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Relatively Strictly Singular Perturbations, Essential Spectra, and Application to Transport Operators

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The stability of essential spectra of a closed, densely defined linear operator A on L_p-spaces, $1 \le p \le \infty$, when A is subjected to a perturbation by a bounded strictly singular operator was discussed in a previous paper by K. Latrach and A. Jeribi (1998, *J. Math. Anal. Appl.* **225,** 461–485). In the present paper we prove the invariance of the Gustafson–Weidmann, Wolf, Schechter, and Browder essential spectra of A under relatively strictly singular (not necessarily bounded) perturbations on these spaces. Further, a precise characterization of the Schechter essential spectrum is given. We show that these results are also valid on $C(\Xi)$ where Ξ is a compact Hausdorff space. The results are applied to the one-dimensional transport equations with anisotropic scattering and abstract boundary conditions.

1. INTRODUCTION AND PRELIMINARIES

Let X and Y be two Banach spaces and let $\mathscr{C}(X,Y)$ be the set of all closed densely defined linear operators from X into Y. We denote by $\mathscr{L}(X,Y)$ the space of all bounded linear operators from X into Y, and $\mathscr{K}(X,Y)$ designates the subspace of all compact operators from X into Y. If $A \in \mathscr{C}(X,Y)$, we write $N(A) \subseteq X$ and $R(A) \subseteq Y$ for the null space and range of A. We set $\alpha := \dim N(A)$, $\beta := \operatorname{codim} R(A)$. Let $A \in \mathscr{C}(X,Y)$ with a closed range. Then A is a Φ_+-operator ($A \in \Phi_+(X,Y)$) if $\alpha(A) < \infty$, and A is a Φ_--operator ($A \in \Phi_-(X,Y)$) if $\beta(A) < \infty$. $\Phi(X,Y) = \Phi_+(X,Y) \cap \Phi_-(X,Y)$ is the class of Fredholm operators while $\Phi_\pm(X,Y)$ denotes the set $\Phi_+(X,Y) \cup \Phi_-(X,Y)$. For $A \in \Phi(X,Y)$, the index of A is defined by $i(A) = \alpha(A) - \beta(A)$. If $X = Y$, then $\mathscr{L}(X,Y)$, $\mathscr{K}(X,Y)$, $\mathscr{C}(X,Y)$, $\Phi_+(X,Y)$, $\Phi_\pm(X,Y)$, and $\Phi(X,Y)$ are

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replaced, respectively, by $\mathscr{L}(X)$, $\mathscr{K}(X)$, $\mathscr{C}(X)$, $\Phi_+(X)$, $\Phi_\pm(X)$, and $\Phi(X)$. Let $A \in \mathscr{L}(X)$. The spectrum of A will be denoted by $\sigma(A)$. The resolvent set of A, $\rho(A)$, is the complement of $\sigma(A)$ in the complex plane. A complex number λ is in Φ_{+A}, Φ_{-A}, $\Phi_{\pm A}$, or Φ_A if $\lambda - A$ is in $\Phi_+(X)$, $\Phi_-(X)$, $\Phi_\pm(X)$, or $\Phi(X)$, respectively. In the next proposition we recall some well known properties of those sets (see, for example, [8, 9, 32]).

PROPOSITION 1.1. (i) *Φ_{+A}, Φ_{-A}, and Φ_A are open,*

(ii) *$i(\lambda - A)$ is constant on any component of Φ_A,*

(iii) *$\alpha(\lambda - A)$ and $\beta(\lambda - A)$ are constant on any component of Φ_A except on a discrete set of points at which they have larger values.*

It is well known that if A is a bounded self-adjoint operator on a Hilbert space, the essential spectrum $\sigma_e(A)$ of A is the set of all points of the spectrum of A that are not isolated eigenvalues of finite algebraic multiplicity (see, for example, [29, 39]). Irrespective of whether A is bounded or not on a Banach space X, there are several definitions of the essential spectrum. At least six of them have been mentioned in the literature (cf. [13, 15, 20, 30, 32, 39]). More precisely, let $A \in \mathscr{C}(X)$. A point $\lambda \in \sigma(A)$ is in the essential spectrum $\sigma_{e1}(A)$ (resp. $\sigma_{e2}(A)$) if $\lambda \notin \Phi_{+A}$ (resp. $\lambda \notin \Phi_{-A}$). $\sigma_{e1}(A)$ and $\sigma_{e2}(A)$ are called Gustafson–Weidmann essential spectra. The point λ is in the Kato essential spectrum, $\sigma_{e3}(A)$, if $\lambda \notin \Phi_{\pm A}$. The Wolf essential spectrum, $\sigma_{e4}(A)$, is $\{\lambda \in \sigma(A), \lambda \notin \Phi_A\}$; the Schechter essential spectrum, $\sigma_{e5}(A)$, is $\mathbb{C} \setminus \Phi_A^0$ where $\Phi_A^0 := \{\lambda \in \Phi_A, i(\lambda - A) = 0\}$; and the Browder essential spectrum, $\sigma_{e6}(A)$, is $\mathbb{C} \setminus \{\lambda \in \Phi_A^0$, such that all scalars near λ are in $\rho(A)\}$. Note that, in general, we have

$$\sigma_{e1}(A) \cap \sigma_{e2}(A) = \sigma_{e3}(A) \subseteq \sigma_{e4}(A) \subseteq \sigma_{e5}(A) \subseteq \sigma_{e6}(A).$$

But if X is a Hilbert space and A is self-adjoint, then all these sets coincide.

An operator $S \in \mathscr{L}(X, Y)$ is said to be strictly singular if for every infinite dimensional subspace M of X, the restriction of S to M is not a homeomorphism. Let $\mathscr{S}(X, Y)$ denote the set of strictly singular operators from X into Y. Note that $\mathscr{S}(X, Y)$ is a closed subspace of $\mathscr{L}(X, Y)$. In general, strictly singular operators are not compact (cf. [9, 10]) and, if $X = Y$, $\mathscr{S}(X)$ is a closed two-sided ideal of $\mathscr{L}(X)$ containing $\mathscr{K}(X)$. If X is a Hilbert space, then $\mathscr{K}(X) = \mathscr{S}(X)$. For basic properties of strictly singular operators we refer to [10, 19].

Essential spectra of closed, densely defined linear operators on L_p-spaces were investigated in [23]. In particular, their invariance under strictly singular perturbations was discussed. The purpose of this work is to continue this analysis. Indeed, we are interested in the behaviour of the essential spectra of these operators under perturbations belonging to a large class of operators (not necessarily bounded) containing, in particular, the ideal of strictly singular operators. To be more precise, let X be a Banach space and let $A \in \mathscr{C}(X)$. For $x \in \mathscr{D}(A)$ (the domain of A), the A-norm $\|\cdot\|_A$ is defined by $\|x\|_A = \|x\| + \|Ax\|$. It follows from the closedness of A that $\mathscr{D}(A)$ endowed with the A-norm is a Banach space, denoted by X_A. Clearly, $A \in \mathscr{L}(X_A, X)$. If S is a linear operator with $\mathscr{D}(A) \subseteq \mathscr{D}(S)$, then S is said to be A-defined. The restriction of S to $\mathscr{D}(A)$ will be denoted by $\hat{S}$. Moreover, if $\hat{S} \in \mathscr{L}(X_A, X)$, we say that S is A-bounded. One checks easily that if S is closed (or closable), then S is A-defined (apply the closed graph theorem [7, 10, 20]).

Definition 1.1 [19]. Let $A \in \mathscr{C}(X)$ and let S be an A-defined linear operator on X. S is called A-strictly singular (or relatively strictly singular with respect to A) if $\hat{S} \in \mathscr{S}(X_A, X)$. We denote by $A\mathscr{S}(X)$ the set of all A-strictly singular operators on X.

Let $L_p := L_p(\Omega, \Sigma, \mu)$, $1 \le p \le \infty$, where (Ω, Σ, μ) is a measure space. In Section 2 we prove that if $X = L_p(\mu)$, then $\Phi_+(X)$ and $\Phi(X)$ are invariant under perturbations in $A\mathscr{S}(X)$. This permits us to show that if $A \in \mathscr{C}(X)$, then $\sigma_{ei}(A) = \sigma_{ei}(A + S)$ for all $S \in A\mathscr{S}(X)$ and $i = 1, 4$, and 5. Also, if the complement of $\sigma_{e5}(A)$ is connected and neither $\rho(A)$ nor $\rho(A + S)$ is empty then $\sigma_{e6}(A) = \sigma_{e6}(A + S)$. The preceding result concerning $\sigma_{e5}(\cdot)$ together with Theorem 5.4 in [32, p. 180] leads to a new refinement of the definition of the Schechter essential spectrum (Theorem 2.2). Our results hold also true on $C(\Xi)$, where Ξ is a compact Hausdorff space.

The question of whether or not $\Phi_-(X)$ and $\Phi_\pm(X)$ are invariant under (bounded) strictly singular perturbations on L_p-spaces, for $p \in \{1\} \cup (2, \infty]$, is not considered in [23] and its solution still seems to be unknown. In the last part of Section 2 we give a positive answer to this question. This enables us to discuss the invariance of $\sigma_{e2}(\cdot)$ and $\sigma_{e3}(\cdot)$ (Proposition 2.1). Further, a practical criterion for the stability of essential spectra of perturbed linear operators is provided (Proposition 2.2). Using arguments due to Pelczynski [27] we observe that these results extend to $C(\Xi)$ where Ξ is a compact Hausdorff space.

In Section 3 we will apply the results obtained in Proposition 2.2 to describe the essential spectra $\sigma_{e2}(\cdot)$ and $\sigma_{e3}(\cdot)$ of the following integro-differential operator

$$\begin{aligned} A_H\psi(x,\xi) &= -\xi\frac{\partial\psi}{\partial x} - \sigma(\xi)\psi(x,\xi) + \int_{-1}^{1}\kappa(x,\xi,\xi')\psi(x,\xi')\,d\xi' \\ &= T_H\psi + K\psi \end{aligned}$$

$x \in [-a, a]$ for a parameter $0 < a < \infty$ and $\xi \in [-1, 1]$. It describes the one-speed neutron transport in a plane parallel domain with a width of $2a$ mean free paths or transfer of unpolarized light in a plane-parallel atmosphere of optical thickness $2a$. The function $\psi(x, \xi)$ represents the number density of gas particles having the position x and the direction cosine of propagation ξ. (The variable ξ may be thought of as the cosine of the angle between the velocity of particles and the x-direction.) $\sigma(\cdot)$ and $\kappa(\cdot, \cdot, \cdot)$ are nonnegative measurable functions called, respectively, the collision frequency and the scattering kernel. The boundary conditions are modeled by

$$\psi_{|\Gamma_-} = H\psi_{|\Gamma_+},$$

where Γ_- (resp. Γ_+) is the incoming (resp. outgoing) part of the phase space boundary, $\psi_{|\Gamma_-}$ (resp. $\psi_{|\Gamma_+}$) is the restriction of ψ to Γ_- (resp. Γ_+), and H is a bounded linear operator from a suitable function space on Γ_- to a similar one on Γ_+. There is a wealth of literature treating the transport equation with different boundary conditions (see, e.g., [2, 4, 11, 18, 25, 34]). The known boundary conditions (vacuum boundary conditions, specular reflections, periodic, diffuse reflections, generalized and mixed type boundary conditions (see the references listed in [21])) are specific examples of our general framework. Our analysis is based essentially on Propositions 2.2, 3.2, and the knowledge of $\sigma_{e2}(T_0)$ and $\sigma_{e3}(T_0)$ where T_0 (i.e., $H = 0$) denotes the streaming operator with vacuum boundary conditions (see, for example, [4, 18, 25]). We give sizable classes of boundary and collision operators for which $\sigma_{e2}(\cdot)$ and $\sigma_{e3}(\cdot)$ of the operators T_0 and A_H coincide. Our results add to those obtained in [23] and extend them to non-homogeneous regular collision operators.

2. MAIN RESULTS

Let (Ω, Σ, μ) be a measure space. By $L_p(\mu) = L_p(\Omega, \Sigma, \mu)$, $1 \leq p < \infty$, we denote the Banach space of equivalence classes of measurable functions on (Ω, Σ, μ) whose pth power is integrable (respectively, which are essentially bounded if $p = \infty$). Throughout this section Ξ designates a compact Hausdorff space while $C(\Xi)$ denotes the Banach space of all continuous scalar-valued functions on Ξ with the supremum norm.

We now generalize the following result given before in [23] to A-strictly singular operators.

THEOREM 2.1. *Let X be one of the spaces $L_p(\mu)$, $1 \leq p \leq \infty$, or $C(\Xi)$ and assume that $A \in \mathscr{C}(X)$. If $S \in A\mathscr{S}(X)$, then*

$$\sigma_{ei}(A) = \sigma_{ei}(A + S), \qquad i = 1, 4, 5.$$

Moreover, if $C\sigma_{e5}(A)$ [*the complement of* $\sigma_{e5}(A)$] *is connected and neither* $\rho(A)$ *nor* $\rho(A+S)$ *is empty, then*

$$\sigma_{e6}(A) = \sigma_{e6}(A+S).$$

In [23], it is shown (Theorem 3.2) that

$$\sigma_{e5}(A) = \bigcap_{S \in \mathcal{S}(L_p(\mu))} \sigma(A+S). \tag{2.1}$$

The next result shows that this result is not optimal and (2.1) may be expressed in terms of A-strictly singular perturbations. More precisely, we have:

THEOREM 2.2. *Assume that X is one of the spaces* $L_p(\mu)$, $1 \leq p \leq \infty$, *or* $C(\Xi)$. *If* $A \in \mathcal{C}(X)$, *then*

$$\sigma_{e5}(A) = \bigcap_{S \in A\mathcal{S}(L_p(\mu))} \sigma(A+S).$$

To prove Theorems 2.1 and 2.2 we will need the following lemma. It extends the items (i) and (iv) in [23, Proposition 3.5] to A-strictly singular perturbations.

LEMMA 2.1. *Let X be one of the spaces* $L_p(\mu)$, $1 \leq p \leq \infty$, *or* $C(\Xi)$ *and assume that* $A \in \mathcal{C}(X)$. *If* $S \in A\mathcal{S}(X)$, *then*

(i) *if* $A \in \Phi_+(X)$, *then* $A + S \in \Phi_+(X)$;

(ii) *if* $A \in \Phi(X)$, *then* $A + S \in \Phi(X)$ *and* $i(A+S) = i(A)$.

Remark 2.1. The first item is an easy consequence of Kato's perturbation theorem [19] and is valid on all Banach spaces.

Proof of Lemma 2.1. As already observed, $\mathcal{D}(A)$ endowed with the norm $\|\cdot\|_A$ is a Banach space denoted by X_A, and A and S as operators from X_A to X (denoted by $\hat{A}$ and $\hat{S}$, respectively) are in $\mathcal{L}(X_A, X)$. Furthermore, we have

$$\begin{cases} \alpha(\hat{A}) = \alpha(A), \beta(\hat{A}) = \beta(A), R(\hat{A}) = R(A), \\ \qquad \alpha(\hat{A} + \hat{S}) = \alpha(A+S), \\ \beta(\hat{A} + \hat{S}) = \beta(A+S) \text{ and } R(\hat{A} + \hat{S}) = R(A+S). \end{cases} \tag{2.2}$$

(i) Clearly, $\hat{A} \in \Phi_+(X_A, X)$ (use (2.2)) and $\hat{S} \in \mathcal{S}(X_A, X)$. Hence, by [19, Theorem 2], we have $\hat{A} + \hat{S} \in \Phi_+(X_A, X)$. Next, the use of (2.2) gives the result.

(ii) Since $A \in \Phi(X)$, it follows from (2.2) that $\hat{A} \in \Phi(X_A, X) \cap \mathscr{L}(X_A, X)$. Hence, by [32, Theorem 2.1, p. 110], there exist $B \in \mathscr{L}(X, X_A)$ and $K \in \mathscr{K}(X)$ such that

$$\hat{A}B = I - K, \qquad \text{on } X. \tag{2.3}$$

This leads to

$$(\hat{A} + \hat{S})B = I - K + \hat{S}B = I - U, \qquad \text{on } X. \tag{2.4}$$

Obviously, (2.3) implies that $\hat{A}B \in \Phi(X)$ and $i(\hat{A}B) = 0$. This together with [32, Theorem 2.6, p. 170] (because $\hat{A} \in \mathscr{L}(X_A, X) \cap \Phi(X_A, X)$) and the Atkinson theorem [32, Theorem 1.3, p. 163] shows that $B \in \Phi(X, X_A)$ and

$$i(\hat{A}) = -i(B). \tag{2.5}$$

On the other hand, since $\mathscr{S}(X)$ is a closed two-sided ideal containing $\mathscr{K}(X)$ we conclude that $U \in \mathscr{S}(X)$.

If $X = L_p(\mu)$, then (2.4) and [23, Lemma 2.2, 24, Theorem 1(b)] imply that $(\hat{A} + \hat{S})B \in \Phi(X)$ and $i((\hat{A} + \hat{S}B) = 0$.

Assume now that $X = C(\Xi)$, it follows from [27, Theorem 1] that U is weakly compact and therefore, by [7, Corollary 5, p. 494], its square is compact. Thus, the use of [23, Lemma 2.2] gives $(\hat{A} + \hat{S})B \in \Phi(X)$ and $i((\hat{A} + \hat{S})B) = 0$.

Accordingly, in both cases, $(\hat{A} + \hat{S})B$ is a Fredholm operator with index equal to 0. Next, since $B \in \Phi(X, X_A)$ (see above), applying [32, Theorem 2.5, p. 169] we conclude that $\hat{A} + \hat{S} \in \Phi(X_A, X)$. Then, it follows from the Atkinson theorem [32, Theorem 1.3, p. 163] that

$$i(\hat{A} + \hat{S}) = -i(B). \tag{2.6}$$

Finally, Eqs. (2.5), (2.6), and (2.2) show that $i(A + S) = i(A)$ and the proof is complete. Q.E.D.

Proof of Theorem 2.1. It is verbatim the proof of Theorem 3.1 in [23]. It suffices only to replace in the text Proposition 3.5 by Lemma 2.1. Q.E.D.

Before proceeding with the proof of Theorem 2.2, we recall the following:

DEFINITION 2.1 [22]. Let X and Y be two Banach spaces and let $F \in \mathscr{L}(X, Y)$. F is called a Fredholm perturbation if $U + F \in \Phi(X, Y)$ whenever $U \in \Phi(X, Y)$. F is called an upper (resp. lower) Fredholm perturbation if $F + U \in \Phi_+(X, Y)$ (resp. $\Phi_-(X, Y)$) whenever $U \in \Phi_+(X, Y)$ (resp. $\Phi_-(X, Y)$).

The sets of Fredholm, upper semi-Fredholm, and lower semi-Fredholm perturbations are denoted by $\mathscr{F}(X,Y)$, $\mathscr{F}_+(X,Y)$, and $\mathscr{F}_-(X,Y)$, respectively. If $X = Y$ we write $\mathscr{F}(X)$, $\mathscr{F}_+(X)$, and $\mathscr{F}_-(X)$ for $\mathscr{F}(X,X)$, $\mathscr{F}_+(X,X)$, and $\mathscr{F}_-(X,X)$.

Let N be a closed subspace of a Banach space X. We denote by π_N the quotient map $X \to X/N$. The codimension of N, $\operatorname{codim}(N)$, is defined to be the dimension of the vector space X/N. An operator $T \in \mathscr{L}(X)$ is said to be strictly cosingular if there exists no closed subspace N of X with $\operatorname{codim}(N) = \infty$ such that $\pi_N T : X \to X/N$ is surjective. Let $C\mathscr{S}(X)$ denote the set of strictly cosingular operators on X. This class of operators was introduced by Pelczynski [27]. It forms a closed two-sided ideal of $\mathscr{L}(X)$ (cf. [33]).

Remark 2.2. Let $\Phi^b(X,Y)$, $\Phi^b_+(X,Y)$, and $\Phi^b_-(X,Y)$ denote the sets $\Phi(X,Y) \cap \mathscr{L}(X,Y)$, $\Phi_+(X,Y) \cap \mathscr{L}(X,Y)$, and $\Phi_-(X,Y) \cap \mathscr{L}(X,Y)$, respectively. If in Definition 2.1 we replace $\Phi(X,Y)$, $\Phi_+(X,Y)$, and $\Phi_-(X,Y)$ by $\Phi^b(X,Y)$, $\Phi^b_+(X,Y)$, and $\Phi^b_-(X,Y)$ we obtain the sets $\mathscr{F}^b(X,Y)$, $\mathscr{F}^b_+(X,Y)$, and $\mathscr{F}^b_-(X,Y)$. These classes of operators were introduced and investigated in [9]. In particular, it is shown that $\mathscr{F}^b(X,Y)$ is a closed subset of $\mathscr{L}(X,Y)$ and $\mathscr{F}^b(X)$ is a closed two-sided ideal of $\mathscr{L}(X)$. In general, we have

$$\mathscr{K}(X,Y) \subseteq \mathscr{S}(X,Y) \subseteq \mathscr{F}^b_+(X,Y) \subseteq \mathscr{F}^b(X,Y), \tag{2.7a}$$

$$\mathscr{K}(X,Y) \subseteq C\mathscr{S}(X,Y) \subseteq \mathscr{F}^b_-(X,Y) \subseteq \mathscr{F}^b(X,Y). \tag{2.7b}$$

The inclusion $\mathscr{S}(X,Y) \subseteq \mathscr{F}^b_+(X,Y)$ is due to Kato [19], whereas the inclusion $C\mathscr{S}(X,Y) \subseteq \mathscr{F}^b_-(X,Y)$ was proved by Vladimirskii [33].

The following identity was established in [22, Lemma 2.3(ii)].

LEMMA 2.2 [22]. *Let X be an arbitrary Banach space. Then*

$$\mathscr{F}(X) = \mathscr{F}^b(X).$$

An immediate consequence of this result is that $\mathscr{F}(X)$ is a closed two sided ideal of $\mathscr{L}(X)$.

Proof of Theorem 2.2. Set $\mathscr{P} = \bigcap_{S \in A\mathscr{S}(X)} \sigma(A + S)$. Obviously the inclusion $\mathscr{S}(X) \subseteq A\mathscr{S}(X)$ implies that $\mathscr{P} \subseteq \sigma_{e5}(A)$. To complete the proof it suffices to show that $\sigma_{e5}(A) \subseteq \mathscr{P}$. To do so, consider $\lambda_0 \notin \mathscr{P}$. Then there exists $S \in A\mathscr{S}(X)$ such that $\lambda_0 \in \rho(A + S)$. Denote by Y (resp. Z) the space $\mathscr{D}(A)$ endowed with the norm $\|x\|_A := \|x\| + \|Ax\|$ (resp. $\|x\|_{A+S} := \|x\| + \|(A + S)x\|$). Let $\varphi \in X$ and let $\psi = (\lambda - A - S)^{-1}\varphi$. It follows from the estimate

$$\begin{aligned}\|\psi\|_{A+S} &= \|\psi\| + \|(A+S)\psi\| = \|\psi\| + \|\varphi - \lambda_0\psi\| \\ &= \left\|\left(\lambda_0 - \hat{A} - \hat{S}\right)^{-1}\varphi\right\| + \left\|\varphi - \lambda_0\left(\lambda_0 - \hat{A} - \hat{S}\right)^{-1}\varphi\right\| \\ &\leq \left(1 + (1+|\lambda_0|)\left\|\left(\lambda_0 - \hat{A} - \hat{S}\right)^{-1}\right\|\right)\|\varphi\|,\end{aligned}$$

that $(\lambda_0 - \hat{A} - \hat{S})^{-1} \in \mathscr{L}(X, Z)$. Moreover, since $\hat{S} \in \mathscr{S}(Y, X)$, we can apply [19, Lemma 461] which ensures that $(\lambda_0 - \hat{A} - \hat{S})^{-1}\hat{S} \in \mathscr{S}(Y, Z)$. Let $\mathscr{I}$ denote the imbedding operator which maps every $\varphi \in Y$ onto the same element φ in Z. Since $R(\mathscr{I}) = Z$ and $N(\mathscr{I}) = \{0\}$, the estimate

$$\begin{aligned}\|\mathscr{I}\varphi\|_{\mathscr{Z}} = \|\varphi\|_{\mathscr{Z}} &\leq \|\varphi\|_X + \|A\varphi\|_X + \|S\varphi\|_X \\ &\leq \left(1 + \|S\|_{\mathscr{L}(Y,X)}\right)\|\varphi\|_Y, \qquad \forall\varphi \in Y\end{aligned}$$

shows that $\mathscr{I} \in \Phi^b(Y, Z)$ and $i(\mathscr{I}) = 0$. Next, making use of the inclusion $\mathscr{S}(Y, X) \subseteq \mathscr{F}(Y, X)$ [10, Theorem 2.1, p. 117] we conclude that

$$\mathscr{I} + \left(\lambda_0 - \hat{A} - \hat{S}\right)^{-1}\hat{S} \in \Phi^b(Y, Z) \qquad \text{and}$$

$$i\left(\mathscr{I} + \left(\lambda_0 - \hat{A} - \hat{S}\right)^{-1}\hat{S}\right) = 0.$$

On the other hand, since $\lambda_0 \in \rho(A + S)$, it follows from (2.2) that

$$\left(\lambda_0 - \hat{A} - \hat{S}\right) \in \Phi^b(Z, X) \qquad \text{and} \qquad i\left(\lambda_0 - \hat{A} - \hat{S}\right) = 0.$$

Hence, by the Atkinson theorem, we have

$$\left(\lambda_0 - \hat{A}\right) = \left(\lambda_0 - \hat{A} - \hat{S}\right)\left(\mathscr{I} + \left(\lambda_0 - \hat{A} - \hat{S}\right)^{-1}\hat{S}\right) \in \Phi^b(Y, X)$$

$$\text{and} \qquad i\left(\lambda_0 - \hat{A}\right) = 0.$$

This together with (2.2) amounts to

$$(\lambda_0 - A) \in \Phi(X) \qquad \text{and} \qquad i(\lambda_0 - A) = 0.$$

Now applying [32, Theorem 5.4, p. 180] we conclude that $\sigma_{e5}(A) \subseteq \mathscr{P}$. This completes the proof. Q.E.D.

As it was already mentioned, the results obtained in [23] dealing with $\sigma_{e2}(\cdot)$ and $\sigma_{e3}(\cdot)$ are open for $p \in \{1\} \cup (2, \infty)$. In the remainder of this section we will show that $\sigma_{e2}(\cdot)$ and $\sigma_{e3}(\cdot)$ are also invariant under strictly singular perturbations on $L_p(\mu)$-spaces for $p \in \{1\} \cup (2, \infty)$. These results will be extended to the space $C(\Xi)$.

PROPOSITION 2.1. *Let X be one of the spaces $L_p(\mu)$, $p \in \{1\} \cup (2, \infty)$, or $C(\Xi)$ and assume that $A \in \mathscr{C}(X)$. If $S \in \mathscr{S}(X)$, then*

$$\sigma_{ei}(A) = \sigma_{ei}(A + S), \qquad i = 2, 3.$$

Remark 2.3. (a) For $p \in (1, 2]$, Proposition 2.1 was already established in [23, Theorem 3.1(b)]. The proof relies on the properties of superprojective Banach spaces [38]. Our proof here uses the identity $\mathscr{S}(L_p(\mu)) = C\mathscr{S}(L_p(\mu))$ (valid for $p \in [1, \infty)$, see [24, 36]) and works for $p \in \{1\} \cup (2, \infty)$ as well as for $p \in (1, 2]$.

(b) Since $\mathscr{S}(X) \subseteq A\mathscr{S}(X)$, it follows from Theorem 2.1 that the result of Proposition 2.1 holds also true for $\sigma_{e1}(\cdot)$, $\sigma_{e4}(\cdot)$, $\sigma_{e5}(\cdot)$, and $\sigma_{e6}(\cdot)$ (see also [23, Theorem 3.1]).

The next result gives a convenient criterion for the invariance of essential spectra of perturbed linear operators.

PROPOSITION 2.2. *Let X be one of the spaces $L_p(\mu)$, $p \in \{1\} \cup (2, \infty)$, or $C(\Xi)$ and let A, $B \in \mathscr{C}(X)$. If, for some $\lambda \in \rho(A) \cap \rho(B)$, we have $(\lambda - A)^{-1} - (\lambda - B)^{-1} \in \mathscr{S}(X)$, then*

$$\sigma_{ei}(A) = \sigma_{ei}(B), \qquad i = 1, 2, 3, 4, \text{ and } 5.$$

Remark 2.4. For $X = L_p(\mu)$ with $p \in [1, \infty)$, the conclusion of Proposition 2.2 concerning $\sigma_{e1}(\cdot)$, $\sigma_{e4}(\cdot)$, and $\sigma_{e5}(\cdot)$ was already established in [23].

The following lemma is essential in proving Propositions 2.1 and 2.2.

LEMMA 2.3. *Let X be one of the spaces $L_p(\mu)$, $p \in \{1\} \cup (2, \infty)$, or $C(\Xi)$. If $S \in \mathscr{S}(X)$, then we have*:

(i) *If $A \in \Phi_-(X)$, then $A + S \in \Phi_-(X)$.*

(ii) *If $A \in \Phi_\pm(X)$, then $A + S \in \Phi_\pm(X)$.*

Remark 2.5. The assertions (i) and (ii) were established in [23] for $X = L_p(\mu)$ with $p \in (1, 2]$. The proofs use the fact that the dual of these spaces is subprojective [38]. Lemma 2.3 completes the results of Proposition 3.5 in [23].

Let X be a Banach space. We say that X is weakly compactly generating (w.c.g) if the linear span of some weakly compact subset is dense in X. For the properties of w.c.g spaces we refer to [5]. In particular, all separable and all reflexive Banach spaces are w.c.g as $L_1(\Omega, d\mu)$ if (Ω, μ) is σ-finite.

In [37] Weis proved that if X is a w.c.g Banach space, then

$$\mathcal{F}_{+}(X) = \mathcal{S}(X) \qquad \text{and} \qquad \mathcal{F}_{-}(X) = C\mathcal{S}(X). \tag{2.8}$$

Let $p \in [1, \infty)$. By the Milman–Weis theorem we have

$$\mathcal{S}\big(L_p(\mu)\big) = C\mathcal{S}\big(L_p(\mu)\big) = \mathcal{F}^b\big(L_p(\mu)\big)$$

(cf. [24; 36, 37, p. 430]). Now putting together (2.8) and Lemma 2.2 we get

$$\mathcal{F}\big(L_p(\mu)\big) = \mathcal{S}\big(L_p(\mu)\big) = \mathcal{F}_{+}\big(L_p(\mu)\big) = C\mathcal{S}\big(L_p(\mu)\big) = \mathcal{F}_{-}\big(L_p(\mu)\big). \tag{2.9}$$

Proof of Lemma 2.3. (i) Let $S \in \mathcal{S}(X)$. If $X = L_p(\mu)$, the assertion follows from (2.9) and Definition 2.1.

Assume now that $X = C(\Xi)$. Then, it follows from [27, Theorem 1] that S is weakly compact. Next, using [27, Proposition 3(b)] we conclude that S' (the dual operator of S) is strictly singular too. Now, taking into account [20, Theorem 5.13, p. 234], it suffices to apply [19, Theorem 2] to $A' + S'$.

(ii) This follows from (i) and Lemma 2.1(i) (because $\mathcal{S}(X) \subseteq A\mathcal{S}(X)$). Q.E.D.

The proof of Proposition 2.1 (resp. Proposition 2.2) may be modeled very closely after that of Theorem 3.1 (resp. Theorem 3.3) in [23]. It suffices solely to replace in the text Proposition 3.5 by Lemma 2.3.

Notes and Remarks. (1) It should be observed that the equalities in (2.9) are also valid for the space $C(\Xi)$ where Ξ is a compact Hausdorff space. Indeed, since $C(\Xi)$ has the Dunford–Pettis property (a Banach space X is said to have the Dunford–Pettis property if for every Banach space Y every weakly compact operator $T: X \to Y$ takes weakly compact sets in X into relatively norm compact sets of Y (cf. [6])), it follows from [27, Theorems 1 and 2] that $\mathcal{S}(C(\Xi)) = C\mathcal{S}(C(\Xi))$. But $C(\Xi)$ is w.c.g; then $\mathcal{S}(C(\Xi)) = \mathcal{F}_{+}(C(\Xi))$ and $C\mathcal{S}(C(\Xi)) = \mathcal{F}_{-}(C(\Xi))$. Next, applying [23, Theorem 1(a)] together with [22, Lemma 2.3(ii)] we get

$$\mathcal{F}(C(\Xi)) = \mathcal{S}(C(\Xi)) = \mathcal{F}_{+}(C(\Xi)) = C\mathcal{S}(C(\Xi)) = \mathcal{F}_{-}(C(\Xi)). \tag{2.10}$$

(2) The proof of the first part of Lemma 2.3 for $X = C(\Xi)$ is now an immediate consequence of (2.10) and Definition 2.1.

(3) The identities (2.9) and (2.10) are not specific to the spaces considered above. There are some Banach spaces X for which $\mathcal{L}(X)$ has only one proper nonzero closed two-sided ideal. Indeed, Calkin [3] proved

that if X is a separable Hilbert space, then $\mathscr{K}(X)$ is the unique proper nonzero closed two-sided ideal of $\mathscr{L}(X)$. Afterwards, Gohberg, Markus, and Feldman [9] obtained the same result for $X = l_p$, $1 \leq p < \infty$, and $X = c_0$. In [16] Herman established this result for a large class of Banach spaces, namely Banach spaces which have perfectly homogeneous block bases and satisfy $(+)$ (for the definition and more information about these spaces we refer to [16]). (Evidently the spaces l_p, $1 \leq p < \infty$, and c_0 belong to this class.) Obviously, if X has perfectly homogeneous block bases which satisfy $(+)$, then using (2.7) and Lemma 2.2 we conclude that

$$\mathscr{K}(X) = \mathscr{S}(X) = \mathscr{F}_+(X) = C\mathscr{S}(X) = \mathscr{F}_-(X) = \mathscr{F}(X).$$

A Banach space X is said to be an h-space if each closed infinite dimensional subspace of X contains a complemented subspace isomorphic to X. Any Banach space isomorphic to an h-space is an h-space; c, c_0, and $l_p (1 \leq p < \infty)$ are h-spaces. In [38, Theorem 6.2], Whitley proved that if X is an h-space, then $\mathscr{S}(X)$ is the greatest proper ideal of $\mathscr{L}(X)$. This together with (2.7) and Lemma 2.2 implies that

$$\mathscr{K}(X) \subseteq \mathscr{F}_+(X) = \mathscr{F}(X) = \mathscr{S}(X) \qquad \text{and}$$
$$\mathscr{K}(X) \subseteq \mathscr{F}_-(X) \subseteq \mathscr{F}(X) = \mathscr{S}(X).$$

Hence the results of this section and those obtained in [23, Sect. 3] are valid when dealing with strictly singular and A-strictly singular perturbations in these classes of spaces.

3. APPLICATION TO TRANSPORT EQUATION

The purpose of this section is to apply Proposition 2.2 to describe the essential spectra of the following integro-differential operator

$$A_H \psi(x, \xi) = -\xi \frac{\partial \psi}{\partial x}(x, \xi) - \sigma(\xi)\psi(x, \xi) + \int_{-1}^{1} \kappa(x, \xi, \xi')\psi(x, \xi')\, d\xi',$$

where $x \in [-a, a]$ and $\xi \in (-1, 1)$. The operator A_H describes the one-speed neutron transport or transfer of unpolarized light (cf. [4, 11, 18, 21, 25, 34]). Let us first make precise the functional setting of the problem.

Let

$$Y_p = L_p(D; dx d\xi),$$

where $D =]-a,a[\times]-1,1[,(a>0)$, and $p \in [1,\infty)$. Define the following sets representing the *incoming* and the *outgoing* boundary of the phase space D:

$$D^i = D_1^i \cup D_2^i = \{-a\} \times]0,1[\cup \{a\} \times]-1,0[,$$
$$D^0 = D_1^0 \cup D_2^0 = \{-a\} \times]-1,0[\cup \{a\} \times]0,1[.$$

Moreover, we introduce the boundary spaces

$$\begin{aligned} Y_p^i &:= L_p(D^i, |\xi| d\xi) \sim L_p(D_1^i, |\xi| d\xi) \oplus L_p(D_2^i, |\xi| d\xi) \\ &:= Y_{1,p}^i \oplus Y_{2,p}^i, \end{aligned}$$

endowed with the norm

$$\begin{aligned} \|\psi^i, Y_p^i\| &= \left(\|\psi_1^i, Y_{1,p}^i\|^p + \|\psi_2^i, Y_{2,p}^i\|^p\right)^{\frac{1}{p}} \\ &= \left[\int_0^1 |\psi(-a,\xi)|^p |\xi|\, d\xi + \int_{-1}^0 |\psi(a,\xi)|^p |\xi|\, d\xi\right]^{\frac{1}{p}}. \end{aligned}$$

$$\begin{aligned} Y_p^0 &:= L_p(D^0, |\xi|\, d\xi) \sim L_p(D_1^0, |\xi|\, d\xi) \oplus L_p(D_2^0, |\xi|\, d\xi) \\ &:= Y_{1,p}^0 \oplus Y_{2,p}^0, \end{aligned}$$

endowed with the norm

$$\begin{aligned} \|\psi^0, Y_p^0\| &= \left(\|\psi_1^0, Y_{1,p}^0\|^p + \|\psi_2^0, Y_{2,p}^0\|^p\right)^{\frac{1}{p}} \\ &= \left[\int_{-1}^0 |\psi(-a,\xi)|^p |\xi|\, d\xi + \int_0^1 |\psi(a,\xi)|^p |\xi|\, d\xi\right]^{\frac{1}{p}}; \end{aligned}$$

where $\sim$ means the natural identification of these spaces.

Let us introduce the boundary operator H,

$$\begin{cases} H : Y_{1,p}^0 \oplus Y_{2,p}^0 \to Y_{1,p}^i \oplus Y_{2,p}^i \\ H\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} := \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \end{cases}$$

with $H_{j,k} : Y_{k,p}^0 \to Y_{j,p}^0$, $H_{j,k} \in \mathscr{L}(Y_{k,p}^0; Y_{j,p}^i)$, $j, k = 1, 2$, defined such that, on natural identification, the boundary conditions can be written as $\psi^i = H\psi^0$. We define, now, the streaming operator T_H with domain including the boundary conditions,

$$\begin{cases} T_H : D(T_H) \subseteq Y_p \to Y_p \\ \qquad \psi \to T_H \psi(x,\xi) = -\xi \dfrac{\partial \psi}{\partial x}(x,\xi) - \sigma(\xi)\psi(x,\xi) \\ D(T_H) = \left\{ \psi \in Y_p, \xi \dfrac{\partial \psi}{\partial x} \in Y_p, \psi_{|D^i} = \psi^i \in Y_p^i, \psi_{|D^0} = \psi^0 \in Y_p^0; \right. \\ \qquad \left. \text{and } \psi^i = H\psi^0 \right\} \end{cases}$$

where $0 \leq \sigma(\cdot) \in L^\infty(-1,1)$, $\psi^0 = (\psi_1^0, \psi_2^0)^\top$, and $\psi^i = (\psi_1^i, \psi_2^i)^\top$ with ψ_1^0, ψ_2^0, ψ_1^i, and ψ_2^i given by

$$\begin{cases} \psi_1^i(\xi) = \psi(-a,\xi), & \xi \in (0,1); \\ \psi_2^i(\xi) = \psi(a,\xi), & \xi \in (-1,0); \\ \psi_1^0(\xi) = \psi(-a,\xi), & \xi \in (-1,0); \\ \psi_2^0(\xi) = \psi(a,\xi), & \xi \in (0,1). \end{cases}$$

Remark 3.1. The derivative of ψ in the definition of T_H is meant in a distributional sense. Note that if $\psi \in D(T_H)$, then it is absolutely continuous with respect to x. Hence the restrictions of ψ to D^i and D^0 are meaningful. Note also that $D(T_H)$ is dense in Y_p because it contains $C_0^\infty[(-a,a) \times (1,1)]$.

Let $\varphi \in Y_p$ and consider the resolvent equation for T_H

$$(\lambda - T_H)\psi = \varphi, \tag{3.1}$$

where λ is a complex number and the unknown ψ must be sought in $D(T_H)$. Let λ^* denote the real defined by

$$\lambda^* := \lim_{|\xi| \to 0} \inf \sigma(\xi).$$

Thus, for Re $\lambda > -\lambda^*$, the solution of (3.1) is formally given by

$$\psi(x,\xi) = \begin{cases} \psi(-a,\xi) e^{\frac{-(\lambda+\sigma(\xi))|a+x|}{|\xi|}} \\ \qquad + \dfrac{1}{|\xi|} \displaystyle\int_{-a}^{x} e^{\frac{-(\lambda+\sigma(\xi))|x-x'|}{|\xi|}} \varphi(x',\xi)\, dx', & \xi \in]0,1[\\ \psi(a,\xi) e^{\frac{-(\lambda+\sigma(\xi))|a-x|}{|\xi|}} \\ \qquad + \dfrac{1}{|\xi|} \displaystyle\int_{x}^{a} e^{\frac{-(\lambda+\sigma(\xi))|x-x'|}{|\xi|}} \varphi(x',\xi)\, dx', & \xi \in]-1,0[\end{cases} \tag{3.2}$$

whereas $\psi(a,\xi)$ and $\psi(-a,\xi)$ are given by

$$\psi(a,\xi) = \psi(-a,\xi)e^{-2a\frac{(\lambda+\sigma(\xi))}{|\xi|}} + \frac{1}{|\xi|}\int_{-a}^{a} e^{\frac{-(\lambda+\sigma(\xi))|a-x|}{|\xi|}}\varphi(x,\xi)\,dx, \tag{3.3}$$

$$\psi(-a,\xi) = \psi(a,\xi)e^{-2a\frac{(\lambda+\sigma(\xi))}{|\xi|}} + \frac{1}{|\xi|}\int_{-a}^{a} e^{\frac{-(\lambda+\sigma(\xi))|a+x|}{|\xi|}}\varphi(x,\xi)\,dx. \tag{3.4}$$

To allow the abstract formulation of Eqs. (3.2), (3.3), and (3.4), let us define the following operators depending on the parameter λ:

$$\begin{cases} M_\lambda : Y_p^i \to Y_p^0,\ M_\lambda u := (M_\lambda^+ u, M_\lambda^- u) \text{ with} \\ (M_\lambda^+ u)(-a,\xi) := u(-a,\xi)\exp\left(-2a\dfrac{(\lambda+\sigma(\xi))}{|\xi|}\right), & 0<\xi<1; \\ (M_\lambda^- u)(a,\xi) := u(a,\xi)\exp\left(-2a\dfrac{(\lambda+\sigma(\xi))}{|\xi|}\right), & -1<\xi<0; \end{cases}$$

$$\begin{cases} B_\lambda : Y_p^i \to Y_p,\ B_\lambda u := \chi_{(-1,0)}(\xi)B_\lambda^- u + \chi_{(0,1)}(\xi)B_\lambda^+ u \text{ with} \\ (B_\lambda^+ u)(-a,\xi) := u(-a,\xi)\exp\left(-\dfrac{(\lambda+\sigma(\xi))|a+x|}{|\xi|}\right), & 0<\xi<1; \\ (B_\lambda^- u)(a,\xi) := u(a,\xi)\exp\left(-\dfrac{(\lambda+\sigma(\xi))|a-x|}{|\xi|}\right), & -1<\xi<0; \end{cases}$$

$$\begin{cases} G_\lambda : Y_p \to Y_p^0,\ G_\lambda\varphi := (G_\lambda^+\varphi, G_\lambda^-\varphi) \text{ with} \\ G_\lambda^+\varphi := \dfrac{1}{|\xi|}\displaystyle\int_{-a}^{a}\exp\left(-\dfrac{(\lambda+\sigma(\xi))|a-x|}{|\xi|}\right)\varphi(x,\xi)\,dx, & 0<\xi<1; \\ G_\lambda^-\varphi := \dfrac{1}{|\xi|}\displaystyle\int_{-a}^{a}\exp\left(-\dfrac{(\lambda+\sigma(\xi))|a+x|}{|\xi|}\right)\varphi(x,\xi)\,dx, & -1<\xi<0; \end{cases}$$

and

$$\begin{cases} C_\lambda : Y_p \to Y_p,\ C_\lambda\varphi := \chi_{(-1,0)}(\xi)C_\lambda^-\varphi + \chi_{(0,1)}(\xi)C_\lambda^+\varphi \text{ with} \\ C_\lambda^+\varphi := \dfrac{1}{|\xi|}\displaystyle\int_{-a}^{a}\exp\left(-\dfrac{(\lambda+\sigma(\xi))|x-x'|}{|\xi|}\right)\varphi(x',\xi)\,dx', & 0<\xi<1; \\ C_\lambda^-\varphi := \dfrac{1}{|\xi|}\displaystyle\int_{x}^{a}\exp\left(-\dfrac{(\lambda+\sigma(\xi))|x-x'|}{|\xi|}\right)\varphi(x',\xi)\,dx', & -1<\xi<0, \end{cases}$$

where $\chi_{(-1,0)}(\cdot)$ and $\chi_{(0,1)}(\cdot)$ denote, respectively, the characteristic functions of the intervals $(-1,0)$ and $(0,1)$.

Let λ_0 denote the real defined by

$$\lambda_0 := \begin{cases} -\lambda^* & \text{if } \|H\| \leq 1 \\ -\lambda^* + \dfrac{1}{2a}\log(\|H\|) & \text{if } \|H\| > 1. \end{cases}$$

Simple calculations using Hölder's inequality show that these operators are bounded on their respective spaces. In fact, for $\operatorname{Re}\lambda > -\lambda^*$, the norms of the operators M_λ, B_λ, G_λ, and C_λ are bounded above, respectively, by $e^{-2a(\operatorname{Re}\lambda+\lambda^*)}$, $[p(\operatorname{Re}\lambda+\lambda^*)]^{-1/p}$, $(\operatorname{Re}\lambda+\lambda^*)^{-1}$ where q denotes the conjugate of p. For the details we refer to [21].

Using these operators and the fact that ψ must satisfy the boundary conditions, Eqs. (3.3) and (3.4) are written in the space Y_p^0 in the operator form

$$\psi^0 = M_\lambda H\psi^0 + G_\lambda\varphi.$$

The solution of this equation reduces to the invertibility of the operator $\mathscr{U}(\lambda) := I - M_\lambda H$ (which is the case if $\operatorname{Re}\lambda > \lambda_0$, see the norm estimate of M_λ). This gives

$$\psi^0 = \{\mathscr{U}(\lambda)\}^{-1}G_\lambda\varphi. \tag{3.5}$$

On the other hand, (3.2) can be rewritten as

$$\psi = B_\lambda H\psi^0 + C_\lambda\varphi.$$

Substituting (3.5) into the above equation we get

$$\psi = B_\lambda H\{\mathscr{U}(\lambda)\}^{-1}G_\lambda\varphi + C_\lambda\varphi.$$

Thus

$$(\lambda - T_H)^{-1} = B_\lambda H\{\mathscr{U}(\lambda)\}^{-1}G_\lambda + C_\lambda.$$

On the other hand, observe that the operator C_λ is nothing else but $(\lambda - T_0)^{-1}$ where T_0 designates the streaming operator with vacuum boundary conditions, i.e., $H = 0$. Obviously, if the operator $I - \{\mathscr{U}(\lambda)\}$ is boundedly invertible (for example, if $\operatorname{Re}\lambda > \lambda_0$), then $\lambda \in \rho(T_H) \cap \rho(T_0)$ and

$$(\lambda - T_H)^{-1} - (\lambda - T_0)^{-1} = \mathscr{R}_\lambda, \tag{3.6}$$

where $\mathscr{R}_\lambda := B_\lambda H\{\mathscr{U}(\lambda)\}^{-1}G_\lambda$.

The essential spectra of T_0 were analyzed in detail in [23, Remark 4.1]. In particular, we have

$$\sigma_{ei}(T_0) = \{\lambda \in \mathbb{C} \text{ such that } \operatorname{Re} \lambda \leq -\lambda^*\} \qquad \text{for } i = 1, \dots, 6. \quad (3.7)$$

We are now ready to prove:

PROPOSITION 3.1. *If the boundary operator H is strictly singular, then*

$$\sigma_{ei}(T_H) = \{\lambda \in \mathbb{C} \text{ such that } \operatorname{Re} \lambda \leq -\lambda^*\}, \qquad i = 1, 2, 3, 4, \text{ and } 5.$$

Remark 3.2. The result of this proposition is new only for $\sigma_{e2}(T_H)$ and $\sigma_{e3}(T_H)$ with $p \in \{1\} \cup (2, \infty)$. The other cases were already established in [23, Theorem 4.1].

Proof of Proposition 3.1. Applying [19, Lemma 461], one sees that $\mathscr{R}_\lambda$ is strictly singular. Now the result follows from (3.6), (3.7), and Proposition 2.2. Q.E.D.

Next we consider the transport operator $A_H = T_H + K$ where K is a bounded operator given by

$$\begin{cases} K : Y_p \to Y_p \\ \qquad \psi \to \int_{-1}^{1} \kappa(x, \xi, \xi')\psi(x, \xi')\, d\xi' \end{cases} \quad (3.8)$$

with $\kappa(\cdot, \cdot, \cdot)$ a measurable function from $[-a, a] \times [-1, 1] \times [-1, 1]$ to $\mathbb{R}^+$.

Observe that the operator K acts only on the velocity ξ', so x may be viewed merely as a parameter in $[-a, a]$. Hence, we may consider K as a function $K(\cdot) : x \in [-a, a] \to K(x) \in Z$ where $Z := \mathscr{L}(L_p([-1, 1], d\xi))$.

In the following we will make the assumptions:

$$\begin{cases} \text{the function } K(\cdot) \text{ is strongly measurable} & (3.9) \\ \text{there exists a compact subset } \mathscr{C} \subseteq Z \text{ such that} & \\ \quad K(x) \in C \text{ a.e. on } [-a, a] & (3.10) \\ \text{and } K(x) \in \mathscr{K}\big(L_p([-1, 1], d\xi)\big) \text{ a.e. on } [-a, a], & (3.11) \end{cases}$$

where $\mathscr{K}(L_p[-1, 1], d\xi))$ denotes the set of all compact operators on $L_p([-1, 1], d\xi)$.

Remark 3.3. Let X and Y be two Banach spaces and let B_X denote the unit ball of X. A subset $\mathscr{N} \subseteq \mathscr{L}(X, Y)$ is collectively compact if and only if the set $\mathscr{N}B_X = \{Nx : N \in \mathscr{N} \text{ and } x \in B_X\}$ has compact closure. Having this in mind, using Theorem 2.5 in [1] one sees easily that the conditions (3.10) and (3.11) imply that the set $\{K(x): x \in [-a, a]\}$ is collectively compact.

It seems that the concept of collective compactness has been already used in transport theory by Belleni-Morante and his students.

Obviously, the hypothesis (3.10) implies that

$$K(\cdot) \in L^{\infty}(]-a, a[, Z). \tag{3.12}$$

Let $\psi \in Y_p$. It is easy to see that $(K\psi)(x, \xi) = (K(x)\psi(x, \xi))(\xi)$ and then, by (3.12), we have

$$\int_{-1}^{1} |(K\psi)(x, \xi)|^p \, d\xi \leq \|K(\cdot)\|^p_{L^{\infty}(]-a, a[, Z)} \int_{-1}^{1} |\psi(x, \xi)|^p \, d\xi$$

and therefore

$$\int_{-a}^{a} \int_{-1}^{1} |(K\psi)(x, \xi)|^p \, d\xi \leq \|K(\cdot)\|^p_{L^{\infty}(]-a, a[, Z)} \int_{-a}^{a} \int_{-1}^{1} |\psi(x, \xi)|^p \, d\xi \, dx.$$

This leads to the estimate

$$\|K\|_{\mathscr{L}(Y_p)} \leq \|K(\cdot)\|_{L^{\infty}(]-a, a[, Z)}. \tag{3.13}$$

In view of these observations, we will make use of the following extensive class of collision operators introduced in [25, 26] and referred to as regular operators.

DEFINITION 3.1 [26]. A collision operator K, in the form (3.8), is said to be regular if it satisfies the assumptions (3.9), (3.10), and (3.11) above.

PROPOSITION 3.2. *Let $p \in [1, \infty)$ and assume that K is a regular operator on Y_p. Let λ be such that the operator $\mathscr{U}(\lambda)$ is boundedly invertible. Then*

(i) $(\lambda - T_H)^{-1}K$ *is compact (resp. weakly compact) on* Y_p, $p \in (1, \infty)$ *(resp.* Y_1*).*

(ii) $K(\lambda - T_H)^{-1}$ *is compact on* Y_p *for* $p \in (1, \infty)$.

Remark 3.4. This proposition extends Theorems 2.1 and 2.2 in [21] to non-homogeneous regular collisions operators. Note also that, for $p = 1$, the operator $K(\lambda - T_H)^{-1}$ in general is not weakly compact [25, Remark 4.2].

To prove this proposition the following lemma is required. It is inspired and adapted from [26, Lemma 2.3].

LEMMA 3.1. *Assume that the collision operator K is regular on Y_p. Then K can be approximated, in the uniform topology, by a sequence $(K_n)_n$ of* *operators with kernels of the form* $\kappa_n(x, \xi, \xi') = \sum_{j \in J} \eta_j(x)\theta_j(\xi)\beta_j(\xi')$ *where* $\eta_j(\cdot) \in L^\infty([-a, a], dx)$, $\theta_j(\cdot) \in L_p([-1, 1], d\xi)$ *and* $\beta_j(\cdot) \in L_q([-1, 1], d\xi)$ (*q denotes the conjugate of p and J is finite*).

Proof. Let $\varepsilon > 0$. By the assumption (3.10) there exists $J_1, \dots, J_n$ such that $\{J_i\}_i \subseteq C$ and $C \subseteq \bigcup_{1 \le i \le n} B(J_i, \varepsilon)$ where $B(J_i, \varepsilon)$ is the open ball, in $\mathscr{K}(L_p([-1, 1], d\xi))$, centered at J_i with radius ε.

Let $A_1 = B(J_1, \varepsilon)$, $A_2 = B(J_2, \varepsilon) \setminus A_1, \dots, A_n = B(J_n, \varepsilon) \setminus A_{n-1}$. Clearly, $A_i \cap A_j = \emptyset$ if $i \neq j$ and $C \subset \bigcup_{1 \le i \le n} A_i$.

Let $1 \le i \le n$ and denote by I_i the set

$$I_i = K^{-1}(A_i) = \{x \in]-a, a[, K(x) \in A_i\}.$$

Hence we have $I_i \cap I_j = \emptyset$ if $i \neq j$ and $]-a, a[= \bigcup_{i=1}^n I_i$.

Consider now the following step function from $]-a, a[$ into Z defined by

$$S(x) = \sum_{i=1}^{n} \chi_{I_i}(x) J_i,$$

where $\chi_{I_i}(\cdot)$ denotes the characteristic function of I_i. Obviously, $S(\cdot)$ satisfies the conditions (3.9), (3.10), and (3.11). Then using (3.12) we get $K - S \in L^\infty(]-a, a[, Z)$. Moreover, an easy calculation leads to

$$\|K - S\|_{L^\infty(]-a, a[, Z)} \le \varepsilon.$$

Now, using (3.13) we obtain

$$\|K - S\|_{\mathscr{L}(Y_p)} \le \|K - S\|_{L^\infty(]-a, a[, Z)} \le \varepsilon.$$

Hence, we infer that the operator K may be approximated (for the uniform topology) by operators of the form $V(x) = \sum_{i=1}^n \eta_i(x) J_i$ where $\eta_j(\cdot) \in L^\infty([-a, a], dx)$ and $J_i \in \mathscr{K}(L_p([-1, 1], d\xi))$. On the other hand, each compact operator J_i on $L_p([-1, 1], d\xi)$ is a limit (in the norm topology) of a sequence of finite rank operators because $L_p([-1, 1], d\xi)$ $(1 \le p < \infty)$ admits a Schauder basis. This ends the proof. Q.E.D.

Proof of Proposition 3.1. Let λ be such that the operator $\mathscr{U}(\lambda)$ is boundedly invertible (for example, $\operatorname{Re} \lambda > \lambda_0$). In view of (3.6), $(\lambda - T_H)^{-1}K$ is given by

$$(\lambda - T_H)^{-1}K = B_\lambda H\{\mathscr{U}(\lambda)\}^{-1} G_\lambda K + C_\lambda K.$$

To prove the compactness of $(\lambda - T_H)^{-1}K$ on $Y_p (1 < p < \infty)$, it suffices to show that the operators $G_\lambda K$ and $C_\lambda K$ are compact on Y_p. But, using [26, Lemma 2.1], one sees that, for K regular, $C_\lambda K$ is compact on Y_p. Thus, it suffices to establish the compactness of the operators G_λ^+K and G_λ^-K. Analogously, for $p = 1$, the proof is reduced to the weak compactness of G_λ^+K and G_λ^-K on Y_1.

According to Lemma 3.1 (and by linearity), it suffices to give a proof for a collision operator K with a kernel of the form $\kappa(x, \xi, \xi') = \eta(x)\theta(\xi)\beta(\xi')$ where $\eta(\cdot) \in L^\infty([-a, a], dx)$, $\theta(\cdot) \in L_p([-1, 1], d\xi)$ and $\beta(\cdot) \in L_q([-1, 1], d\xi)$ where q denotes the conjugate of p.

Let $\varphi \in Y_p$,

$$\begin{cases} (G_\lambda^+ K\varphi)(\xi) = \displaystyle\int_{-1}^{1}\int_{-a}^{a} \frac{\eta(x)\theta(\xi)}{|\xi|} \exp\left(-\frac{(\lambda + \sigma(\xi))|a - x|}{|\xi|}\right) \\ \qquad \times \beta(\xi')\varphi(x, \xi')\, dx\, d\xi', \qquad 0 < \xi < 1; \\ \qquad = J_\lambda U; \end{cases}$$

where U and J_λ denote the bounded operators

$$\begin{cases} U : Y_p \to L_p([-a, a], dx) \\ \qquad \varphi \to (U\varphi)(x) := \displaystyle\int_{-1}^{1} \beta(\xi)\varphi(x, \xi)\, d\xi \end{cases}$$

$$\begin{cases} J_\lambda : L_p([-a, a], dx) \to Y_{2,p}^0 \\ \qquad \psi \to \displaystyle\int_{-a}^{a} \frac{\eta(x)\theta(\xi)}{|\xi|} \exp\left(-\frac{(\lambda + \sigma(\xi))|a - x|}{|\xi|}\right)\psi(x)\, dx. \end{cases}$$

We first consider the case $p \in (1, \infty)$. It is then sufficient to check that J_λ is compact. This will follow from [17, Theorem 11.6, p. 275] if we show

$$\int_{-1}^{1}\left[\int_{-a}^{a}\left|\frac{\eta(x)\theta(\xi)}{|\xi|} \exp\left\{-\frac{(\lambda + \sigma(\xi))|a - x|}{|\xi|}\right\}\right|^q dx\right]^{\frac{p}{q}} |\xi|\, d\xi < +\infty$$

(J_λ is then a Hille–Tamarkin operator). To do so, let us first observe that we have

$$\begin{aligned} &\int_{-a}^{a}\left|\frac{\eta(x)\theta(\xi)}{|\xi|} \exp\left\{-\frac{(\lambda + \sigma(\xi))|a - x|}{|\xi|}\right\}\right|^q dx \\ &\quad \le \|\eta\|_\infty^q \frac{|\theta(\xi)|^q}{|\xi|} \int_{-a}^{a} \exp\left\{-q\frac{(\operatorname{Re}\lambda + \sigma(\xi))|a - x|}{|\xi|}\right\} dx \\ &\quad \le \|\eta\|_\infty^q \frac{|\theta(\xi)|^q}{q(\operatorname{Re}\lambda + \lambda^*)|\xi|^{q-1}} \end{aligned}$$

which leads to

$$\left[\int_{-a}^{a}\left|\frac{\eta(x)\theta(\xi)}{|\xi|}\exp\left\{-\frac{(\lambda+\sigma(\xi))|a-x|}{|\xi|}\right\}\right|^{q} dx\right]^{\frac{p}{q}}$$

$$\leq \frac{\|\eta\|_{\infty}^{p}|\theta(\xi)|^{p}}{[q(\operatorname{Re}\lambda+\lambda^{*})]^{\frac{p}{q}}|\xi|^{\frac{p}{q}-1}}.$$

Integrating in ξ from 0 to 1 we obtain

$$\int_{0}^{1}\left[\int_{-a}^{a}\left|\frac{\eta(x)\theta(\xi)}{|\xi|}\exp\left\{-\frac{(\lambda+\sigma(\xi))|a-x|}{|\xi|}\right\}\right|^{q} dx\right]^{\frac{p}{q}}|\xi|\,d\xi$$

$$\leq \int_{0}^{1}\|\eta\|_{\infty}^{p}\frac{|\theta(\xi)|^{p}}{[q(\operatorname{Re}\lambda+\lambda^{*})]^{\frac{p}{q}}}\,d\xi$$

$$\leq \|\eta\|_{\infty}^{p}\frac{|\theta(\xi)|^{p}}{[q(\operatorname{Re}\lambda+\lambda^{*})]^{\frac{p}{q}}} < \infty.$$

Consider now the case $p = 1$,

$$\int_{0}^{1}|J_{\lambda}\psi|\,|\psi|\,d\xi$$

$$\leq \|\eta\|_{\infty}\int_{0}^{1}\int_{-a}^{a}|\theta(\xi)|\exp\left\{-\frac{(\operatorname{Re}\lambda+\sigma(\xi))|a-x|}{|\xi|}\right\}|\psi(x)|\,dx$$

$$\leq \|\eta\|_{\infty}\int_{0}^{1}|\theta(\xi)|\,d\xi\int_{-a}^{a}|\psi(x)|\,dx$$

$$\leq \|\eta\|_{\infty}\|\theta\|_{L_1}\|\psi\|_{L_1}.$$

This amounts to

$$\|J_{\lambda}\| \leq \|\eta\|_{\infty}\|\theta\|_{L_1}.$$

This inequality shows that J_{λ} depends continuously (in the uniform topology) on $\theta(\cdot) \in L_1(-1,1)$. But the set of bounded functions which vanish in a neighborhood of $\xi = 0$ is dense in $L_1(-1,1)$, so J_{λ} is a limit, in the uniform topology, of integral operators with bounded kernels. Hence J_{λ} is a weakly compact operator (see [7, Definition 1.4, p. 482, and Corollary 11, p. 294]).

A similar reasoning allows us to reach the same result for the operator $G_{\lambda}^{-}K$. This completes the proof of the first assertion.

(ii) To prove the compactness of $K(\lambda - T_H)^{-1}$, it suffices to observe that $K(\lambda - T_H)^{-1} = [(\lambda - T_H)^* K^*]^*$. Accordingly, using the same strategy as in the proof of [21, Theorem 2.2] and arguing as above we infer the compactness of $(\lambda - T_H^*)K^*$. Next, applying the Schauder theorem [7, Theorem 2, p. 485] we get the desired result. Q.E.D.

Now we turn to the investigation of the invariance of the essential spectra of transport operators.

THEOREM 3.1. *Let $p \in [1, \infty)$ and assume that the collision operator K is regular. Then*

$$\sigma_{ei}(A_H) = \sigma_{ei}(T_H), \qquad \text{for } i = 1, \dots, 5.$$

Moreover, if H is a strictly singular boundary operator, then

$$\sigma_{ei}(A_H) = \{\lambda \in \mathbb{C} \text{ such that } \operatorname{Re}\lambda \le -\lambda^*\}, \qquad \text{for } i = 1, \dots, 5.$$

Remark 3.5. Note that the statement concerning $\sigma_{e1}(\cdot)$, $\sigma_{e4}(\cdot)$, and $\sigma_{e5}(\cdot)$ was already considered in [23] for homogeneous regular collision operators (i.e., the scattering kernel $\kappa(\cdot,\cdot,\cdot)$ is independent of the space variable x). The proofs use the compactness results obtained in [21, Theorems 2.1 and 2.2]. On the other hand, $\sigma_{e2}(\cdot)$ and $\sigma_{e3}(\cdot)$ were discussed only for $p \in (1, 2]$. Theorem 3.1 may be regarded as an extension of [23, Theorem 4.2] to inhomogeneous regular collision operators, while the items concerning $\sigma_{e2}(\cdot)$ and $\sigma_{e3}(\cdot)$ for $p \in \{1\} \cup (2, \infty)$ are new.

Proof of Theorem 3.1. Let $\lambda \in \rho(T_H)$ be such that $r_\sigma[(\lambda - T_H)^{-1}K] < 1$. Then $\lambda \in \rho(T_H + K)$ and

$$(\lambda - A_H)^{-1} - (\lambda - T_H)^{-1} = \sum_{n \ge 1} \left[(\lambda - T_H)^{-1}K\right]^n (\lambda - T_H)^{-1}. \tag{3.14}$$

Next, since K is regular, we know by Proposition 3.2(i) that $(\lambda - A_H)^{-1} - (\lambda - T_H)^{-1}$ is compact (resp. weakly compact) on Y_p with $1 < p < \infty$ (resp. Y_1). Therefore, it follows from the inclusion $\mathscr{K}(Y_p) \subseteq \mathscr{S}(Y_p)$ for $p \in (1, \infty)$ and the fact that the set of weakly compact operators and that of strictly singular ones coincide on L_1 spaces (cf. [28, Theorem 1]) that $(\lambda - A_H)^{-1} - (\lambda - T_H)^{-1} \in \mathscr{S}(Y_p)$. Hence, Proposition 2.2 gives the first assertion. Analogously, writing (3.14) in the form (use (3.6))

$$(\lambda - A_H)^{-1} - (\lambda - T_0)^{-1} = \mathscr{R}_\lambda + \sum_{n \ge 1} \left[(\lambda - T_H)^{-1}K\right]^n (\lambda - T_H)^{-1}$$

and arguing as above we obtain the second item. Q.E.D.

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REFERENCES

1. P. M. Anselone and T. W. Palmer, Collectively compact sets of linear operators, *Pacific J. Math.* **25** (1968), 417–422.
2. R. Beals and V. Protopopescu, Abstract time-dependent transport equations, *J. Math. Anal. Appl.* **121** (1987), 370–405.
3. J. W. Calkin, Two-sided ideals and congruences in the ring of bounded operators in Hilbert spaces, *Ann. Math.* **42** (1941), 839–873.
4. R. Dautray and J. L. Lions, "Analyse Mathématique et Calcul Numérique," Vol. 9, Masson, Paris, 1988.
5. J. Diestel, "Geometry of Banach Spaces—Selected Topics," Lectures Notes in Mathematics, Vol. 485, Springer-Verlag, New York, 1975.
6. J. Diestel, A survey of results related to Dunford–Pettis property, *in* "Proc. of Conf. on Integration, Topology, and Geometry in Linear Spaces," Contemp. Math., Vol. 2, pp. 15–60, Amer. Math. Soc., Providence, 1980.
7. N. Dunford and J. T. Schwartz, "Linear Operators. Part I. General Theory," Wiley–Interscience, New York, 1958.
8. I. C. Gohberg and M. G. Krein, Fundamental theorems on deficiency numbers, root numbers and indices of linear operators, *in* Amer. Math. Soc. Transl. Ser. 2, Vol. 13, pp. 185–264, Amer. Math. Soc., Providence, 1960.
9. I. C. Gohberg, A. Markus, and I. A. Feldman, Normally solvable operators and ideals associated with them, *in* Amer. Math. Soc. Transl. Ser. 2, Vol. 61, pp. 63–84, Amer. Math. Soc., Providence, 1967.
10. S. Goldberg, "Unbounded Linear Operators," McGraw-Hill, New York, 1966.
11. W. Greenberg, C. Van der Mee, and V. Protopopescu, "Boundary Value Problems in Abstract Kinetic Theory," Birkhäuser, Basel, 1987.
12. K. Gustafson, Compact-like operators and the Eberlein theorem, *Amer. Math. Monthly* **75** (1968), 958–964.
13. K. Gustafson, On algebraic multiplicity, *Indiana Univ. Math. J.* **25** (1976), 769–781.
14. K. Gustafson, Spectral states, *Comput. Math. Appl.* **34** (1997), 467–508.
15. K. Gustafson and J. Weidmann, On the essential spectrum, *J. Math. Anal. Appl.* **25** (1969), 121–127.
16. R. H. Herman, On the uniqueness of the ideals of compact and strictly singular operators, *Studia Math.* **29** (1968), 161–165.
17. K. Jörgens, "Linear Integral Operators," Pitman, London, 1982.
18. H. G. Kaper, C. G. Lekkerkerker, and J. Hejtmanek, "Spectral Methods in Linear Transport Theory," Birkhäuser, Basel, 1982.
19. T. Kato, Perturbation theory for nullity, deficiency and other quantities of linear operators, *J. Anal. Math.* **6** (1958), 261–322.
20. T. Kato, "Perturbation Theory for Linear Operators," Springer-Verlag, New York, 1966.
21. K. Latrach, Compactness properties for linear transport operator with abstract boundary conditions in slab geometry, *Transport Theory Statist. Phys.* **22** (1993), 39–65.
22. K. Latrach and A. Dehici, Fredholm, semi-Fredholm perturbations and essential spectra, preprint.

23. K. Latrach and A. Jeribi, Some results on Fredholm operators, essential spectra and application, *J. Math. Anal. Appl.* **255** (1998), 461–485.
24. V. D. Milman, Some properties of strictly singular operators, *Funct. Anal. Appl.* **3** (1969), 77–78.
25. M. Mokhtar-kharroubi, "Mathematical Topics in Neutron Transport Theory, New Aspects," Adv. Math. Appl. Sci., Vol. 46, World Scientific, Singapore, 1997.
26. M. Mokhtar-kharroubi, Time asymptotic behaviour and compactness in neutron transport theory, *European J. Mech. B. Fluids* **11** (1992), 39–68.
27. A. Pelczynski, On strictly singular and cosingular operators. I. Strictly singular and cosingular operators on $C(S)$-spaces, *Bull. Acad. Polon. Sci.* **13** (1965), 31–36.
28. A. Pelczynski, On strictly singular and cosingular operators. II. Strictly singular and cosingular operators on $L(\nu)$-spaces, *Bull. Acad. Polon. Sci.* **13** (1965), 37–41.
29. F. Riesz and B. S. Nagy, "Leçons d'Analyse Fonctionnelle," Akad. Kiadó, Budapest, 1952.
30. M. Schechter, On the essential spectrum of an arbitrary operator, *J. Math. Anal. Appl.* **13** (1966), 205–215.
31. M. Schechter, Riesz operators and Fredholm perturbations, *Bull. Amer. Math. Soc.* **74** (1968), 1139–1144.
32. M. Schechter, "Principles of Functional Analysis," Academic Press, New York, 1971.
33. J. I. Vladimirskii, Strictly cosingular operators, *Soviet Math. Dokl.* **8** (1967), 739–740.
34. J. Voigt, "Functional Analytic Treatment of the Initial Boundary Value Problem for Collisionless Gases," Habilitationsschrift, München, 1981.
35. J. Voigt, Spectral properties of the neutron transport equation, *J. Math. Anal. Appl.* **106** (1985), 140–153.
36. L. Weis, On perturbations of Fredholm operators in $L_p(\mu)$-spaces, *J. Amer. Math. Soc.* **67** (1978), 287–292.
37. L. Weis, Perturbation classes of semi-Fredholm operators, *Math. Z.* **178** (1981), 429–442.
38. R. J. Whitley, Strictly singular operators and their conjugates, *Trans. Amer. Math. Soc.* **18** (1964), 252–261.
39. F. Wolf, On the essential spectrum of partial differential boundary problems, *Comm. Pure Appl. Math.* **12** (1959), 211–228.