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Some spectral properties in Banach spaces and application to transport theory

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Abstract In this paper, by using the notion of the Bochner measurability, we establish some properties concerning some classes of C_0-semigroups in Banach spaces. After, we will apply them in the framework of the transport theory in order to obtain compactness properties giving a good comprehension of the time asymptotic behavior of the solutions for the associated Cauchy problems. Moreover, by using the concept of the Hausdorff measure of noncompactness, we obtain some results incoming within the framework of the Fredholm theory. Also, a fine description of the Schechter essential spectrum of a closed densely defined operators is given.

Keywords Transport equation · Bochner measurable function · C_0-semigroup · Boundary operator · Remainder term of the Dyson–Phillips expansion · Schechter essential spectrum

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1 Introduction and notations

Let X be a complex Banach space and $\mathcal{L}(X)$ the algebra of bounded linear operators on X. Consider the following abstract Cauchy problem on X

$$\begin{cases} \dfrac{du}{dt} = Au, \\ u(0) = u_0 \in X \end{cases} \tag{1.1}$$

It is well known that if A satisfies the conditions of the Hille–Yosida theorem (see [8], page 363), then the problem (1.1) admits a unique solution given by $u(t) = V(t)u_0$, $(t \geq 0)$

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where $(V(t), t \geq 0)$ is the c_0-semigroup generated by A. The significant framework which will interest us here is when the operator A can be written under the form $A = T + B$ where T generates a c_0-semigroup $(U(t); t \geq 0)$ and $B \in \mathcal{L}(X)$. This attraction comes owing to the fact that several phenomena of applied sciences are modeled by such a type of problems, like that of the Boltzmann equation and its derivatives. In this case, the perturbation theorem (see [15], page 14) asserts that A generates a c_0-semigroup given by the Dyson-Phillips expansion

$$V(t) = \sum_{j=0}^{\infty} U_j(t),$$

where

$$U_0(t) = U(t),\ U_j(t) = \int_0^t U(s)KU_{j-1}(t-s)ds\ (j \geq 1)$$

Let T be a closed linear densely defined operator on X. By $\sigma(T)$ and $\rho(T)$ we denote respectively the spectrum and the resolvent set of T while the resolvent of T, $(\lambda - T)^{-1}$ will be designated by $R(\lambda, T)$.

In the case where T is a advection transport operator and K the collision operator (local with respect to the positions) which models the physical scattering and the production of the particles (fission), it is proved that in this situation the time asymptotic behavior of the solutions of the problem (1.1) is connected directly to the compactness results which take account of course of the parameters of the equation and the geometry of the phase space. The approaches most used in this direction are due respectively, to Mokhtar-Kharroubi [16] and Jörgens-Vidav [9,19]. The first approach consists to express the solution as an inverse Laplace transform of the resolvent of the transport operator $A = T + K$ like that was made for particular models [14]. It is based on two arguments:

(i) $\exists\, m \in \mathbb{N}$ such that $[(\lambda - T)^{-1}K]^m$ is compact for $Re\lambda > \eta$
(ii) $|Im\lambda| \overset{\lim}{\longrightarrow} +\infty ||[(\lambda - T)^{-1}K]^m|| = 0$ uniformly on the set $\{\lambda \in \mathbb{C}/Re\lambda \geq \omega > \eta\}$.

where η is the type of the semigroup $(e^{tT}; t \geq 0)$. Thus if $(\imath)$ and $(\imath\imath)$ are satisfied, it is possible to derive the time asymptotic behavior of the solutions (for t sufficiently large) for regular initial data. The second approach, said of semigroup is based on the compactness of a remainder term of the Dyson–Phillips expansion $R_n(t)$. It has the advantage of not imposing any condition on the initial data. In the case of the vacuum boundary conditions, this technique was systematically used by several authors. Let us quote, for example, Jorgens [9], Vidav [19], Voigt [20,21], Greiner [7], Takac [18]. The compactness of such remainder asserts that the part of the spectrum of the operator $V(t)$ outside the spectral disc $\{\alpha \in \mathbb{C}/|\alpha| \leq e^{\omega t}\}$ consist at most of eigenvalues of finite algebraic multiplicities. In particular, for every $\upsilon > \omega, \sigma(T + K) \bigcap \{\lambda \in \mathbb{C} : Re\lambda > \upsilon\}$ consists, at most, of a finite number of eigenvalues $\{\lambda_1, \ldots\ldots, \lambda_n\}$. If P_i and D_i denotes ,respectively, the spectral projection and the nilpotent operator associated to the value λ_i, $1 \leq i \leq n$, then

$$V(t) = (I - P)V(t) + \sum_{i=1}^{n} e^{\lambda_i t} e^{D_i t} P_i$$

with $\|(I - P)V(t)\| = o(e^{(\lambda' - \epsilon)t})$ $(t \longrightarrow +\infty)$, where $P = \sum_{i=1}^{n} P_i$, $\lambda' = \min\{Re\lambda_i, 1 \leq i \leq n\}$ and $\epsilon > 0$.

Knowing that the boundary conditions in the transport phenomena can be of very complex nature what returns their mathematical formulation very controversy, even in the case where these conditions are given by a bounded boundary operator acting on suitably selected spaces of traces, a major problem appears, it consists to see if the associated advection transport operator generates or not a c_0-semigroup, even in the positive case, in general, it is almost impossible to establish its explicit formula to substitute it in the calculus. We notice that the study of transport equations with abstract boundary conditions has the advantage to cover those used in the kinetic theory of gases (vacuum boundary, specular reflections, diffuse reflections, periodic and mixed type boundary conditions).

Our study in this paper is organized as follows:

In the first section, we start by the use of the S. Bochner result's relating to the concentration of the Lebesgue measure, we will establish compactness results concerning some classes of regularized c_0-semigroups in the case of Banach spaces. They will be illustrated better in the case of the transport theory. In Sect. 2, the approaches discussed are based essentially on the concept of Haussdorff measure of noncompactness. First, we give some general results in Fredholm theory including the case of Riesz operators. After, they will be applied to establish a fine characterization of Schechter essential spectrum of closed densely defined operators in Banach spaces what makes it possible to generalize well known results in the literature.

1.1 Bochner measurability and compactness results in Banach spaces

In the remainder of this paper, if X and Y are two Banach spaces, then the Banach space of all bounded linear operators from X to Y will be denoted by $\mathcal{L}(X, Y)$.

We start this section by the following fundamental theorem.

Theorem 1.1 *Let X be a complex Banach space and let T an infinitesimal generator of a c_0-semigroup $(U(t); t \geq 0)$ on X and $K \in \mathcal{L}(X)$ such that $R(\lambda, T)K$ (resp. $KR(\lambda, T)$) is compact for $\lambda \in \rho(T)$. Assume that the mapping $t \in]0, +\infty[\longrightarrow U(t)K \in \mathcal{L}(X)$ (resp. $t \in]0, +\infty[\longrightarrow KU(t) \in \mathcal{L}(X)$) is Bochner measurable, then for every $t \in]0, +\infty[$, $U(t)K$ is compact on X (resp. $KU(t)$ is compact on X) and hence $t \in]0, +\infty[\longrightarrow U(t)K \in \mathcal{L}(X)$ (resp. $t \in]0, +\infty[\longrightarrow KU(t) \in \mathcal{L}(X)$) is uniformly continuous.*

Proof We denote by ω the type of the semigroup $(U(t), t \geq 0)$, the expression of the resolvent of the generator as a Laplace transform of the semigroup gives that

$$R(\lambda, T) = \int_0^{+\infty} e^{-\lambda t} U(t) dt \quad (Re\lambda > \omega). \tag{1.2}$$

By composing by K at the right of each one of the two terms in (1.2), it follows that

$$R(\lambda, T)K = \int_0^{+\infty} e^{-\lambda t} U(t) K dt \quad (Re\lambda > \omega). \tag{1.3}$$

From our hypothesis, $\int_0^{+\infty} e^{-\lambda t} U(t) K dt$ is compact, this gives that the operator $U(t_0) \int_0^{+\infty} e^{-\lambda t} U(t) K dt = e^{\lambda t_0} \int_{t_0}^{+\infty} e^{-\lambda t} U(t) K dt$ $(t_0 \geq 0)$ is compact on X and consequently $\int_{t_0}^{+\infty} e^{-\lambda t} U(t) K dt$ $(t_0 \geq 0)$ is compact (since $e^{\lambda t_0}$ is a scalar) , this implies that $\int_0^{t_0} e^{-\lambda t} U(t) K dt = \int_0^{+\infty} e^{-\lambda t} U(t) K dt - \int_{t_0}^{+\infty} e^{-\lambda t} U(t) K dt$ is compact, which allows us to conclude the compactness of the operator $\int_{t_1}^{t_2} e^{-\lambda t} U(t) K dt$ $(0 < t_1 < t_2)$ on X. Now the use of ([22], Theorem 2, page 134) shows that for almost every $t \in]0, +\infty[$, we obtain that $\frac{1}{2\epsilon} \int_{t-\epsilon}^{t+\epsilon} e^{-\lambda t} U(t) K dt \longrightarrow e^{-\lambda t} U(t) K$ (in $\mathcal{L}(X)$), hence, we get the compactness of the operator $U(t)K$ almost every where $t \in]0, +\infty[$ and by the same arguments, we study the case of the operator $KU(t)$. On the other hand, Let S be the set of the Lebesgue points, in other words the set of points $t \in]0, +\infty[$ for which $U(t)K$ (resp. $KU(t)$) is not compact on X, S is empty, indeed, let $t_0 \in S$, then there exists $t < t_0$ such that $U(t)K$ (resp. $KU(t)$) is compact, this implies that $U(t_0)K = U(t_0 - t)U(t)K$ (resp. $KU(t_0) = KU(t_0 - t)U(t)$) is compact and contradicts the fact that $t_0 \in S$. Finally, the uniform continuity of $U(t)K$ (resp. $KU(t)$) follows directly from Lemma 2.1 in [11].

Theorem 1.2 *Under the hypotheses of Theorem 1.1, we have $R_n(t)$ is compact for every $n \geq 1$ and $t \geq 0$.*

Proof If $t = 0$, then $R_1(0) = 0_{\mathcal{L}(X)}$ is trivially compact.
Now, let $t > 0$ and $n = 1$, then $R_1(t) = \int_0^t U(s)KU(t-s)ds$. Theorem 1.1 asserts that $U(t)K$ is compact for every $s \in]0, t]$ and the use of ([15], Theorem 2.2, page 12) completes the proof for the case $R_1(t)$. Now, the compactness of $R_n(t)$, $n > 1$ follows from the formula $R_n(t) = \int_0^t U(s)KR_{n-1}(t-s)ds$ and ([15], Theorem 2.2, page 12).

Before stating of a crucial result which asserts the compactness of the remainder term $R_n(t)(t \geq 0)$ in the general framework of Banach lattices and positive semigroups, we will give some recalls.

Definition 1.1 A Banach lattice is said to have order continuous norm if any increasing net which has a supremum is convergent.

It is easy to observe that $L_p(\mu)$, $(1 \leq p < \infty)$ spaces have order continuous norm.

In the case of positivity, to establish some compactness results, we will use a comparison results, we only need a particular case of them.

Proposition 1.1 (see [3]) *Let (Ω, Σ, μ) be a positive measure space and $X = L_p(\Omega, \Sigma, \mu)$ with $p \in [1, +\infty[$. Let A and B in $\mathcal{L}(X)$ such that $0 \leq B \leq A$.*

(i) *If $p = 1$ and A is weakly compact, then B is weakly compact.*
(ii) *(ii) If $1 < p < +\infty$ and A is compact, then B is compact on X.*

Theorem 1.3 *Let $X = L_p(\mu)(1 \leq p < \infty)$ and let $(U(t); t \geq 0)$ a positive c_0-semigroup generated by T and $B \geq 0$ (in the lattice sence). Assume that the map $t \in]0, +\infty[\longrightarrow U_n(t) \in \mathcal{L}(X)$ is Bochner measurable. If $[(\lambda - T)^{-1}B]^n(\lambda - T)^{-1}$ is compact $(1 < p < +\infty)$ (resp. weakly compact on $L_1(\mu)$), then $R_{n+1}(t)$ is compact for $t \in [0, +\infty[$ on $L_p(\mu)(1 < p < +\infty)$ (resp. weakly compact on $L_1(\mu)$).*

Proof Let $1 < p < +\infty$. We have

$$\int_0^{+\infty} e^{-\lambda t} U_n(t) dt = [(\lambda - T)^{-1}B]^n (\lambda - T)^{-1}.$$

Let $s > 0$. Then

$$[(\lambda - T)^{-1}B]^n(\lambda - T)^{-1} \geq \int_{s-\epsilon}^{s+\epsilon} e^{-\lambda t}U_n(t)dt$$

By the use of Proposition 1.1, we obtain the compactness of the operator $\int_{s-\epsilon}^{s+\epsilon} e^{-\lambda t}U_n(t)dt$ on $L_p(\mu)(1 < p < \infty)$. On the other hand ([22], Theorem 2, p. 134) shows that

$$\frac{1}{2\epsilon}\int_{s-\epsilon}^{s+\epsilon} e^{-\lambda t}U_n(t)dt \longrightarrow e^{-\lambda s}U_n(s) \text{ in } \mathcal{L}(X)(\epsilon \longrightarrow 0).$$

Thus, $U_n(s)$ is compact for almost every where $s \in]0, +\infty[$. Moreover, we have $U_{n+1}(t) = \int_0^t U(s)KU_n(t-s)ds$, then $U_{n+1}(t)$ is compact for $t \geq 0$ ([15], Theorem 2.2, page 12). Now, the compactness of $R_{n+1}(t)$ follows directly from ([15], Theorem 2.6, page 16). If $p = 1$, by the same arguments above, we obtain the weak compactness of $U_n(s)$ for almost every where $s \in]0, +\infty[$ which implies the weak compactness of $U_{n+1}(s)$ for $s \geq 0$ ([15], Theorem 2.3, page 12). Finally, the weak compactness of $R_{n+1}(t)$ follows from ([15], Remark 2.1, page 16).

1.2 Generation results and compactness in transport theory

We are interested by generation results of c_0-semigroups and compactness of the following integro-differential operator on $L_p(D \times V; dxd\mu)$.

$$\begin{aligned} A_H\psi(x,v) &= -v.\nabla_x\psi(x,v) - \sigma(v)\psi(x,v) + \int_V \kappa(x,v,v')\psi(x,v')d\mu(v') \\ &= T_H\psi + K\psi, \end{aligned}$$

where $(x, v) \in D \times V$. D is a smooth open subset of $\mathbb{R}^n$, $\mu(.)$ is a positive Radon measure on $\mathbb{R}^n$ such that $\mu(\{0\}) = 0$ and V (admissible velocity space) denotes the support of μ. This operator models the transport of particles (neutrons, photons, molecules of gas, ...) in the domain D. The function $\psi(x, v)$ is the number (or probability) density of gas particles having the position x and the velocity v. The functions $\sigma(.)$ and $\kappa(., ., .)$ are called, respectively, the collision frequency and the scattering kernel.

Here, the boundary conditions which represent the interaction between the particles and ambient medium are given by a boundary bounded operator H satisfying

$$\psi_- = H(\psi_+)$$

where ψ_- (resp. ψ_+) is the restriction of ψ to Γ_- (resp. Γ_+) with Γ_- (resp. Γ_+) is the incoming (resp. outcoming) part of the boundary phase space (for more details, see ([12])).

1.3 Notations and preliminaries

Let $(x, v) \in \overline{D} \times V$. We set $t^{\pm}(x, v)$ the positive real

$$t^{\pm}(x,v) = \sup\{s > 0; x \pm sv \in D, \forall\, 0 < s < t\},$$

We denote by $\Gamma_{\pm}$ the set

$$\Gamma_{\pm} = \{(x,v) \in \partial D \times V; \pm v.n_x \geq 0\},$$

where n_x is the outer unit normal vector at $x \in \partial D$.

Let $1 \leq p < \infty$, we introduce the functional spaces

$$W_p = \{\psi \in X_p \text{ such that } v.\nabla_x \psi \in X_p\},$$

where

$$X_p := L_p(D \times V; dx d\mu(v)),$$

The spaces of traces are $L_p^{\pm} := L_p(\Gamma_{\pm}; |v.n_x| d\gamma(x) d\mu(v))$. Here $d\gamma(.)$ being the Lebesgue measure on ∂D.

Recall that for every $\psi \in W_p$, we can define the traces $\psi_{\pm}$ on $\Gamma_{\pm}$, unfortunately, theses traces do not belong to $L_p^{\pm}$. The traces lie only in $L_{p,loc}^{\pm}$, or precisely in a certain weighted L_p space (see [6], for details).

Define

$$\widetilde{W_p} = \{\psi \in W_p; \psi_{\pm} \in L_p^{\pm}\}$$

In this case $H \in \mathcal{L}(L_p^+, L_p^-)(1 \leq p < \infty)$ and the associated advection operator T_H is given as follows:

$$\begin{cases} T_H : D(T_H) \subseteq X_p \longrightarrow X_p, \\ \varphi \longrightarrow T_H \varphi = -v.\nabla_x \varphi(x, v) - \sigma(v)\varphi(x, v). \end{cases}$$

with domain

$$D(T_H) = \{\psi \in \widetilde{W_p} \text{ such that } \psi_- = H(\psi_+)\}$$

where the collision frequency $\sigma(.) \in L^{\infty}(V)$. Let $\lambda \in \mathbb{C}$, consider the boundary value problem

$$\begin{cases} \lambda\psi(x, v) + v.\nabla_x \psi(x, v) + \sigma(v)\psi(x, v) = \varphi(x, v), \\ \psi_- = H(\psi_+). \end{cases} \tag{1.4}$$

where $\varphi \in X_p$ and the unknown ψ must belong to $D(T_H)$. Let

$$\lambda^{\star} := \mu - ess \inf_{v \in V} \sigma(v)$$

For $Re\lambda + \lambda^{\star} > 0$, the Eq. (1.4) can be solved formally by

$$\psi(x, v) = \psi(x - t^-(x, v)v, v)e^{-(\lambda+\sigma(v))t^-(x,v)} + \int_0^{t^-(x,v)} e^{-(\lambda+\sigma(v))s} \varphi(x - sv, v) ds \tag{1.5}$$

Moreover, if $(x, v) \in \Gamma_+$, the Eq. (1.4) becomes

$$\psi_+(x, v) = \psi_- e^{-(\lambda+\sigma(v))\tau(x,v)} + \int_0^{\tau(x,v)} e^{-(\lambda+\sigma(v))s} \varphi(x - sv, v) ds \tag{1.6}$$

where $\tau(x, v) = t^+(x, v) + t^-(x, v)$. On the other hand, for every $(x, v) \in \overline{\Omega} \times V$, we have $(x - t^-(x, v)v, v) \in \Gamma_-$ (for more details on the time numbers t^+, t^- and τ, see [20]).

For the abstract formulation of (1.5) and (1.6), we define the following operators depending on the parameter λ.

$$M_\lambda : L_p^- \longrightarrow L_p^+, u \longrightarrow M_\lambda u := ue^{-(\lambda+\sigma(v))\tau(x,v)};$$

$$B_\lambda : L_p^- \longrightarrow X_p, u \longrightarrow B_\lambda u := ue^{-(\lambda+\sigma(v))t^-(x,v)};$$

$$\begin{cases} G_\lambda : X_p \longrightarrow L_p^+, \\ \varphi \longrightarrow \int_0^{\tau(x,v)} \quad e^{-(\lambda+\sigma(v))s}\varphi(x-sv, v)ds; \end{cases}$$

and

$$\begin{cases} C_\lambda : X_p \longrightarrow X_p, \\ \varphi \longrightarrow \int\limits_0^{\tau(x,v)} \quad e^{-(\lambda+\sigma(v))s}\varphi(x-sv, v)ds. \end{cases}$$

Straightforward calculations using Hölder's inequality show that all these operators are bounded on their respective spaces. More precisely, we have for $Re\lambda > -\lambda^\star$,

$$\|M_\lambda\| \leq 1, \qquad \|B_\lambda\| \leq (p(Re\lambda + \lambda^\star))^{-\frac{1}{p}},$$
$$\|G_\lambda\| \leq (q(Re\lambda + \lambda^\star))^{-\frac{1}{q}}, \quad \|C_\lambda\| \leq \frac{1}{Re\lambda + \lambda^\star} \, (\frac{1}{p} + \frac{1}{q} = 1).$$

For a boundary operator $H \in \mathcal{L}(L_p^+, L_p^-)$, we denote by Ξ_H the set $\{\lambda \in \mathbb{R}/r_\sigma(M_\lambda H) < 1\}$.

The following generation theorem for c_0-semigroups shows how comparison results for the positive boundary operators can induce a generation results for c_0-semigroups.

Theorem 1.4 *Let $H_1, H_2 \in \mathcal{L}(L_p^+, L_p^-)$ with $0 \leq H_2 \leq H_1$. Assume that $[\omega, +\infty[\subseteq \Xi_{H_1}$ and T_{H_1} generates a c_0-semigroup $(U_{H_1}(t); t \geq 0)$ of type ω. Then T_{H_2} generates a c_0-semigroup with the same type ω.*

The application of the preceding theorem is illustrated better in the case of slab geometry with integral boundary conditions. Indeed, let $D =]-a, a[$, $V = [-1, 1]$ and $X_1 = L_1(D \times V, dxd\xi)$. Without loss of generality, we take $\sigma = 0$, the free advection transport operator is given by

$$T_H\psi(x, \xi) = -\xi\frac{\partial\psi}{\partial x}(x, \xi); x \in]-a, a[, \ \xi \in [-1, 1].$$

The boundary conditions here are given as follows:

$$\xi\psi(-a, \xi) = \gamma_- \int\limits_{-1}^{0} |\xi'|\psi(-a, \xi')d\xi', \qquad \xi \in [0, 1] \tag{1.7}$$

$$|\xi|\psi(-a, \xi) = \gamma_+ \int\limits_{0}^{1} \xi'\psi(-a, \xi')d\xi', \qquad cm\xi \in [-1, 0] \tag{1.8}$$

Here, $\partial D = \{-a, a\}$ and the outer unit normal vector is $n_x = (\pm 1, 0)$. Then

$$D_- := \{-a\} \times [0, 1] \bigcup \{a\} \times [-1, 0]$$
$$D_+ := \{-a\} \times [-1, 0] \bigcup \{a\} \times [0, 1]$$

We also define

$$\mathcal{Z}_+(h) = \begin{cases} (-1, 0) \text{ for } h = -a, \\ (0, 1) \text{ for } h = a \end{cases}$$

and

$$\mathcal{Z}_-(h) = \begin{cases} (0, 1) \text{ for } h = -a, \\ (-1, 0) \text{ for } h = a \end{cases}$$

With theses notations, these boundary conditions can be modeled by the following operator:

$$H\psi(h, \xi) = \frac{\alpha(h)}{|\xi|} \int_{\mathcal{Z}_+(h)} |\xi'| \psi(h, \xi') d\xi', \quad \psi \in L_1(D_+, |\xi| d\xi)$$

If $(h, \xi) \in D_-$, we have $\alpha(h) = \gamma_\pm$ for $h = \pm a$.

Proposition 1.2 *For all γ_+, γ_-, we have T_H generates a c_0-semigroup on $X_1 = L_1(D \times V, dxd\xi)$.*

Proof See ([2], Theorem 10.48, page 359).

1.4 Compactness results in transport theory

In the remainder of this paper, if X is a given Banach space, the ideal of compact operators on X will be denoted by $\mathcal{K}(X)$.

The collision operator K given as a perturbation of the advection transport operator T_H is defined on X_p by

$$\begin{aligned} K : X_p &\longrightarrow X_p \\ \psi &\longrightarrow \int_V \kappa(x, v, v')\psi(x, v')d\mu(v'), \end{aligned}$$

Note that the operator K is local in x so it can be viewed as a mapping

$$K(.) : x \in D \longrightarrow K(x) \in \mathcal{L}(L_p(V)).$$

We assume that $K(.)$ is strongly measurable.

$$x \in D \longrightarrow K(x)\varphi \in L_p(V) \text{ is measurable for any } \varphi \in L_p(V)$$

and bounded

$$ess \sup_{x \in D} \|K(x)\|_{\mathcal{L}(L_p(V))} < \infty.$$

It follows that K defines a bounded operator on the space $L_p(D \times V)$ according to the formula

$$\varphi \in L_p(D \times V)$$

$(L_p(D \times V) \simeq L_p(D; L_p(V)))$ and

$$\|K(x)\|_{\mathcal{L}(L_p(D\times V))} \leq ess \sup_{x\in D} \|K(x)\|_{\mathcal{L}(L_p(V))}$$

The final assumption on K is

$$K(x) \in \mathcal{K}(L_p(V)) \text{ almost every where,}$$

Now, we give the concept of regular collision operators introduced by M. Mokhtar-Kharroubi [15].

Definition 1.2 A collision operator

$$K(.) : x \in D \longrightarrow K(x) \in \mathcal{L}(L_p(V)).$$

is said to be regular if $K(x) \in \mathcal{K}(L_p(V))$ almost everywhere on D and

$$K(.) : x \in D \longrightarrow \mathcal{L}(L_p(V))$$

is Bochner measurable.

In the sequel, we denote by $\mathcal{K}_r(L_p(D \times V))$ the set of all regular operators on $L_p(D \times V; dxd\mu)$. The interest of this class lies in the following lemma

Lemma 1.1 *A regular collision operator K can be approximated, in the uniform topology, by a sequence $\{K_n\}$ of collision operators with kernels of the form*

$$\sum_{i\in I} f_i(x)g_i(\xi)h_i(\xi'),$$

where $f_i \in L^\infty(D)$, $g_i \in L_p(V)$ and $h_i \in L_q(V)(\frac{1}{p} + \frac{1}{q} = 1)$ (I is finite).

From now, we will assume that the measure μ satisfies

$$(H') \quad \mu\{v \in \mathbb{R}^n, v.e = 0\} = 0, \quad e \in S^{n-1}.$$

where S^{n-1} denotes the unit sphere on $\mathbb{R}^n$.

Our first result in this section is given by the following theorem

Theorem 1.5 *Let $(1 < p < +\infty)$. Assume that (H') is satisfied, let D be a convex bounded subset of $\mathbb{R}^n$ and $H \in \mathcal{L}(L_p^+, L_p^-)(H \geq 0)$ such that T_H generates a c_0-semigroup $(U_H(t); t \geq 0)$ and $K \in \mathcal{K}_r(L_p(D\times V))(K \geq 0)$. If the map $t \in]0, +\infty[\longrightarrow U_n(t) \in \mathcal{L}(X)$ is Bochner measurable, then $R_{n+1}(t)$ is compact on X_p for all $t \in [0, +\infty[$.*

Proof If $1 < p < +\infty$, we have $(\lambda - T_H)^{-1}K$ is compact on X_p (see [13], Theorem 3.1) and the result follows immediately from Theorem 1.3.

In what follows, we will establish some compactness results for the third remainder of the Dyson–Phillips expansion in the abstract framework $(H \geq 0)$. To do it, we need some notations which we will introduce here

Let $[\mathcal{U}(\lambda)]_H = I - M_\lambda H$. Along this section, we assume that $[\mathcal{U}(\lambda)]_H^{-1}$ exists for $\{\lambda \in \mathbb{C}/Re\lambda > -\lambda^\star\}$

We denote by $(U_0(t); t \geq 0)$ the c_0-neutronique semigroup generated by the advection transport operator associated to $(H = 0)$. Recall that its explicit formula is given by

$$U_0(t)\psi(x, v) = \begin{cases} e^{-t\sigma(v)}\psi(x - tv, v) & \text{if } t < s(x, v); , \\ 0 & \text{otherwise} \end{cases}$$

where $s(x, v) = \inf\{s > 0; x - sv \notin D\}$.

Let $H \in \mathcal{L}(L_p^+, L_p^-)$. Assume that the inverse Laplace transform of the family $B_\lambda H[\mathcal{U}(\lambda)]_H^{-1} G_\lambda$ exists and will be denoted by $\Im_H^{-1}(t)(t \geq 0)$, we have

Proposition 1.3 *Let $H \in \mathcal{L}(L_p^+, L_p^-)$, then T_H generates a c_0-semigroup $(U_H(t); t \geq 0)$ if and only if $\Im_H^{-1}(t)(t \geq 0)$ satisfies the following conditions:*

(i) $\Im_H^{-1}(0) = 0_{\mathcal{L}(X_p)}$;
(ii) $\Im_H^{-1}(t_1 + t_2) = \Im_H^{-1}(t_1)\Im_H^{-1}(t_2) + \Im_H^{-1}(t_1)U_0(t_2) + U_0(t_1)\Im_H^{-1}(t_2), \quad t_1, t_2 \in [0, +\infty[$;
(iii) *For all $\varphi \in X_p$, we have $t \overset{\lim}{\longrightarrow} 0^+ \Im_H^{-1}(t)\varphi = 0$.*

Proof The result follows immediately from the properties satisfied by c_0-semigroups (observe that in this case, $(\Im_H^{-1}(t))_{t\geq 0}$ is nothing else but the family of bounded operators $(U_H(t) - U_0(t))_{t\geq 0)}$.

For $H \in \mathcal{L}(L_p^+, L_p^-)(H \geq 0)$, we denote by $U_{(2,i)}^H(t))_{(1\leq i\leq 8)}$ the family of the following bounded operators.

$$U_{(2,1)}^H(t) = \int_0^t \Im_H^{-1}(s)K\left[\int_0^{t-s} \Im_H^{-1}(h)K\Im_H^{-1}(t-s-h)dh\right]ds;$$

$$U_{(2,2)}^H(t) = \int_0^t \Im_H^{-1}(s)K\left[\int_0^{t-s} U_0(h)KU_0(t-s-h)dh\right]ds;$$

$$U_{(2,3)}^H(t) = \int_0^t \Im_H^{-1}(s)K\left[\int_0^{t-s} \Im_H^{-1}(h)KU_0(t-s-h)dh\right]ds;$$

$$U_{(2,4)}^H(t) = \int_0^t U_0(s)K\left[\int_0^{t-s} \Im_H^{-1}(h)K\Im_H^{-1}(t-s-h)dh\right]ds;$$

$$U_{(2,5)}^H(t) = \int_0^t U_0(s)K\left[\int_0^{t-s} U_0(h)K\Im_H^{-1}(t-s-h)dh\right]ds;$$

$$U_{(2,6)}^H(t) = \int_0^t U_0(s)K\left[\int_0^{t-s} U_0(h)KU_0(t-s-h)dh\right]ds;$$

$$U_{2,7}^H(t) = \int_0^t U_0(s)K\left[\int_0^{t-s} \Im_H^{-1}KU_0(t-s-h)dh\right]ds;$$

$$U_{(2,8)}^H(t) = \int_0^t U_0(s)K\left[\int_0^{t-s} U_0(h)KU_0(t-s-h)dh\right]ds.$$

Let us give now the result which asserts the compactness of the third remainder of the Dyson–Phillips expansion. For this, we assume that the measure μ satisfies that for every bounded set $D \subset \mathbb{R}^n$

$$(H'') \int_D e^{-ix.\omega} d\mu(x) \longrightarrow 0 \text{ as } |\omega| \longrightarrow \infty.$$

Proposition 1.4 *Let D be a bounded convex open set in $\mathbb{R}^n$ and let $H \in \mathcal{L}(L_p^+, L_p^-)(H \geq 0)$ such that T_H generates a c_0-semigroup $(U_H(t); t \geq 0)$ on X_p $(1 < p < \infty)$. Assume that the collision operator K is regular and positif. If each one of the maps $t \in]0, +\infty[\longrightarrow U^H_{(2,i)}(t) \in \mathcal{L}(X)(1 \leq i \leq 7)$ is Bochner measurable then $R_3^H(t)$ is compact for all $t \in [0, +\infty[$.*

Proof First of all, we have $U_2^H(t) = \sum_{i=1}^8 U^H_{(2,i)}(t)$. Our assumption on each map $t \in]0, +\infty[\longrightarrow U^H_{(2,i)}(t) \in \mathcal{L}(X)(1 \leq i \leq 7)$ shows that $t \in]0, +\infty[\longrightarrow \sum_{i=1}^7 U^H_{(2,i)}(t)$ is Bochner measurable. On the other hand, $t \in]0, +\infty[\longrightarrow R_2^0(t) \in \mathcal{L}(X)$ is uniformly continuous (see [15], Theorem 4.7) which implies the uniform continuity of the map $t \in]0, +\infty[\longrightarrow U^H_{(2,8)}(t) \in \mathcal{L}(X)$ (see [15], Theorem 2.7) and consequently its Bochner measurability. Moreover, the mapping $t \in]0, +\infty[\longrightarrow U_2^H(t) \in \mathcal{L}(X)$ is Bochner measurable (as the sum of finite number of Bochner measurable maps), the use of Theorem 1.5 gives the desired result.

2 Haussdorff measure of noncompactness and Fredholm theory

Let X be a Banach space and let $\mathcal{C}(X)$ the set of all closed densely defined operators on X. If $A \in \mathcal{C}(X)$, we write $N(A) \subseteq X$ and $R(A) \subseteq X$ for the null space and range of A. We set $\alpha(A) := \dim N(A)$, $\beta(A) := \text{codim } R(A)$. Let $A \in \mathcal{C}(X)$ with a closed range. Then A is a Φ_+-operator ($A \in \Phi_+(X)$) if $\alpha(A) < \infty$, and A is a Φ_--operator ($A \in \Phi_-(X)$) if $\beta(A) < \infty$, $\Phi(X) = \Phi_+(X) \bigcap \Phi_-(X)$ is the class of Fredholm operators while $\Phi_\pm(X) = \Phi_+(X) \bigcup \Phi_-(X)$. For $A \in \Phi(X)$, the index of A is defined by $i(A) = \alpha(A) - \beta(A)$. Let $A \in \mathcal{C}(X)$. A complex number λ is in Φ_{+A}, Φ_{-A}, $\Phi_{\mp A}$, or Φ_A if $\lambda - A$ is in $\Phi_+(X)$, $\Phi_-(X)$, $\Phi_\mp(X)$ or $\Phi(X)$, respectively. In the sequel, $\Phi_0(X)$ will denote the set of Fredholm operators with indices 0 (for more details, we refer to [4,5]).

Let $A \in \mathcal{C}(X)$. A point $\lambda \in \sigma(A)$ is in the Wolf essential spectrum, $\sigma_e(A)$ if $\lambda \notin \Phi_A$; the Schechter (or Weyl) essential spectrum $\sigma_w(A)$, is $\mathbb{C} \backslash \varphi_A^0$ where $\varphi_A^0 := \{\lambda \in \mathbb{C}, \lambda - A \in \Phi_0(X)\}$.

We introduce here the notion of Haussdorff measure of noncompactness, a positive function which measure the degree of noncompactness of sets.

Let X be a complex Banach space and let $\mathcal{P}(X)$ the set of all sets of X, we denote by $B(x, r)$ and $\overline{B}(x, r)$ the open and closed ball of centers x and radius $r > 0$. By means of the concept of Haussdorff measure of noncompactness, we will establish some results in Fredholm theory which represents one of the tools for the resolution of the equations $u - Tu = f$ (Fredholm alternative).

Definition 2.1 The Haussdorff measure of noncompactness $\chi(\Omega)$ of $\Omega \in \mathcal{P}(X)$ is defined as the infimum of the numbers $\epsilon > 0$ such that Ω has a finite ϵ-net in X.

Recall that a set $S \subset X$ is called an ϵ-net of Ω if $\Omega \subset S + \epsilon\overline{B}(0, 1) = \{s + \epsilon b : s \in S, b \in \overline{B}(0, 1)\}$

If $T \in \mathcal{L}(X)$, $\chi(T)$ is defined as being the number $\chi(T(\overline{B}(0, 1)))$, for more details on the notion of Haussdorff measure of noncompactness and its properties, we can refer to [1].

2.1 Preparatory results

According to the properties satisfied by the function χ, we can establish the following for bounded operators on X.

1) If $T \in \mathcal{L}(X)$, then χ is a seminorm on $\mathcal{L}(X)$, moreover, $\chi(T) \leq ||T||$;
2) If $S \in \mathcal{L}(X)$, we have $\chi(ST) \leq \chi(S)\chi(T)$ (monotonicity);
3) If $K \in \mathcal{K}(X)$, then $\chi(T+K) = \chi(T)$;
4) If T^* denotes the dual of the operator T in $\mathcal{L}(X^*)$, then

$$\frac{1}{2}\chi(T^*) \leq \chi(T) \leq 2\chi(T^*).$$

Our first result in this direction is given by the following theorem:

Theorem 2.1 *Let X be a Banach space, $A \in \mathcal{L}(X)$, we assume that there exist two complex polynomials P and Q such that $P(\lambda_0) \neq 0$, Q divides $P - P(\lambda_0)$ and $\chi(P(A)) < |P(\lambda_0)|$, then $Q(A) \in \Phi(X)$. Moreover if $\chi(P(tA)) < |P(\lambda_0)|\ \forall t \in [0, 1]$ and $Q(0) \neq 0$, then $Q(A)$ is of index 0.*

Proof Without loss of generality, we can take $P(\lambda_0) = 1$.

a) Let us start by showing that $\alpha(Q(A)) < \infty$. To do it, it suffices to establish that the set $N(Q(A)) \bigcap \overline{B}(0, 1)$ is compact. More precisely, we will show that if M is a compact set in X then $B = \{x \in \overline{B}(0, 1) : Q(A)(x) \in M\}$ is or empty or it forms a compact set in X. Indeed, let us assume that B is not empty. According to the hypothesis Q divides $P - 1$, there exists H a complex polynomial such that $P = HQ + 1$. Consider $z \in M$ and $x \in \overline{B}(0, 1)$ with $Q(A)(x) = z$, then, we get, $H(A)Q(A)(x) + x = H(A)(z) + x$, which imply $x = P(A)x - H(A)(z)$. Since the range of a compact set by a bounded linear operator is compact, it follows that:

$$\widetilde{A} = \{-H(A)(z) : z \in M\}.$$

is a compact as well. Obviously, $B \subseteq P(A)B + \widetilde{A}$. Then by using the monotonicity property, it comes

$$\chi(B) \leq \chi(P(A)B + \widetilde{A}) \leq \chi(P(A)B) \leq \chi(P(A))\chi(B)$$

By assumption we have $\chi(P(A)) < 1$, thus $\chi(B) = 0$ which shows that B is a compact set in X. To establish the result concerning the set $N(Q(A)) \bigcap \overline{B}(0, 1)$, it suffices to take $M = \{0\}$.

To complete the proof, first we will establish the closedness of the range $R(Q(A))$ in X. Since $N(Q(A))$ is a finite dimensional space, then there exists a closed subspace Y of X such that $X = N(Q(A)) \bigoplus Y$. Now we will establish an inequality of the form $r||Q(A)(x)|| \geq ||x||$ for all $x \in Y$ with $r > 0$. Let assume that the converse is satisfied, then for all $n \in \mathbb{N}$, there exists $x_n \in Y$ of norm 1 satisfying $||Q(A)(x_n)|| \leq \frac{1}{n}$, it follows that $Q(A)(x_n) \longrightarrow 0 (n \longrightarrow +\infty)$. Afterwards, by taking $M = \{Q(A)(x_n) : n \in \mathbb{N}\} \bigcup \{0\}$ and according to the first part, it follows that the sequence $\{x_n\}_n$ admits a subsequence $\{x_{nk}\}_n$ which converges to $x_0 \in Y$. In addition, it is easy to see that $||x_0|| = 1$ and $Q(A)(x_0) = 0$. This is a contradiction, therefore there exists $r > 0$ such that $r||Q(A)(x)|| \geq ||x||$ which establish the closeness of the range $R(Q(A))$.

b) Since $\chi(P(A)) < 1$, then $n \overset{\lim}{\longrightarrow} +\infty \chi[(P(A))^n] = 0$ which implies that there exists $k \in \mathbb{N}$ such that $\chi[(P(A))^k] < \frac{1}{2}$. Using the property (4), one has $\chi[((P(A))^k)^*] = \chi(P^k(A^*)) \leq 2\chi(P(A))^k < 1$ where $((P(A))^k)^*$ stands for the dual of the operator $(P(A))^k$. It is clear that if Q divides $P-1$, then Q divides also $P^k - 1$ and consequently by proceeding as in the proof of the first part of the proof of the theorem for the polynomials P^k,Q and the operator A^* , we get $\alpha((Q(A))^*) = \alpha(Q(A^*)) = \beta(Q(A)) < \infty$ and consequently $Q(A) \in \Phi_-(X) \bigcap \Phi_+(X) = \Phi(X)$.

Now if $Q(0) \neq 0$, we establish that $Q(tA)$ is a Fredholm operator for every $t \in [0, 1]$, the stability of the index by small perturbations and the compactness of [0, 1] imply that $i(Q(tA)) = i(Q(0A)) = i(Q(A)) = 0$.

Corollary 2.1 *Let X be a Banach space and $A \in \mathcal{L}(X)$. If $\chi(A^m) < 1$ for some $m > 0$ then $T = I - A$ is a Fredholm operator with index 0.*

Proof $I - A \in \Phi_0(X)$ follows immediately from Theorem 2.1 by taking $Q(z) = 1 - z$ and $P(z) = z^n$ and $\lambda_0 = 1$.

2.2 Riesz operators and essential spectrum

Let us start this section by giving the definition of Riesz operators.

Definition 2.2 *Let X be Banach space and let $R \in \mathcal{L}(X)$, we say that R is a Riesz operator if $\forall \lambda \in \mathbb{C}^*$, $\lambda I - A \in \Phi(X)$.*

The class of Riesz operators $\mathcal{R}(X)$ has been introduced by Ruston [17] who defined such operators by considering as axioms some of the spectral properties of compact operators. Successively it has been studied and investigated by several authors (see, for example [10]). In this section, by using the concept of Haussdorff measure of noncompactness, we will show that powers of these operators have sufficiently small measure. Indeed, we have

Proposition 2.1 *Let X be a Banach space and let $R \in \mathcal{R}(X)$, then for all $1 \geq \epsilon > 0$, there exists $n(\epsilon) \geq 1$ such that $\chi(R^{n(\epsilon)}) < \epsilon$.*

Proof Recall that $R \in \mathcal{R}(X)$ satisfies the Riesz–Schauder theory, this shows that the set of points $\lambda \in \sigma(R)$ which satisfy the inequality $|\lambda| > \frac{\epsilon}{2}$ consists of finite number $\{\lambda_1, \lambda_2, ..., \lambda_{m_\epsilon}\}$. Let P_i be projections of X on the subspaces $N(\lambda_i I - R)$ in the decomposition of X under the form $X = N(\lambda_i I - R) \bigoplus H(\lambda_i)(1 \leq i \leq m_\epsilon)$. We denote by V the operator $R - \sum_{i=1}^{m_\epsilon} R \circ P_i$. It is easy to see that $\sum_{i=1}^{m_\epsilon} R \circ P_i$ is a finite rank operator thus compact, moreover, we have $r_\sigma(V) = n \overset{\lim}{\longrightarrow} +\infty ||V^n||^{\frac{1}{n}} \leq \frac{\epsilon}{2}$, which shows the existence of an integer n_ϵ such that $||V^{n_\epsilon}|| < \epsilon$. On the other hand, we can write $R^{n_\epsilon} = V^{n_\epsilon} + F_{n_\epsilon}$ where F_{n_ϵ} is a compact operator, which implies that $\chi(R^{n_\epsilon}) = \chi(V^{n_\epsilon}) < \epsilon$.

Remark 2.1 As an immediate consequence of this result, we obtain the classical property of Riesz operators saying that if X is a Banach space and $A \in \mathcal{R}(X)$, then $i(\lambda I - A) = 0 \, \forall \lambda \in \mathbb{C}^*$ or alternatively equivalent to $\sigma_e\{0\}$.

Let X be a Banach space and A a closed densely defined operator on X. Recall that the Schechter essential spectrum of A can also be defined as follows:

$$\sigma_\omega(A) = \bigcap_{K \in \mathcal{K}(X)} \sigma(A + K)$$

Let us note by $\mathcal{F}(X)$ the greatest closed two-sided ideal included in $\mathcal{R}(X)$. This ideal is also known under the name of Fredholm perturbations (see [4]) and satisfies $\mathcal{K}(X) \subseteq \mathcal{F}(X) \subseteq \mathcal{R}(X)$. In [13], the authors gave a characterization of Weyl essential spectrum by using some classes in $\mathcal{F}(X)$.

Our objective here is to establish a rather general functional framework which enable us to unify many well known results in this direction.

Let A be a closed densely defined operator on X and let us define the following set:

$$\mathcal{M}_A(X) = \{M \in \mathcal{L}(X) : \forall \lambda \in \rho(A+M), \exists\, m(\lambda) > 0 : \; \chi[(\lambda - A - M)^{-1}M]^{m(\lambda)} < 1\}$$

Let us give now the following characterization of $\sigma_\omega(A)$ which extend that established in [13].

Theorem 2.2 *Let A be a closed densely defined operator on X, then we have:*

$$\sigma_\omega(A) = \bigcap_{M \in \mathcal{M}_A(X)} \sigma(A+M) \tag{2.1}$$

Proof Since $\chi(K) = 0 < 1$ for every compact operator K on X, we have $\mathcal{K}(X) \subseteq \mathcal{M}_A(X)$ and consequently $\bigcap_{M \in \mathcal{M}_A(X)} \sigma(A+M) \subseteq \sigma_\omega(A)$. Let us now show the opposite inclusion. Consider $\lambda \in \sigma_\omega(A)$ and assume that $\lambda \notin \bigcap_{M \in \mathcal{M}_A(X)} \sigma(A+M)$, this implies the existence of an operator $M_0 \in \mathcal{M}_A(X)$ such that $\lambda \in \rho(A+M_0)$. On the other hand, we can write

$$\lambda - A = (\lambda - A - M_0)[I + (\lambda - A - M_0)^{-1}M_0]$$

Following Corollary 2.1, we have $(I + (\lambda - A - M_0)^{-1}M_0) \in \Phi(X)$ and $i(I + (\lambda - A - M_0)^{-1}M_0) = 0$. Afterwards, we have $\lambda - A = (\lambda - A - M_0)[I + (\lambda - A - M_0)^{-1}M_0] \in \Phi(X)$ and $i(\lambda - A) = 0$ (see [5], Theorem IV. 2. 7, page 103) which implies that $\lambda \notin \sigma_\omega(A)$. This is a contradiction, hence $\sigma_\omega(A) \subseteq \bigcap_{M \in \mathcal{M}_A(X)} \sigma(A+M)$, which achieves the proof.

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