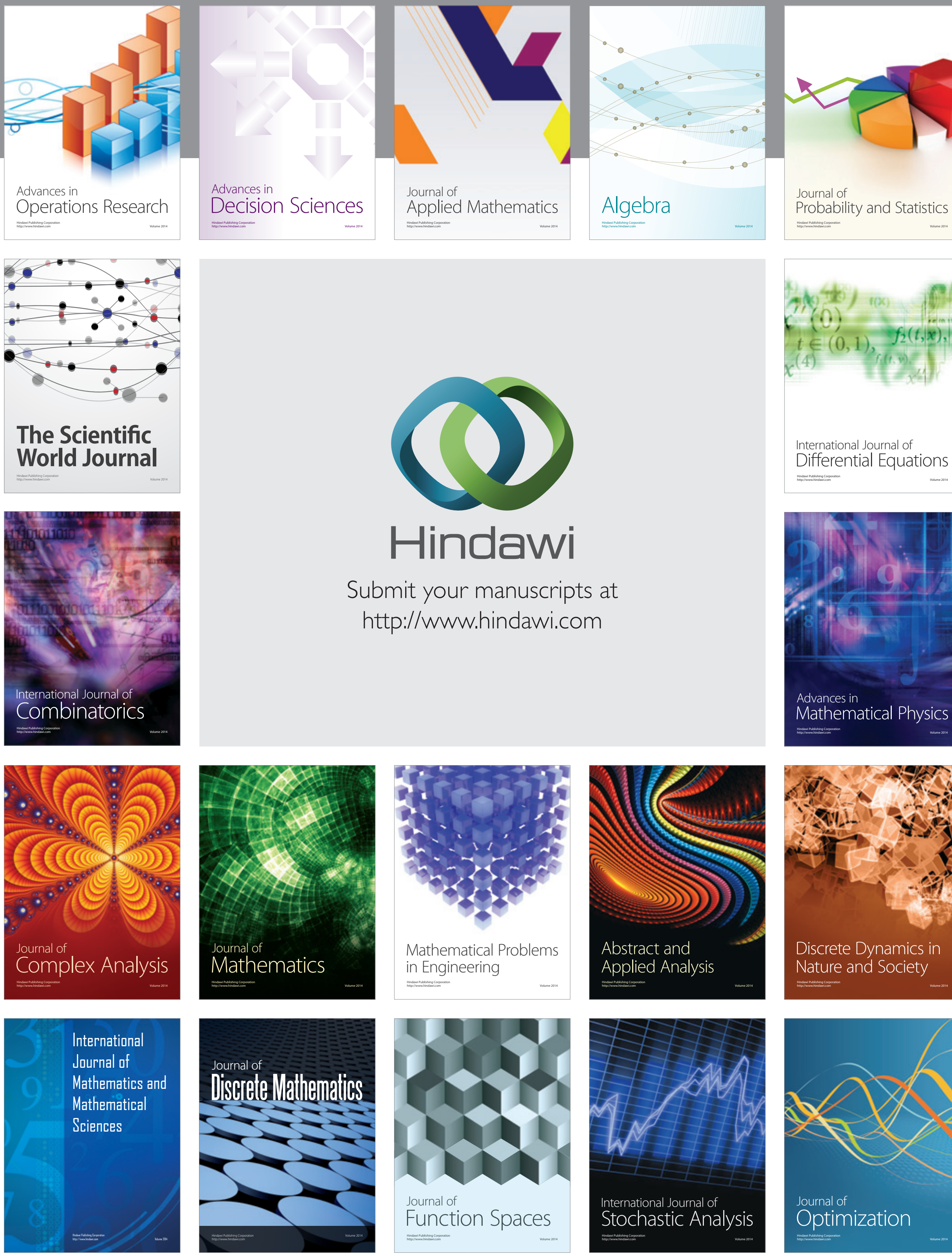
[49] N. Dunford and T. T. Schwartz, *Linear Operators, General Theory Part I*, Interscience, New York, NY, USA, 1958.

[50] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer, New York, NY, USA, 1983.

[51] J. Voigt, "Spectral properties of the neutron transport equation," *Journal of Mathematical Analysis and Applications*, vol. 106, no. 1, pp. 140–153, 1985.

[52] M. A. Krasnoselskii, "On a theorem of M. Riesz," *Soviet Mathematics Doklady*, vol. 1, pp. 229–231, 1960.

[53] E. B. Davies, *Heat Kernels and Spectral Theory*, vol. 92, Cambridge University Press, Cambridge, UK, 1989.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