

# Multi-objective $H_-/H_\infty$ fault detection observer design for Takagi–Sugeno fuzzy systems with unmeasurable premise variables: descriptor approach

S. Aouaouda<sup>1</sup>, M. Chadli<sup>2,\*</sup>,<sup>†</sup>, V. Cocquempot<sup>3</sup> and M. Tarek Khadir<sup>1</sup>

<sup>1</sup>*Laboratoire de Gestion Électronique de Document (LabGED), Annaba University, BP 12, 23200, Sidi-Amar, Annaba, Algeria*

<sup>2</sup>*MIS (E.A.4290), University of Picardie Jules Verne, 33 rue Saint-Leu, 80039 Amiens, France*

<sup>3</sup>*LAGIS UMR CNRS 8219 Lille 1 University, Villeneuve d'Ascq, France*

## SUMMARY

This paper investigates the problem of robust fault detection observer design for nonlinear Takagi–Sugeno models with unmeasurable premise variables subject to sensor faults and unknown bounded disturbance. The main idea is to synthesize a robust fault detection observer by means of a mixed  $H_-/H_\infty$  performance index. The considered observer is used to estimate jointly states and faults. Using the technique of descriptor system representation, we proposed a new less-conservative approach in term of a linear matrix inequality (LMI) by considering the sensor fault as an auxiliary state variable. A solution of the problem is obtained by using an iterative LMI procedure. Copyright © 2012 John Wiley & Sons, Ltd.

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## 1. INTRODUCTION

Model-based fault diagnosis techniques have shown their interest in the industrial domain. Recently, a considerable attention is given to this context in order to cope with diverse damages resulting in faults occurrence (see [1–6] and references therein). Faults can cause unacceptable economic loss or hazards to human operators and can lead to catastrophic consequences on the system itself or its environment. Therefore, it is important to provide on-line operating information by using a monitoring system [7–11]. In the literature, many results on fault detection observer have been reported for linear systems [2, 12] and nonlinear ones [13–15]. Two main criteria dealing with the above observer design must be considered. The first one is that the fault detection observers have to be robust, that is, insensitive to disturbances. The second guarantees the sensitivity to faults. For the two cases, a suitable performance index has to be optimized. For this purpose, several performance indexes are considered in the literature such as  $H_\infty$  [12, 16, 17],  $H_-$  [13], and mixed  $H_-/H_\infty$  criteria [2, 13, 18]. In recent years, the Takagi–Sugeno (T–S) fuzzy representation has attracted a growing interest because it is a powerful solution that bridges the gap between linear and nonlinear control systems. The important advantage of the T–S fuzzy model is its universal approximation of any smooth nonlinear function by a ‘blending’ of some local linear system models, which greatly facilitates observer/controller synthesis for complex nonlinear systems. Many results on fault detection observer design for T–S fuzzy systems have been reported in the literature [9, 19]. These works generally considered that the weighting functions depend on measurable premise variables [1, 2, 20].

\*Correspondence to: M. Chadli, MIS (E.A.4290), University of Picardie Jules Verne, 33 rue Saint-Leu, 80039 Amiens, France.

<sup>†</sup>E-mail: mohammed.chadli@u-picardie.fr

In the field of diagnosis, this assumption forces to design observers with weighting functions depending on the input  $u(t)$ , for the detection of the sensors faults, and on the output  $y(t)$ , for the detection of actuator faults. Indeed, if the decision variables are the inputs, for example, in a bank of observer, even if the  $i$ th observer is not controlled by the input  $u_i$ , this input appears indirectly in the weighting function and it cannot be eliminated. For this reason, it is interesting to consider the case of weighting functions depending on unmeasurable premise variables, such as the state of the system. This case makes it possible to handle a large class of physical systems [21–23].

Using descriptor approach, this work dealt with the problem of fault detection observer for Takagi–Sugeno (T-S) model affected by both sensor faults and bounded disturbances. Although many papers have dealt with the problem of observer design for descriptor systems, only a few works have been carried out for simultaneous disturbance rejection and fault detection algorithms [1]. Compared with existing fault estimation schemes [24, 25], the given descriptor observer approach leads to more suitable observer design, which is applicable to diagnosis of more general faults. The proposed procedure has the advantage, over the ones proposed on [26, 27], to estimate different faults types, whereas the proposed method in [26] is only able to estimate step faults. The problem formulation in a descriptor form allows also to estimate state and sensor faults simultaneously.

This paper aims to extend the results proposed in [4] to T-S models with unmeasurable premise variables. The present work illustrates the design of a fault detection observer for T-S model affected by sensor faults and unknown bounded disturbances. The observer gains and the residual weighting matrix are obtained through the minimization of an  $H_\infty$  norm and the maximization of an  $H_-$  norm. The main objective is to design a fault detection observer such that the resulting residual has the best robustness to disturbances and the best sensitivity to faults. Sufficient conditions are expressed in terms of linear matrix inequalities (LMIs), and an iterative algorithm is provided to get the solution. This algorithm can be solved effectively using numerical optimization techniques.

This paper is organized as follows. In the next section, the class of studied systems and the T-S fuzzy descriptor observer are presented. In Section 3, the problem of residual generation and disturbance attenuation is expressed. Section 4 is devoted to the robustness conditions on the fault detection observer, whereas the fault sensitivity conditions are presented in Section 5. The multi-objective  $H_-/H_\infty$  fault detection observer is then detailed in Section 6, and an iterative LMI algorithm is proposed. In the last section, a numerical example and a bioreactor model are considered to illustrate the efficiency of the proposed approach.

### Notation

The following notations are considered.  $H(P)$  denotes the Hermitian of the matrix  $P$ , that is,  $H(P) = P + P^T$ .  $I_n$  is the identity matrix of dimension  $n \times n$ , and the symbol \* indicates the transposed element in the symmetric positions of a matrix.

## 2. T-S FUZZY MODEL

Let us consider the following T-S structure model:

$$\begin{cases} \dot{x}_n(t) = \sum_{i=1}^r \mu_i(\xi(t))(A_i x_n(t) + B_i u(t)) \\ y_n(t) = C x_n(t) \end{cases} \quad (1)$$

where  $x_n(t) \in \mathbb{R}^n$ ,  $y_n(t) \in \mathbb{R}^p$ , and  $u(t) \in \mathbb{R}^{n_u}$  represent respectively the nominal state, the measured nominal output, and the bounded input vectors.  $\{A_i, B_i\}$  are the submodels matrices with appropriate dimensions. All  $A_i$  matrices are supposed to be stable.  $r$  is the number of submodels, and  $\mu_i(\xi(t))$  are the weighting functions depending on the variables  $\xi(t)$ , which can be measurable (as the input or the output of the system) or non-measurable variables (as the state of the system). These functions verify the convex sum property

$$\begin{cases} 0 \leq \mu_i(\xi(t)) \leq 1 \\ \sum_{i=1}^r \mu_i(\xi(t)) = 1 \quad \forall i \in \{1, 2, \dots, r\} \end{cases} \quad (2)$$

In this work, the considered premise variables  $\xi(t)$  can be partially or completely unavailable for measurement. We consider the continuous-time T-S fuzzy model to be affected by sensor faults and unknown bounded disturbances. Then the T-S fuzzy system (1) becomes

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^r \mu_i(\xi(t))(A_i x(t) + B_i u(t)) + B_d d(t) \\ y(t) = Cx(t) + D_f f(t) \end{cases} \quad (3)$$

where  $f(t) \in \mathbb{R}^s$  is the sensor fault vector, and  $d(t) \in \mathbb{R}^{n_d}$  is the unknown bounded disturbance vector. Matrices  $B_d$  and  $D_f$  are of appropriate dimensions, and  $D_f$  is assumed to be of full column rank. To ensure the estimation of both the state and sensor fault vectors, we first constructed an augmented system using the descriptor technique. The faulty system given by (3) can be rewritten as follows:

$$\begin{cases} \bar{E} \dot{\bar{x}}(t) = \sum_{i=1}^r \mu_i(\xi(t)) (\bar{A}_i \bar{x}(t) + \bar{B}_i u(t)) + \bar{B}_d d(t) + \bar{D}_h h(t) \\ y(t) = \bar{C} \bar{x}(t) = C_0 \bar{x}(t) + h(t) \end{cases} \quad (4)$$

where

$$h(t) = D_f f(t) \in \mathbb{R}^p, \quad \bar{x}(t)^T = [x(t)^T h(t)^T] \in \mathbb{R}^{n+p} \quad (5a)$$

$$\bar{E} = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{A}_i = \begin{bmatrix} A_i & 0 \\ 0 & -I_p \end{bmatrix}, \quad \bar{B}_i = \begin{bmatrix} B_i \\ 0 \end{bmatrix}, \quad \bar{B}_d = \begin{bmatrix} B_d \\ 0 \end{bmatrix} \quad (5b)$$

$$\bar{D}_h = \begin{bmatrix} 0 \\ I_p \end{bmatrix}, \quad C_0 = [C \quad 0], \quad \bar{C} = [C \quad I_p] \quad (5c)$$

We consider an observer under the usual form:

$$\begin{cases} E \dot{z}(t) = \sum_{i=1}^r \mu_i(\hat{\xi}(t)) (F_i z(t) + \bar{B}_i u(t)) \end{cases} \quad (6a)$$

$$\begin{cases} \hat{x}(t) = z(t) + Ly(t) \end{cases} \quad (6b)$$

$$\begin{cases} \hat{y}(t) = C_0 \hat{x}(t) = C \hat{x}(t) \end{cases} \quad (6c)$$

where  $z(t) \in \mathbb{R}^{n+p}$  is the auxiliary state vector of the observer, and  $\hat{x}(t) \in \mathbb{R}^{n+p}$  is the estimate state.  $\hat{\xi}(t)$  is the unmeasured premise variable depending partially or completely on the estimated state  $\hat{x}(t)$ .  $F_i$ ,  $E$ , and  $L$  are the observer gains to be determined.

### 3. RESIDUAL GENERATION AND DISTURBANCE ATTENUATION

Let us define the state error  $e(t)$  and the residual signal  $r(t)$ :

$$r(t) = V(y(t) - \hat{y}(t)) \quad (7a)$$

$$e(t) = \bar{x}(t) - \hat{x}(t) \quad (7b)$$

where  $V$  is a weighting matrix.

#### Definition 3.1

Given the fuzzy system (3), two scalars  $\gamma > 0$  and  $\beta > 0$ . The observer (6) is called an  $H_-/H_\infty$  fault detection observer if (6) is asymptotically stable, and the following inequalities are satisfied:

$$\int_0^\infty r^T(t)r(t)dt \leq \gamma^2 \int_0^\infty d^T(t)d(t)dt \quad (8)$$

$$\int_0^\infty r^T(t)r(t)dt \geq \beta^2 \int_0^\infty f^T(t)f(t)dt \tag{9}$$

The goal is to find an admissible observer (6) to minimize  $\gamma$  and maximize  $\beta$ , that is, an observer that generates residual signals that have the best robustness to disturbances ( $d(t)$ ) and a maximal sensitivity to faults ( $f(t)$ ). In the following, we give conditions to design the fuzzy observer, and we also give a bound for the estimation error.

To cope with the difficulty of expressing the augmented state estimation error dynamic in a tractable way, Equation (4), is rewritten, on the basis of the property (2),

$$\begin{aligned} \bar{E}\dot{\hat{x}}(t) &= \sum_{i=1}^r \sum_{j=1}^r \mu_i(\xi(t))\mu_j(\hat{\xi}(t)) ((\bar{A}_i + \bar{A}_j - \bar{A}_j)\bar{x}(t) + (\bar{B}_i + \bar{B}_j - \bar{B}_j)u(t)) \\ &+ \bar{B}_d d(t) + \bar{D}_h h(t) \end{aligned} \tag{10}$$

Using (6a)–(6c), we get

$$\bar{E}\dot{\hat{x}}(t) - E\dot{\hat{x}}(t) = \bar{E}\dot{\hat{x}}(t) - E(\dot{z}(t) + L\bar{C}\dot{\hat{x}}(t))$$

It follows

$$(\bar{E} + EL\bar{C})\dot{\hat{x}}(t) - E\dot{\hat{x}}(t) = \bar{E}\dot{\hat{x}}(t) - E\dot{z}(t)$$

Then taking account (10) and (6a), we get

$$\begin{aligned} (\bar{E} + EL\bar{C})\dot{\hat{x}}(t) - E\dot{\hat{x}}(t) &= \sum_{i=1}^r \sum_{j=1}^r \mu_i(\xi(t))\mu_j(\hat{\xi}(t)) ((\bar{A}_i + \bar{A}_j - \bar{A}_j)\bar{x}(t) + (\bar{B}_i + \bar{B}_j - \bar{B}_j)u(t)) \\ &+ \bar{B}_d d(t) + \bar{D}_h h(t) - \sum_{i=1}^r \mu_i(\hat{\xi}(t)) (F_i z(t) + \bar{B}_i u(t)) \end{aligned}$$

which is equivalent to

$$\begin{aligned} (\bar{E} + EL\bar{C})\dot{\hat{x}}(t) - E\dot{\hat{x}}(t) &= \sum_{i=1}^r \sum_{j=1}^r \mu_i(\xi(t))\mu_j(\hat{\xi}(t)) [\bar{A}_j e(t) + (\bar{A}_i - \bar{A}_j)\bar{x}(t) \\ &+ \bar{B}_d d(t) - (F_j - \bar{A}_j)\hat{\hat{x}}(t) + F_j L C_0 \bar{x}(t) \\ &+ (\bar{B}_i - \bar{B}_j)u(t) + (F_j L + \bar{D}_h)h(t)] \end{aligned} \tag{11}$$

Consider the following matrices  $F_j \in \mathbb{R}^{(n+p).(n+p)}$ ,  $L \in \mathbb{R}^{n.p}$ , and  $E \in \mathbb{R}^{(n+p).(n+p)}$

$$F_j = \begin{bmatrix} A_j & 0 \\ -C & -I_p \end{bmatrix}, L = \begin{bmatrix} 0 \\ I_p \end{bmatrix}, E = \begin{bmatrix} I_n + QC & Q \\ RC & R \end{bmatrix} \tag{12}$$

where  $Q \in \mathbb{R}^{n.p}$  and  $R \in \mathbb{R}^{p.p}$  are chosen as non-singular, and we have

$$E = \bar{E} + EL\bar{C}, F_j = \bar{A}_j + F_j L C_0, F_j L = -\bar{D}_h \tag{13}$$

Then from (11), we obtain:

$$\begin{aligned} E\dot{e}(t) &= \sum_{i=1}^r \sum_{j=1}^r \mu_i(\xi(t))\mu_j(\hat{\xi}(t)) [(F_j L C_0 + \bar{A}_j)e(t) \\ &+ (\bar{A}_i - \bar{A}_j)\bar{x}(t) + (\bar{B}_i - \bar{B}_j)u(t) + \bar{B}_d d(t)] \end{aligned} \tag{14}$$

also

$$(\bar{A}_i - \bar{A}_j)\bar{x}(t) = \begin{bmatrix} A_i - A_j \\ 0 \end{bmatrix} x(t) \tag{15}$$

Therefore, (14) is equivalent to

$$\dot{e}(t) = \sum_{i=1}^r \sum_{j=1}^r \mu_i(\xi(t)) \mu_j(\hat{\xi}(t)) [S_j e(t) + \tilde{A}_{ij} x(t) + \tilde{B}_{ij} u(t) + G d(t)] \quad (16)$$

where

$$E^{-1} = \begin{bmatrix} I_n & -QR^{-1} \\ -C & R^{-1} + CQR^{-1} \end{bmatrix} \quad (17)$$

$$S_j = E^{-1} F_j = \begin{bmatrix} A_j + QR^{-1}C & QR^{-1} \\ -CA_j - (R^{-1} + CQR^{-1})C & -R^{-1} - CQR^{-1} \end{bmatrix} \quad (18)$$

$$\tilde{A}_{ij} = E^{-1} \begin{bmatrix} A_i - A_j \\ 0 \end{bmatrix} = \begin{bmatrix} A_i - A_j \\ -C(A_i - A_j) \end{bmatrix} \quad (19)$$

$$\tilde{B}_{ij} = E^{-1} [\bar{B}_i - \bar{B}_j] = \begin{bmatrix} B_i - B_j \\ -C(B_i - B_j) \end{bmatrix} \quad (20)$$

$$G = E^{-1} \bar{B}_d = \begin{bmatrix} B_d \\ -CB_d \end{bmatrix} \quad (21)$$

Consequently, the augmented state estimation error responds to the following nonlinear system:

$$r(t) = V(C_0 e(t) + h(t)) \quad (22a)$$

$$\begin{bmatrix} \dot{e}(t) \\ \dot{x}(t) \end{bmatrix} = \sum_{i=1}^r \sum_{j=1}^r \mu_i(\xi(t)) \mu_j(\hat{\xi}(t)) \times \left\{ \begin{bmatrix} S_j & \tilde{A}_{ij} \\ 0 & A_i \end{bmatrix} \cdot \begin{bmatrix} e(t) \\ x(t) \end{bmatrix} + \begin{bmatrix} \tilde{B}_{ij} & G \\ B_i & B_d \end{bmatrix} \cdot \begin{bmatrix} u(t) \\ d(t) \end{bmatrix} \right\} \quad (22b)$$

$$e(t) = \begin{bmatrix} I_{n+s} & 0 \end{bmatrix} \begin{bmatrix} e(t) \\ x(t) \end{bmatrix} \quad (22c)$$

*Remark 1*

$R$  and  $Q$  are free matrices, which must be chosen to ensure the non-singularity of matrix  $E$ . In addition, the dynamics of the residual signal depends not only on fault  $f(t)$  but also on the state  $x(t)$ , input  $u(t)$ , and disturbance  $d(t)$ . Thus the problem of designing the observer can be described as designing matrices  $R$ ,  $Q$  (i.e., finding the observer gain  $E$ ) and  $V$  such that

- the generated residual  $r(t)$  is as sensitive as possible to fault  $f(t)$  and as robust as possible to unknown disturbance  $d(t)$ ;
- the  $\mathcal{L}_2$  gain from the input  $u(t)$  to the state and fault estimation error  $e(t)$  is minimized; and
- the matrices  $\begin{bmatrix} S_j & \tilde{A}_{ij} \\ 0 & A_i \end{bmatrix}$  are quadratically asymptotically stable.

When  $d(t) = 0$ , we have

$$r_f(t) = V(C_0 e_f(t) + D_f f(t)) \quad (23a)$$

$$\begin{bmatrix} \dot{e}_f(t) \\ \dot{x}(t) \end{bmatrix} = \sum_{i=1}^r \sum_{j=1}^r \mu_i(\xi(t)) \mu_j(\hat{\xi}(t)) \times \left\{ \begin{bmatrix} S_j & \tilde{A}_{ij} \\ 0 & A_i \end{bmatrix} \cdot \begin{bmatrix} e_f(t) \\ x(t) \end{bmatrix} + \begin{bmatrix} \tilde{B}_{ij} \\ B_i \end{bmatrix} \cdot u(t) \right\} \quad (23b)$$

and when  $f(t) = 0$ , we have

$$r_d(t) = VC_0 e_d(t) \tag{24a}$$

$$\begin{bmatrix} \dot{e}_d(t) \\ \dot{x}(t) \end{bmatrix} = \sum_{i=1}^r \sum_{j=1}^r \mu_i(\xi(t)) \mu_j(\hat{\xi}(t)) \times \left\{ \begin{bmatrix} S_j & \tilde{A}_{ij} \\ 0 & A_i \end{bmatrix} \cdot \begin{bmatrix} e_d(t) \\ x(t) \end{bmatrix} + \begin{bmatrix} \tilde{B}_{ij} & G \\ B_i & B_d \end{bmatrix} \cdot \begin{bmatrix} u(t) \\ d(t) \end{bmatrix} \right\} \tag{24b}$$

In the following two sections, expressions (23) and (24) will be independently used to study the problems of robustness and sensitivity.

#### 4. $H_\infty$ ROBUSTNESS CONDITIONS

In this section, only robustness against disturbance is studied by considering the  $H_\infty$  performance index.

##### Lemma 1

If there exist symmetric positive definite matrices  $P_1 \in \mathbb{R}^{(n+s) \times (n+s)}$  and  $P_2 \in \mathbb{R}^{(n_u+n_d) \times (n_u+n_d)}$  for a given constant  $\lambda > 0$ , such that the following conditions are satisfied for  $i, j = 1, \dots, r$

$$\begin{bmatrix} Z_j & * & * & * \\ \tilde{A}_{ij}^T P_1 & \mathcal{H}(P_2 A_i) & * & * \\ \tilde{B}_{ij}^T P_1 & B_i^T P_2 & -\lambda^2 I_{n_u} & * \\ G^T P_1 & B_d^T P_2 & 0 & -\gamma^2 I_{n_d} \end{bmatrix} < 0 \tag{25}$$

with

$$Z_j = \mathcal{H}(P_1 S_j) + C_0^T V^T V C_0 + I_n \tag{25a}$$

Then the system (24) is stable with  $\gamma$ -disturbance attenuation and the  $\mathcal{L}_2$ -gain from  $u(t)$  to  $e_d(t)$  is bounded by  $\lambda$ .

##### Proof

Let  $V(e_d(t), x(t))$  denote the following candidate Lyapunov function:

$$V(e_d(t), x(t)) = \begin{bmatrix} \dot{e}_d(t) \\ \dot{x}(t) \end{bmatrix}^T \times \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \begin{bmatrix} e_d(t) \\ x(t) \end{bmatrix} \tag{26}$$

where  $P_1, P_2$  are symmetric positive definite matrices. By respecting the criterion (8), that is,  $\int_0^\infty r_d^T(t) r_d(t) \leq \gamma^2 \int_0^\infty d^T(t) d(t) dt$ , the  $\mathcal{L}_2$  gain from  $u(t)$  to  $e_d(t)$  is bounded by  $\lambda$  if

$$\dot{V}(e_d(t), x(t)) + e_d^T(t) e_d(t) + r_d^T(t) r_d(t) - \lambda^2 u^T(t) u(t) - \gamma^2 d^T(t) d(t) < 0 \tag{27}$$

Considering the Lyapunov function (26) and the trajectory of  $e_d(t)$  defined by (24b), the inequality (27) can be written as

$$\sum_{i=1}^r \sum_{j=1}^r \mu_i(\xi(t)) \mu_j(\hat{\xi}(t)) \begin{bmatrix} e_d(t) \\ x(t) \\ u(t) \\ d(t) \end{bmatrix}^T \times \begin{bmatrix} Z_j & * & * & * \\ \tilde{A}_{ij}^T P_1 & \mathcal{H}(P_2 A_i) & * & * \\ \tilde{B}_{ij}^T P_1 & B_i^T P_2 & -\lambda^2 I_{n_u} & * \\ G^T P_1 & B_d^T P_2 & 0 & -\gamma^2 I_{n_d} \end{bmatrix} \begin{bmatrix} e_d(t) \\ x(t) \\ u(t) \\ d(t) \end{bmatrix} < 0 \tag{28}$$

It follows that (28) is satisfied if the LMI (25) holds. The following result derived in LMI terms guarantees the robustness against the disturbance.  $\square$

**Theorem 1**

Consider the system (3) with observer (6), system (24) is asymptotically stable satisfying (8), for a given constant  $\gamma$ , and minimizing the  $\mathcal{L}_2$  gain  $\lambda$  if there exist some symmetric positive definite matrices  $P_{11}, P_{12}, P_2$ , matrices  $N_1, N_2$ , and  $V$  such that the following LMI are satisfied:

$$\mathcal{M}_{1ij} \leq 0 \text{ for } i, j = 1, \dots, r \quad (29)$$

where  $\mathcal{M}_{1ij}$  is defined by

$$\mathcal{M}_{1ij} = \begin{bmatrix} \Delta_{1j} & * & * & * & * & * \\ \Delta_{2j} & -\mathcal{H}(N_2) & * & * & * & * \\ \tilde{A}_{ij}^T P_{11} & -\tilde{A}_{ij}^T C^T P_{12} & \mathcal{H}(P_2 A_i) & * & * & * \\ \mathcal{K}_{ij} & -\tilde{B}_{ij}^T C^T P_{12} & B_i^T P_2 & -\lambda^2 I_{n_u} & * & * \\ B_d^T P_{11} & -B_d^T C^T P_{12} & B_d^T P_2 & 0 & -\gamma^2 I_{n_d} & * \\ VC & 0 & 0 & 0 & 0 & -I_n \end{bmatrix} \quad (30)$$

with

$$\begin{cases} \Delta_{1j} = \mathcal{H}(P_{11} A_j) + \mathcal{H}(N_1 C) + I_n \\ \Delta_{2j} = N_1^T - P_{12}^T C A_j - N_2 C \end{cases} \quad (31a)$$

$$\begin{cases} N_1 = P_{11} Q R^{-1} \\ N_2 = P_{12} (R^{-1} + C Q R^{-1}) \end{cases} \quad (31b)$$

and

$$\begin{cases} \tilde{A}_{ij}^T = A_i^T - A_j^T, \tilde{B}_{ij}^T = B_i^T - B_j^T \\ \mathcal{K}_{ij} = (B_i^T - A_j^T) P_{11} \end{cases} \quad (31c)$$

**Proof**

On the basis of Lemma 1 and by using the Schur complement, we get

$$\begin{bmatrix} \mathcal{H}(P_1 S_j) + I_n & * & * & * & * \\ \tilde{A}_{ij}^T P_1 & \mathcal{H}(P_2 A_i) & * & * & * \\ \tilde{B}_{ij}^T P_1 & B_i^T P_2 & -\lambda^2 I_{n_u} & * & * \\ G^T P_1 & B_d^T P_2 & 0 & -\gamma^2 I_{n_d} & * \\ VC_0 & 0 & 0 & 0 & -I \end{bmatrix} < 0 \quad (32)$$

By considering  $P_1 = \text{diag} [P_{11} \ P_{12}]$ , and Equations (18)–(21), the inequality (32) becomes

$$\begin{bmatrix} \Theta_j^{1,1} & * & * & * & * & * \\ \Theta_j^{2,1} & \Theta_j^{2,2} & * & * & * & * \\ \tilde{A}_{ij}^T P_{11} & -\tilde{A}_{ij}^T C^T P_{12} & \Theta_j^{3,3} & * & * & * \\ \mathcal{K}_{ij} & -\tilde{B}_{ij}^T C^T P_{12} & B_i^T P_2 & -\lambda^2 I_{n_u} & * & * \\ B_d^T P_{11} & -B_d^T C^T P_{12} & B_d^T P_2 & 0 & -\gamma^2 I_{n_d} & * \\ VC & 0 & 0 & 0 & 0 & -I \end{bmatrix} < 0 \quad (33)$$

where

$$\begin{cases} \Theta_j^{1,1} = \mathcal{H}(P_{11} A_j) + \mathcal{H}(P_{11} Q R^{-1} C) + I_n \\ \Theta_j^{2,1} = (Q R^{-1})^T P_{11} - P_{12}^T C A_j - P_{12} (R^{-1} + C Q R^{-1}) C \\ \Theta_j^{2,2} = -\mathcal{H}(P_{12} (R^{-1} + C Q R^{-1})) \\ \tilde{A}_{ij}^T = A_i^T - A_j^T, \tilde{B}_{ij}^T = B_i^T - B_j^T \\ \mathcal{K}_{ij} = (B_i^T - A_j^T) P_{11} \\ \Theta_j^{3,3} = \mathcal{H}(P_2 A_i) \end{cases} \quad (34)$$

Unfortunately, conditions (33) are not jointly convex because of the coupling of  $P_{11}$ ,  $P_{12}$ ,  $Q$ , and  $R$ . By introducing the variable changes defined by (31b), we obtain LMI (29).  $\square$

*Remark 2*

As  $D_f$  is assumed to be of full column rank, the sensor faults estimation can be obtained by

$$\hat{f}(t) = \left( D_f^T D_f \right)^{-1} D_f^T \hat{h}(t) \tag{35}$$

This assumption seems to be too strong in some cases, and the condition can be omitted using finite frequency domain method [28, 29].

### 5. $H_-$ FAULT SENSITIVITY CONDITIONS

This section is devoted to the sensitivity problem of the residual  $r(t)$  with respect to fault  $f(t)$ . In fact, our objective is to make the residual as sensitive as possible to fault. To achieve this goal, the  $H_-$  index is used hereafter.

*Theorem 2*

Consider the system (3) with observer (6), system (24) is asymptotically stable satisfying (9), for a given positive constant  $\beta$ , and minimizing the  $\mathcal{L}_2$  gain  $\lambda$  if there exist some symmetric positive definite matrices  $P_{11}$ ,  $P_{12}$ ,  $P_2$ , matrices  $N_1$ ,  $N_2$ , and  $V$  such that the following LMI are satisfied:

$$\bar{\mathcal{M}}_{1ij} \leq 0 \text{ for } i, j = 1, \dots, r \tag{36}$$

where  $\bar{\mathcal{M}}_{1ij}$  is defined by

$$\bar{\mathcal{M}}_{1ij} = \begin{bmatrix} \Delta_{1j} & * & * & * & * & * \\ \Delta_{2j} & \mathcal{H}(N_2) & * & * & * & * \\ \tilde{A}_{ij}^T P_{11} & -\tilde{A}_{ij}^T C^T P_{12} & -\mathcal{H}(P_2 A_i) & * & * & * \\ \mathcal{K}_{ij} & -\tilde{B}_{ij}^T C^T P_{12} & B_i^T P_2 & -\lambda^2 I_{n_u} & * & * \\ L & 0 & 0 & 0 & M & * \\ VC & 0 & 0 & 0 & 0 & -I \end{bmatrix} \tag{37}$$

with  $\Delta_{1j}$  and  $\Delta_{2j}$  are defined by (31a) and  $N_1$  and  $N_2$  are given by (31b).

*Proof*

Let  $V(e_f(t), x(t))$  denote the following candidate Lyapunov function:

$$V(e_f(t), x(t)) = \begin{bmatrix} e_f(t) \\ \dot{x}(t) \end{bmatrix}^T \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \begin{bmatrix} e_f(t) \\ x(t) \end{bmatrix} \tag{38}$$

where  $P_1$ ,  $P_2$  are symmetric positive definite matrices. Note that we seek to minimize the  $\mathcal{L}_2$ -gain of the transfer from  $u(t)$  to the estimation error vector  $e_f(t)$ ; this is formulated by:

$$\|u\|_2 \neq 0, \frac{\|e_f\|_2}{\|u\|_2} < \lambda^2 \tag{39}$$



Then we get by considering (23)

$$\begin{aligned}
 J_- &= \int_0^\infty r_f^T r_f d\tau - \beta^2 \int_0^\infty f^T f d\tau = \int_0^\infty \left[ r_f^T r_f - \beta^2 f^T f - \frac{d(V(e_f, x))}{dt} \right] d\tau + V(e_f, x) \\
 &= \int_0^\infty \left( [C^* e_f + D_f f]^T V^T V [C^* e_f + D_f f] - \beta^2 f^T f \right. \\
 &\quad \left. - \sum_{i=1}^r \sum_{j=1}^r \mu_i(\xi(t)) \mu_j(\hat{\xi}(t)) \times \left\{ \begin{bmatrix} e_f \\ x \end{bmatrix}^T \mathcal{H} \left( \begin{bmatrix} S_j & \tilde{A}_{ij} \\ 0 & A_i \end{bmatrix}^T \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \right) \begin{bmatrix} e_f \\ x \end{bmatrix} \right. \right. \\
 &\quad \left. \left. + \begin{bmatrix} e_f \\ x \end{bmatrix}^T \mathcal{H} \left( \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}^T \begin{bmatrix} \tilde{B}_{ij} \\ B_i \end{bmatrix} \cdot u(t) \right) \begin{bmatrix} e_f \\ x \end{bmatrix} \right\} \right) d\tau + V(e_f, x) \\
 &= \int_0^\infty \left( \sum_{i=1}^r \sum_{j=1}^r \mu_i(\xi(t)) \mu_j(\hat{\xi}(t)) \times \left\{ \begin{bmatrix} e_f \\ x \\ u \\ f \end{bmatrix}^T \mathfrak{F}_{ij} \begin{bmatrix} e_f \\ x \\ u \\ f \end{bmatrix} \right\} \right) d\tau + V(e_f, x)
 \end{aligned} \tag{40}$$

where

$$\mathfrak{F}_{ij} = \begin{bmatrix} \bar{Z}_j & * & * & * \\ -\tilde{A}_{ij}^T P_1 & -\mathcal{H}(P_2 A_i) & * & * \\ -\tilde{B}_{ij}^T P_1 & -B_i^T P_2 & \lambda^2 I_{n_u} & * \\ L & 0 & 0 & M \end{bmatrix} \tag{41}$$

with

$$L = (VD_f)^T (VC_0), \quad \bar{Z}_j = -\mathcal{H}(P_1 S_j) + C_0^T V^T V C_0 + I_n, \quad M = -\beta^2 I_s + (VD_f)^T (VD_f)$$

Hence if  $\mathfrak{F}_{ij} \geq 0$ , it follows that  $J_- \geq 0$ , that is,

$$\begin{bmatrix} -\bar{Z}_j & * & * & * \\ -\tilde{A}_{ij}^T P_1 & \mathcal{H}(P_2 A_i) & * & * \\ -\tilde{B}_{ij}^T P_1 & -B_i^T P_2 & -\lambda^2 I_{n_u} & * \\ L & 0 & 0 & -M \end{bmatrix} < 0 \tag{42}$$

By considering  $P_1 = \text{diag}[P_{11} \ P_{12}]$ , and Equations (18)–(21), the Schur complement and variable changes defined by (41b), the conditions (36) are fulfilled where  $\Delta_{1j}$  and  $\Delta_{2j}$  are defined by (41a), which achieves the proof.  $\square$

## 6. MULTI-OBJECTIVE $H_-/H_\infty$ FAULT DETECTION OBSERVER DESIGN

In this section, we propose to mix  $H_-$  and  $H_\infty$  performances where the goal is to design a robust fault detection observer, which generates residual signals that have the best robustness to disturbances and the best sensitivity to fault. The following theorem is proposed:

### Theorem 3

Consider the system (3) with observer (6), and system (7) is asymptotically stable satisfying (8) and (9) and minimizing the  $\mathcal{L}_2$  gain  $\lambda$  if there exist some symmetric positive definite matrices  $P_{11}, P_{12}, P_2$ , matrices  $N_1, N_2$ , and  $V$  such that LMI  $\mathcal{M}_{1ij}$  and  $\tilde{\mathcal{M}}_{1ij}$ , respectively defined by (30) and (37), hold.

The objective is to find  $Q, R$ , and  $V$ , which satisfy the performances (8) and (9). However, as stated earlier, the mixed  $H_-$  and  $H_\infty$  performances given by Theorem 3 lead to a nonlinear problem in  $V$ . To solve this problem, consider the following variable changes:

$$V_f^k = V^{k-1} D_f \tag{43a}$$

$$V_c^k = V^{k-1}C \tag{43b}$$

According to Theorem 3, using the Schur complement theorem,  $\bar{\mathcal{M}}_{1ij}$  is substituted by the following matrix:

$$\begin{bmatrix} \psi & * & * & * & * & * \\ \Delta_{2j} & \mathcal{H}(N_2) & * & * & * & * \\ \tilde{A}_{ij}^T P_{11} & -\tilde{A}_{ij}^T C^T P_{12} & \mathcal{H}(P_2 A_i) & * & * & * \\ \mathcal{K}_{ij} & -\tilde{B}_{ij}^T C^T P_{12} & -B_i^T P_2 & -\lambda^2 I_{n_u} & * & * \\ 0 & 0 & 0 & 0 & \mathcal{Q} & * \\ VC & 0 & 0 & 0 & VD_f & -I_n \end{bmatrix} \tag{44}$$

where

$$\psi = \Delta_{1j} + 2G_1(V, V_c^K), \quad \mathcal{Q} = \beta^2 I_s + 2G_2(V, V_f^K) \tag{45a}$$

$$G_1(V, V_c^K) = (V_c^K)^T V_c^K - (V_c^K)^T VC - C^T V^T V_c^K \tag{45b}$$

$$G_2(V, V_f^K) = (V_f^K)^T V_f^K - (V_f^K)^T VD_f - D_f^T V^T V_f^K \tag{45c}$$

Considering a starting point  $V^0$ , an iterative algorithm can be used to solve such problem. Thus, to find a suitable initial value  $V^0$ , a solution consists in solving  $\mathcal{M}_{1ij}$  or  $\mathcal{M}_{2ij}$  for given values of  $\gamma$  and  $\beta$ . The following algorithm summarizes the method:

- (1) Fix a value of  $\beta > 0$ .
- (2) Solve LMI  $\mathcal{M}_{1ij}$  or  $\mathcal{M}_{2ij}$  to find feasible solutions  $P_{11} P_{12}$  or  $X_{11} X_{12}$ , matrices  $P_2, N_1, N_2, \lambda$  and  $V^{k-1}, k = 1$ .
- (3) Include  $V^{k-1}$  into LMI  $\bar{\mathcal{M}}_{1ij}$  or  $\bar{\mathcal{M}}_{2ij}$  and set  $V_f^k = V^{k-1} D_f V_c^k = V^{k-1} C$  to find a feasible solution  $P_{11}, P_{12}$  or  $X_{11}, X_{12}$ , matrices  $P_2 N_1 N_2 \lambda, \gamma$ , and  $V^k$ .
- (4) Increase  $\beta, k = k + 1$  and go to step 3 if a feasible solution cannot be found, then stop.

Recall that when  $\gamma$  and  $\beta$  cannot be improved, we deduce from (31b) the gains of the observer (6) satisfying the multi-objective  $H_-/H_\infty$  performance as follows:

- (1)  $R = (P_{12}^{-1} N_2 - C P_{11}^{-1} N_1)^{-1}$ ;
- (2)  $Q = P_{11}^{-1} N_1 R$ ;
- (3) then  $F_i, L$ , and  $E$  are computed from (12).

*Remark 3*

Note that the proposed approach cannot guarantee an optimal solution because of interactions between disturbance and faults. Coupling terms introduce a sub-optimality of the result that it seems difficult to consider without knowledge of the evolution of disturbances and faults. However, by minimizing and maximizing  $H_\infty/H_-$  performances, we try to obtain a high sensitivity to faults (but not the strongest) and low sensitivity to disturbance (but not the lowest). The given examples illustrate this objective.

## 7. SIMULATION EXAMPLE

### 7.1. Example 1—numerical example

Let us consider the following T-S model:

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^2 \mu_i(\xi(t)) (A_i x(t) + B_i u(t)) + B_d d(t) \\ y(t) = Cx(t) + D_f f(t) \end{cases} \tag{46}$$

with

$$A_1 = \begin{bmatrix} -5 & 3 \\ 1 & -3 \end{bmatrix}, A_2 = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, C = \begin{bmatrix} 0.1 & -0.1 \\ -0.1 & 0 \end{bmatrix},$$

$$D_f = \begin{bmatrix} 4 & 0 \\ 0 & 8 \end{bmatrix}, B_d = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}$$

The weighting functions depend only on the first state variable  $x_1(t)$ . They are defined by the following membership functions:

$$\begin{aligned} \mu_1(x_1(t)) &= (1 - \tanh(0.5 - x_1(t)))/2 \\ \mu_2(x_1(t)) &= 1 - \mu_1(x_1(t)) \end{aligned} \quad (47)$$

Let us consider the following fault signal  $f(t) = (f_1(t) f_2(t))^T$  affecting the system behavior and described as follows:

$$f_1(t) = \sin(5(t - 0.4))e^{1.6t-14} \text{ occurs at } 6 \text{ s} \leq t \leq 9 \text{ s} \quad (48)$$

$$f_2(t) = \begin{cases} 0.02(t - 1) & 12 \text{ s} \leq t < 14 \text{ s} \\ 0.01(t - 1) & 14 \text{ s} \leq t \leq 16 \text{ s} \\ 0 & \text{otherwise} \end{cases} \quad (49)$$

An unknown disturbance  $d(t)$  with band-limited white noise as given by Figure 1 is considered. Thus the simulation results are illustrated.

To show the sensitivity of the residual signal  $r(t)$  to the faults  $f_1(t)$ ,  $f_2(t)$ , we perform two simulations: In the first one, robustness against disturbance is considered by applying the  $H_\infty$  conditions in Theorem 1. In this case, Figure 2 shows the residual trajectories of  $r_1(t)$  and  $r_2(t)$ . The second case concerns the multi-objective  $H_-/H_\infty$  observer design where Theorem 3 applied. As a result, when  $\gamma$  is reduced to 0.6 and  $\beta$  is increased to 4, we compute  $V$ ,  $R$ , and  $Q$

$$V = \begin{bmatrix} 37.2858 & 30.6721 \\ -55.6939 & 121.7611 \end{bmatrix}, R = \begin{bmatrix} 7.3800 & -12.4450 \\ 0.1919 & -0.0312 \end{bmatrix}, Q = \begin{bmatrix} 2.0546 & 6.5354 \\ -2.7236 & 8.9490 \end{bmatrix}$$

The corresponding observer gain matrices and residual weighing matrix are

$$F_1 = \begin{bmatrix} -5 & 3 & 0 & 0 \\ 1 & -3 & 0 & 0 \\ -0.1 & 0.1 & -1 & 0 \\ 0.1 & 0 & 0 & -1 \end{bmatrix}, F_2 = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -0.1 & 0.1 & -1 & 0 \\ 0.1 & 0 & 0 & -1 \end{bmatrix}, L = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^T$$

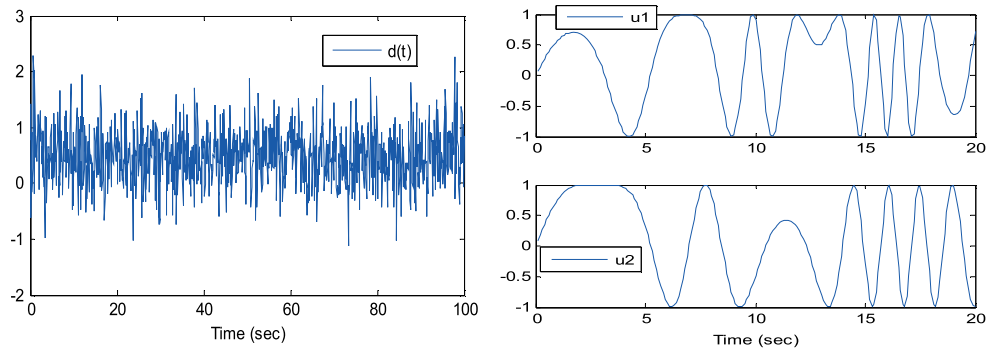


Figure 1. Disturbance  $d(t)$  (left) and inputs (right).

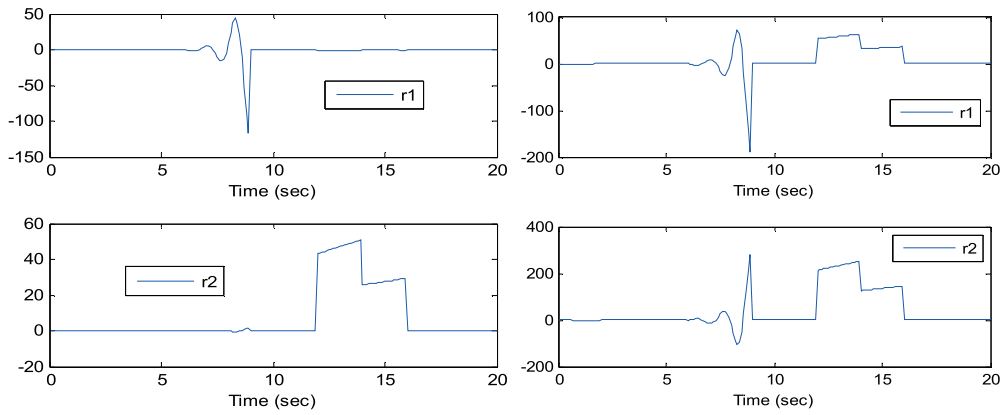


Figure 2. Generated residuals  $r_1, r_2$  Theorem 1(left) and Theorem 3(right).

$$E = \begin{bmatrix} 0.5519 & -0.2055 & 2.0546 & 6.5354 \\ -1.1673 & 1.2724 & -2.7236 & 8.9490 \\ 1.9825 & -0.7380 & 7.3800 & -12.4450 \\ 0.0223 & -0.0192 & 0.1919 & -0.0312 \end{bmatrix}$$

Figure 2 illustrates the residual trajectories of  $r_1(t)$  and  $r_2(t)$  generated from Theorems 1 and 3. Figure 3 shows good estimation of both state and fault sensor affecting the system and Figure 4 the good estimation of the output signals  $y(t)$ .

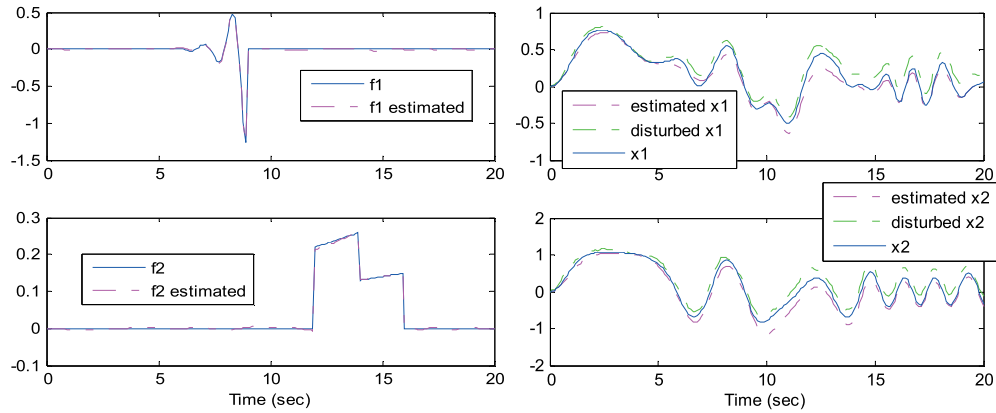


Figure 3. Faults ( $f_1, f_2$ ) and their estimates (left); states ( $x_1, x_2$ ) and their estimates (right).

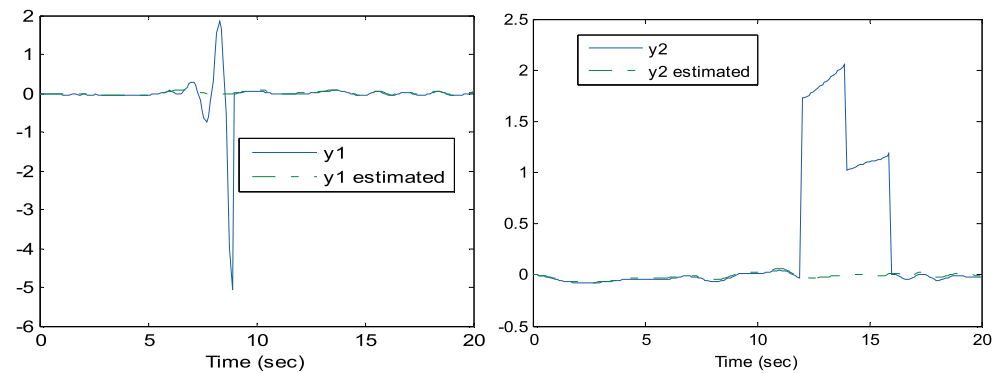


Figure 4. Outputs ( $y_1, y_2$ ) and their estimates.

## 7.2. Example 2—bioreactor model

In this section, the effectiveness and the applicability of the proposed approach are illustrated on a reduced bioreactor model. This system is widely used in wastewater treatment plant. The state estimation problem is then an important task in monitoring the operation of the process in order to respond to a failure. The bioreactor under consideration can be represented by the following nonlinear system:

$$\begin{cases} \dot{x}_1(t) = \frac{ax_1(t)x_2(t)}{x_2(t)+b} - x_1(t)u(t) \\ \dot{x}_2(t) = -\frac{cax_1(t)x_2(t)}{x_2(t)+b} + (d - x_2(t))u(t) \end{cases} \quad (50)$$

where  $x_1(t)$  represents the biomass concentration,  $x_2(t)$  is the substrate concentration, and  $u(t)$  is the dilution rate. The following parameters are given:  $a = 0.5$ ,  $b = 0.07$ ,  $c = 0.7$ ,  $d = 2.5$ . Using the well-known sector nonlinearity approach [30], we obtained a T-S model structure where the input and state vectors are considered as ‘premise variables’ and denoted  $\xi_j(\cdot)$  ( $j = 1, \dots, q$ ). For  $q$  premise variables,  $r = 2^q$  submodels will be obtained. The preceding model is composed of three nonlinearities:

$$\xi_1(t) = -u(t)$$

$$\xi_2(x) = \frac{ax_1(t)}{x_2(t) + b} \quad (51)$$

$$\xi_3(x) = -u(t) - \frac{cax_1(t)}{x_2(t) + b}$$

A T-S model with eight submodels is then obtained in a compact state space leading to define the intervals variations of  $\xi_1(t)$ ,  $\xi_2(x)$ , and  $\xi_3(x)$  by  $\xi_1(t) \in [-1, -0.2]$ ,  $\xi_2(t) \in [0.004, 15]$ ,  $\xi_3(t) \in [-1.72, -0.2]$ . The derived T-S model is

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^8 \mu_i(\xi(t))(A_i x(t) + Bu(t)) + B_d d(t) \\ y(t) = Cx(t) + D_f f(t) \end{cases} \quad (52)$$

where

$$A_1 = \begin{bmatrix} -0.2 & 15 \\ -0.2 & -0.2 \end{bmatrix}, A_2 = \begin{bmatrix} -0.2 & 15 \\ -1.72 & -0.2 \end{bmatrix}, A_3 = \begin{bmatrix} -0.2 & 0.004 \\ -0.2 & -0.2 \end{bmatrix}, A_4 = \begin{bmatrix} -0.2 & 0.004 \\ -1.72 & -0.2 \end{bmatrix},$$

$$A_5 = \begin{bmatrix} -1 & 15 \\ -0.2 & -1 \end{bmatrix}, A_6 = \begin{bmatrix} -1 & 15 \\ -1.72 & -1 \end{bmatrix}, A_7 = \begin{bmatrix} -1 & 0.004 \\ -0.2 & -1 \end{bmatrix}, A_8 = \begin{bmatrix} -1 & 0.004 \\ -1.72 & -1 \end{bmatrix}$$

$$B_i = B = \begin{bmatrix} 0 \\ 2.5 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, D_f = \begin{bmatrix} 4 & 0 \\ 0 & 8 \end{bmatrix}, B_d = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}$$

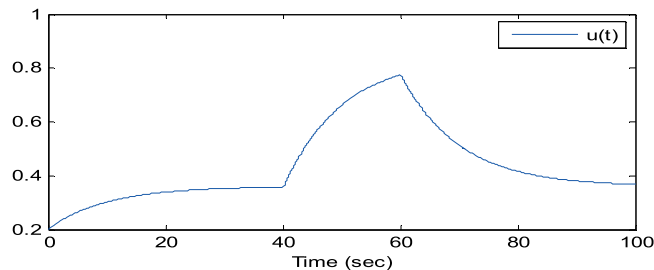


Figure 5. Time evolution of input.

A fault signal  $f(t) = (f_1(t), f_2(t))^T$  affecting the system behavior is considered and described in Figure 7. An unknown disturbance  $d(t)$  with band-limited white noise is also considered (the same as in example 1, plotted in Figure 1). The time evolution of input vector  $u(t)$  is plotted in Figure 5 and membership function evolution in Figure 8.

First, robustness against disturbance is considered by applying the  $H_\infty$  conditions (Theorem 1). The second case concerns the multi-objective  $H_-/H_\infty$  observer design where Theorem 3 is applied. As a result, when  $\gamma$  is reduced to 0.8 and  $\beta$  is increased to 3, we compute  $V$ ,  $R$ , and  $Q$ :

$$V = \begin{bmatrix} 81.3443 & -3.6069 \\ -24.4508 & 50.3752 \end{bmatrix}, R = \begin{bmatrix} -5.8908 & 1.3030 \\ -7.9326 & 1.6555 \end{bmatrix}, Q = \begin{bmatrix} 24.9392 & -6.3961 \\ 3.3261 & -0.8377 \end{bmatrix}$$

The corresponding observer gain matrices and residual weighing matrix are

$$F_1 = \begin{bmatrix} -0.2 & 15 & 0 & 0 \\ 0 & -0.2 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ -1 & 0 & 0 & -1 \end{bmatrix}, F_2 = \begin{bmatrix} -0.2 & 15 & 0 & 0 \\ 0 & -1.72 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ -1 & 0 & 0 & -1 \end{bmatrix}$$

$$F_3 = \begin{bmatrix} -0.2 & 0.004 & 0 & 0 \\ 0 & -0.2 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ -1 & 0 & 0 & -1 \end{bmatrix}, F_4 = \begin{bmatrix} -0.2 & 0.004 & 0 & 0 \\ 0 & -1.72 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ -1 & 0 & 0 & -1 \end{bmatrix}$$

$$F_5 = \begin{bmatrix} -1 & 15 & 0 & 0 \\ 0 & -0.2 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ -1 & 0 & 0 & -1 \end{bmatrix}, F_6 = \begin{bmatrix} -1 & 15 & 0 & 0 \\ -1.72 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ -1 & 0 & 0 & -1 \end{bmatrix}$$

$$F_7 = \begin{bmatrix} -1 & 0.004 & 0 & 0 \\ -0.2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ -1 & 0 & 0 & -1 \end{bmatrix}, F_8 = \begin{bmatrix} -1 & 0.004 & 0 & 0 \\ -1.72 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ -1 & 0 & 0 & -1 \end{bmatrix}$$

$$L = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^T, E = \begin{bmatrix} -5.3961 & 24.9392 & 24.9392 & -6.3961 \\ -0.8377 & 4.3261 & 3.3261 & -0.8377 \\ 1.3030 & -5.8908 & -5.8908 & 1.3030 \\ 1.6555 & -7.9326 & -7.9326 & 1.6555 \end{bmatrix}$$

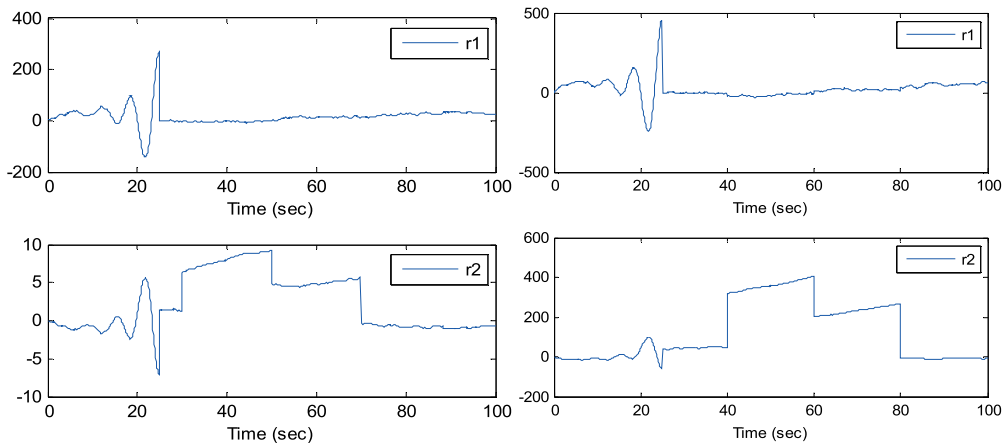


Figure 6. Generated residuals  $r_1, r_2$  Theorem 1(left) and Theorem 3(right).

Figure 6 illustrates the residual trajectories of  $r_1(t)$  and  $r_2(t)$  generated from Theorems 1 and 3. Figure 7 shows the good estimation of both state and sensor faults affecting the system and Figure 9 the good estimation of the output signals  $y(t)$ .

Comparing the right and left simulations of Figures 2 and 6, we can conclude that the sensitivity of the residual  $r(t)$  to the fault  $f(t)$  and the robustness against the disturbance  $d(t)$  are significantly improved with the multi-objective observer  $H_-/H_\infty$ .

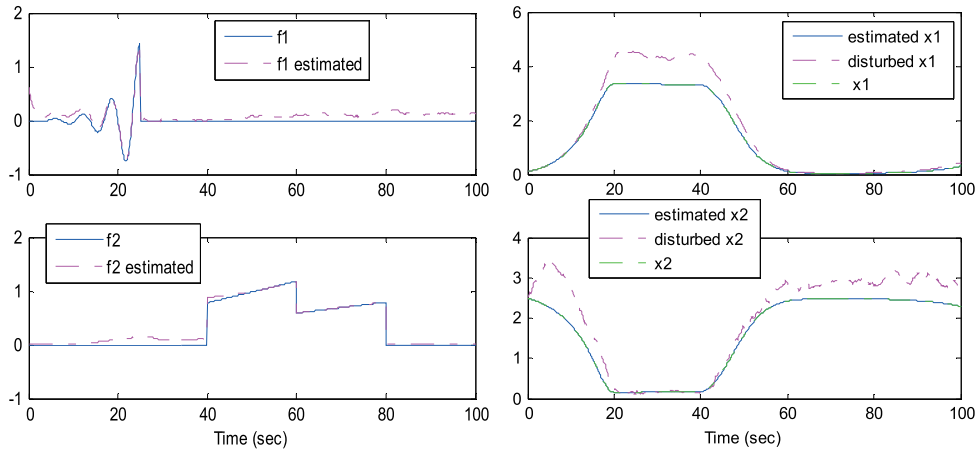


Figure 7. Faults ( $f_1, f_2$ ) and their estimates (left), states ( $x_1, x_2$ ) and their estimates (right).

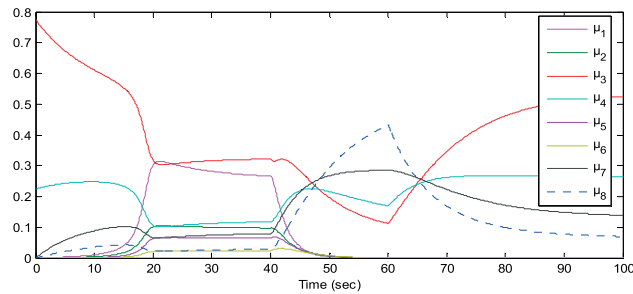


Figure 8. Membership function evolution.

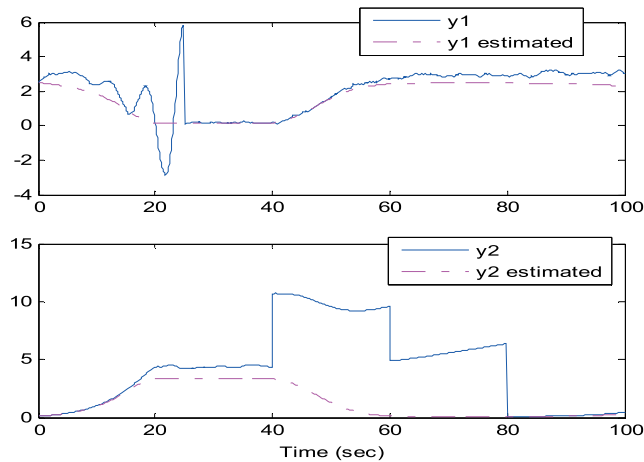


Figure 9. Outputs ( $y_1, y_2$ ) and their estimates.

As a result, the effect of the disturbance on the residuals is weak (because of the minimization of the  $H_\infty$  performance index). Furthermore, the maximization of  $H_-$  performance index leads to have a high sensitivity to faults. However, remember that such minimization/maximization of  $H_\infty/H_-$  performances does not lead to the strongest sensitivity to faults nor to the lowest sensitivity to disturbance.

## 8. CONCLUSION

In this paper, a multi-objective  $H_-/H_\infty$  fault detection observer has been designed for T-S fuzzy model with unmeasurable premise variables. A robust sensor fault detection observer using descriptor theory has been designed using a T-S model with unknown bounded disturbances. Sufficient conditions for the existence of such observer are given in terms of LMIs, and an iterative algorithm is proposed to obtain a solution. At last two examples are given to show the effectiveness of the proposed approach.

On the basis of these results, interesting future studies are planned such as considering uncertainties and delay problems with more relaxed design conditions.

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