Some Spectral Properties of Linear Operators on Exotic Banach Spaces

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Abstract—In this work, we present some results concerning the operators defined on various classes of exotic Banach spaces, containing in particular those studied respectively by V. Ferenczi [7, 8] and T. Gowers with B. Maurey [15, 16]. We show that, on hereditarily indecomposable or quotient hereditarily indecomposable Banach space X, the set of Fredholm operators is dense in $\mathcal{L}(X)$, this gives that the boundary of Fredholm operators is nothing else but the ideal of strictly singular operators if X is hereditarily indecomposable Banach space (resp. the ideal of strictly cosingular operators if X is quotient hereditarily indecomposable Banach space). On the other hand, a comparison between sufficiently rich and exotic Banach spaces is given via some properties of the two maps spectra and essential spectra.

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1. INTRODUCTION AND NOTATIONS

Let X and Y be two infinite dimensional complex Banach spaces and let $\mathcal{C}(X,Y)$ the set of all closed linear densely defined operators from X into Y. We denote by $\mathcal{L}(X,Y)$ the space of all bounded linear operators from X into Y while $\mathcal{K}(X,Y)$ designates the subspace of all compact operators from X into Y. If $A \in \mathcal{C}(X,Y)$, we write $N(A) \subseteq X$ and $R(A) \subseteq Y$ for the null space and range of A. We set $\alpha(A) := \dim N(A)$, $\beta(A) := \operatorname{codim} R(A)$. Let $A \in \mathcal{C}(X,Y)$ with a closed range. Then A is a Φ_+-operator ($A \in \Phi_+(X,Y)$) if $\alpha(A) < \infty$, and A is a Φ_--operator ($A \in \Phi_-(X,Y)$) if $\beta(A) < \infty$, $\Phi(X,Y) = \Phi_+(X,Y) \bigcap \Phi_-(X,Y)$ is the class of Fredholm operators while $\Phi_\pm(X,Y) = \Phi_+(X,Y) \bigcup \Phi_-(X,Y)$. For $A \in \Phi(X,Y)$, the index of A is defined by $i(A) = \alpha(A) - \beta(A)$. If $X = Y$, then $\mathcal{L}(X,Y)$, $\mathcal{K}(X,Y)$, $\mathcal{C}(X,Y)$, $\Phi_+(X,Y)$, $\Phi_\mp(X,Y)$ and $\Phi(X,Y)$ are replaced respectively by $\mathcal{L}(X)$, $\mathcal{K}(X)$, $\mathcal{C}(X)$, $\Phi_+(X)$, $\Phi_\mp(X)$ and $\Phi(X)$. Let $A \in \mathcal{C}(X)$. The spectrum of A will be denoted dy $\sigma(A)$. The resolvent set of A, $\rho(A)$, is the complement of $\sigma(A)$ in the complex plane. A complex number λ is in φ_{+A}, φ_{-A}, $\varphi_{\mp A}$, or φ_A if $\lambda - A$ is in $\Phi_+(X)$, $\Phi_-(X)$, $\phi_\mp(X)$ or $\Phi(X)$, respectively. In the sequel, $\Phi_0(X)$ will denote the set of Fredholm operators with indices 0, we recall now some properties of those sets (for more details, we refer to [9, 10]).

Proposition 1.1 (i) $\varphi_{+A}, \varphi_{-A}$ and φ_A are open,

(ii) $i(\lambda - A)$ is constant on any component of φ_A,

(iii) $\alpha(\lambda - A)$ and $\beta(\lambda - A)$ are constant on any component of φ_A except on a discrete set of points at which they have larger values.

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Let $A \in \mathcal{C}(X)$. A point $\lambda \in \sigma(A)$ is in the Wolf essential spectrum, $\sigma_e(A)$ if $\lambda \notin \varphi_A$; the Schechter (or Weyl) essential spectrum $\sigma_w(A)$, is $\mathbb{C}\backslash\varphi_A^0$ where $\varphi_A^0 := \{\lambda \in \varphi_A, \lambda - A \in \Phi_0(X)\}$.

Definition 1.1. Let X, Y be two Banach spaces. An operator $S \in \mathcal{L}(X, Y)$ is called strictly singular if the restriction of S to any closed infinite-dimensional subspace of X is not an isomorphism. Let $\mathcal{S}(X, Y)$ denote the set of strictly singular operators from X into Y.

For a detailed study of the properties of strictly singular operators we refer to [20, 17]. Note that $\mathcal{S}(X) = \mathcal{S}(X, X)$ is a closed two-sided ideal of $\mathcal{L}(X)$. In general, strictly singular operators are not compact and the strict singularity is not preserved under conjugation. Let X be a Banach space. If N is a closed subspace of X, we denote by π_N the quotient map $X \longrightarrow X/N$. The codimension of N, $codim(N)$, is defined to be the dimension of the vector space X/N.

Definition 1.2. Let X, Y be two Banach spaces and $S \in \mathcal{L}(X, Y)$. S is said to be strictly cosingular if there exists no closed subspace N of Y with $codim(N) = \infty$ such that $\pi_N S : X \longrightarrow Y/N$ is surjective. Let $\mathcal{CS}(X, Y)$ denote the set of strictly cosingular operators on X.

This class of operators was introduced by A. Pelczynski [23]. If $X = Y$, $\mathcal{CS}(X, X) = \mathcal{CS}(X)$ forms a closed two-sided ideal of $\mathcal{L}(X)$.

Definition 1.3. Let $F \in \mathcal{L}(X, Y)$. F is called a Fredholm perturbation if $U + F \in \Phi(X, Y)$ whenever $U \in \Phi(X, Y)$. F is called a upper (resp. lower) semi-Fredholm perturbation if $F + U \in \Phi_+(X, Y)$ (resp. $\Phi_-(X, Y)$) whenever $U \in \Phi_+(X, Y)$ (resp. $\Phi_-(X, Y)$).

Remark 1.1. Let $\Phi^b(X, Y)$, $\Phi^b_+(X, Y)$, $\Phi^b_-(X, Y)$ denote the sets $\Phi(X, Y) \bigcap \mathcal{L}(X, Y)$, $\Phi_+(X, Y) \bigcap \mathcal{L}(X, Y)$ and $\Phi_-(X, Y) \bigcap \mathcal{L}(X, Y)$, respectively. If in Definition 1.3 we replace $\Phi(X, Y)$, $\Phi_+(X, Y)$ and $\Phi_-(X, Y)$ by $\Phi^b(X, Y)$, $\Phi^b_+(X, Y)$ and $\Phi^b_-(X, Y)$, we obtain the sets $\mathcal{F}^b(X, Y)$, $\mathcal{F}^b_+(X, Y)$, $\mathcal{F}^b_-(X, Y)$. These classes of operators were introduced and investigated in [9]. In particular, it is shown that $\mathcal{F}^b(X, Y)$, $\mathcal{F}^b_+(X, Y)$, $\mathcal{F}^b_-(X, Y)$, $\mathcal{F}_+(X, Y)$ and $\mathcal{F}_-(X, Y)$ are closed, moreover if $X = Y$, then $\mathcal{F}^b(X)$, $\mathcal{F}^b_+(X)$, $\mathcal{F}^b_-(X)$ are closed two-sided ideals of $\mathcal{L}(X)$.

In general, we have

$$\mathcal{K}(X, Y) \subseteq \mathcal{S}(X, Y) \subseteq \mathcal{F}_+(X, Y) \subseteq \mathcal{F}^b_+(X, Y) \subseteq \mathcal{F}^b(X, Y),$$
$$\mathcal{K}(X, Y) \subseteq \mathcal{CS}(X, Y) \subseteq \mathcal{F}_-(X, Y) \subseteq \mathcal{F}^b_-(X, Y) \subseteq \mathcal{F}^b(X, Y),$$

The containment $\mathcal{S}(X, Y) \subseteq \mathcal{F}_+(X, Y)$ is due to Kato [17] while the inclusion $\mathcal{CS}(X, Y) \subseteq \mathcal{F}_-(X, Y)$ was proved by Vladimirskii [29].

An operator $R \in \mathcal{L}(X)$ is said to be Riesz operator if $\Phi_R = \mathbb{C}\backslash\{0\}$. For further information on the family of Riesz operators, $\mathcal{R}(X)$, we can see [3] and the references therein. We recall that Riesz operators satisfy the Riesz-Schauder theory of compact operators and $\mathcal{R}(X)$ is not an ideal of $\mathcal{L}(X)$. In [26], it is proved that $\mathcal{F}^b(X)$ is the largest ideal of $\mathcal{L}(X)$ contained in $\mathcal{R}(X)$. Hence, the inclusions above imply that $\mathcal{K}(X)$, $\mathcal{S}(X)$, $\mathcal{CS}(X)$, $\mathcal{F}^b_-(X)$, $\mathcal{F}^b_+(X)$ are contained in $\mathcal{R}(X)$.

Let X and Y be two Banach spaces and $A \in \mathcal{C}(X, Y)$. For every $x \in D(A)$ (the domain of A), we write

$$||x||_A := ||x|| + ||Ax|| \quad \text{(graph norm)}$$

As already observed, $D(A)$ endowed with the norm $||.||_A$ is a Banach space denoted by X_A and A as operator from X_A into Y designated by $\widehat{A}$ is bounded. If $D(A) \subseteq D(J)$, then J is A-defined. Furthermore, we have

$$\alpha(\widehat{A}) = \alpha(A), \quad \beta(\widehat{A}) = \beta(A), R(\widehat{A}) = R(A), \quad \alpha(\widehat{A} + \widehat{J}) = \alpha(A + J),$$
$$\beta(\widehat{A} + \widehat{J}) = \beta(A + J), R(\widehat{A} + \widehat{J}) = R(A + J).$$

It is clear that the relations given above lead to

$$A \in \Phi_+(X, Y) \Longleftrightarrow \widehat{A} \in \Phi^b_+(X_A, Y);$$
$$A \in \Phi_-(X, Y) \Longleftrightarrow \widehat{A} \in \Phi^b_-(X_A, Y);$$

$$A \in \Phi(X,Y) \Longleftrightarrow \widehat{A} \in \Phi^b(X_A,Y).$$

Given a complex Banach space X and an operator $T \in \mathcal{L}(X)$, we define:

$$\sigma_+(T) = \mathbb{C}\backslash\varphi_{+T}; \quad \sigma_-(T) = \mathbb{C}\backslash\varphi_{-T};$$

It is well known that $\sigma_e(T)$ is a non-empty compact set of the complex plane $\mathbb{C}$ because it coincides with the spectrum of the image of T in the Calkin algebra $\mathcal{L}(X)/\mathcal{K}(X)$ (see [21, 53]). On the other hand, it is clear that

$$\sigma_+(T) \bigcup \sigma_-(T) \subseteq \sigma_e(T) \tag{1.1}$$

Moreover, the stability of the index of a semi-Fredholm operator under small perturbations [18, Proposition 2, c. 9] provides the inclusions

$$\begin{aligned} \partial(\sigma_e(T)) &\subseteq \sigma_+(T) \\ \partial(\sigma_e(T)) &\subseteq \sigma_-(T) \end{aligned} \tag{1.2}$$

where $\partial(\sigma_e(T))$ denotes the boundary of the set $\sigma_e(T)$.

Definition 1.4. Let X be a complex Banach space and let $A \in \mathcal{L}(X)$, the punctual spectrum of A, $\sigma_p(A)$ is the set of scalars $\lambda \in \mathbb{C}$ such that $\lambda I - A$ is not one-to-one (the set of eigenvalues of A). the residual spectrum of A, $\sigma_r(A)$ is the set of scalars $\lambda \in \mathbb{C}$ such that $\lambda I - A$ is one-to-one and $R(A)$ is not dense in X while the continuous spectrum $\sigma_c(A)$ is the set of scalars $\lambda \in \mathbb{C}$ such that $\lambda I - A$ is one-to-one and $R(A)$ is dense but not closed in X.

Definition 1.5. We say that two Banach spaces X and Y are essentially incomparable if $\mathcal{L}(X,Y) = \mathcal{F}^b(X,Y)$.

Definition 1.6. We say that two Banach spaces X and Y are totally incomparable if $\mathcal{L}(X,Y) = \mathcal{S}(X,Y)$.

Remark 1.2. Recall that the definition of essentially incomparable Banach spaces is symmetric, in other words, $\mathcal{L}(X,Y) = \mathcal{F}^b(X,Y)$ if and only if $\mathcal{L}(Y,X) = \mathcal{F}^b(Y,X)$ [13]. Also, we can easily observe that for two Banach spaces X and Y, $\mathcal{L}(X,Y) = \mathcal{F}^b(X,Y)$ if and only if $\Phi^b(X,Y) = \emptyset$. On the other hand, it is clear that two totally incomparable Banach spaces are essentially incomparable but the converse is not true in general, it suffices to take a hereditarily Banach space Y (see Definition 2.2) and $X = Y \times Y$, we get that $\Phi^b(Y \times Y, Y) = \emptyset$, moreover, the projection operator $\mathcal{P}_r : (Y \times Y, Y) \longrightarrow Y$ given by $\mathcal{P}_r(x,y) = x$ is bounded from $Y \times Y$ to Y, however, it is not strictly singular.

It is well known that the theory of Banach spaces and that of the operators on which it is dependent play a central role in modern theoretical and applied mathematics. The evolution in this context is always of topicality. Indeed, as far as our days, the specialists could not find criteria allowing to give an ideal classification of Banach spaces, this in spite of the striking results discovered these last years which cleared up and gave answers to questions remained open for a long time. Before the nineties, the mathematicians raise the question whether there exists or not an indecomposable Banach space, if yes, it admits or not at least an infinite dimensional decomposable subspace? this does not come from the chance, indeed, they know well that practically all spaces to which they are accustomed are decomposable and sufficiently rich, in other words having an infinite number of nontrivial projections, for example l_p spaces, $C([0,1])$, L_p-spaces, this question was solved negatively by T. Gowers and B. Maurey (1991) [14, 15] who have constructed a reflexive (then separable) Banach space such that nor him, nor each one of its closed infinite dimensional subspaces are decomposable. This discovery made it possible to solve famous problems like: the unconditional basic problem, the problem of scalar identity plus Fredholm perturbation and the Banach's hyperplane problem (a Banach space is isomorphic or not to its finite codimensional subspaces). It is announced that the source of this discovery is the Tsirelson space constructed by B.S. Tsirelson in 1974 giving an example of a reflexive Banach space which neither an l_p space nor a c_0 space can be embedded. Starting from this space, T. Schlumprecht established his famous example of an arbitrary distortable Banach space [26]. After, T. Gowers and B. Maurey could construct their hereditarily indecomposable Banach space and others exotic having strange properties. As it was already mentioned in [16, 19], the principle of the construction of exotic Banach spaces is sample: given a relatively small semi-group of operators on the space of scalar sequences (for example, the semi-group generated by the right and left shifts), then the construction of Banach spaces such that every bounded linear operator on theses spaces are strictly singular or almost strictly singular perturbations of elements in the algebra generated by the given semi-group is established. Let us give now some examples of exotic not hereditarily indecomposable Banach spaces with their surprising properties:

I. Shift space X_s

1. The space X_s is prime;
2. All the subspaces of X_s of a fixed codimension are isomorphic;
3. The space X_s is indecomposable but not H.I;
4. As a consequence of 3 is that X_s does not have a non-trivial projections;
5. Up to strictly singular perturbations, any two operators on X_s commute.

II. Double shift space X_d

1. The space X_d is isomorphic to its subspaces of even codimension while not being isomorphic to those of odd codimensions. In particular, it is isomorphic to its subspaces of codimensions 2 but not to its hyperplans;
2. As a consequence of 1 is that every Fredholm operator is of an even index;
3. The space X_d has exactly two infinite dimensional complemented subspaces, up to isomorphism;
4. X_d is isomorphic to no subspace of infinite codimension.

III. Ternary space X_t

The ternary space X_t is isomorphic to its cub $X_t^3 = X_t \bigoplus X_t \bigoplus X_t$ but not to its square $X_t^2 = X_t \bigoplus X_t$.

IV. Banach space $\mathcal{E}$

Let $\mathcal{T}$ the ternary tree $\bigcup_{n=0}^{\infty}\{0,1,2\}^n$, Y_{00} the vector space of finitely supported scalar sequences induced by $\mathcal{T}$ and the canonical bases for Y_{00} is denoted by $(e_t)_{t\in\mathcal{T}}$, let $Y = l_1(\mathcal{T})$ be the completion of Y_{00} equipped with the l_1 norm and let $\mathcal{E}$ denote the norm closure of a convenable algebra in $\mathcal{L}(Y)$.

Every Fredholm operator in $\mathcal{E}$ has index 0. Moreover, every Fredholm operator $T : Y^n \longrightarrow Y^n$ has index 0 (for more details concerning these spaces, we refer to [15]).

By which criterion, the richness of the algebra of bounded operators on Banach spaces is measured. By misuse of language, a rich algebra of bounded operators on a certain Banach space is connected to the fact that for example the space is decomposed and an infinite number of its subspaces have an infinite number of nontrivial projections, this can be seen of another angle: let $\mathcal{K}(\mathbb{C})$ be the family of all nonempty compact sets of the complex plane, a rich Banach space is a space for which the mappings spectra and essential spectra defined by

$$\begin{array}{ll} \sigma : \mathcal{L}(X) \longrightarrow \mathcal{K}(\mathbb{C}) & \sigma_e : \mathcal{L}(X) \longrightarrow \mathcal{K}(\mathbb{C}) \\ \quad A \longrightarrow \sigma(A) & \quad A \longrightarrow \sigma_e(A) \end{array}$$

are surjective. We will prove that this property is satisfied if X is a Hilbert space. Unfortunately, it is not the case of exotic Banach spaces, since we will observe that in the case of these spaces, $\mathcal{K}(\mathbb{C})$ is replaced by compact (connected or not) sets with empty interiors.

In this work, via this vision, several comparisons between rich and exotic Banach spaces are established.

2. MAIN RESULTS

Definition 2.1. *A Banach space X is said to be decomposable if it is the topological direct sum of two closed infinite dimensional subspaces. A Banach space is said to be hereditarily indecomposable (in short H.I) if it does not contain any decomposable subspace.*

Definition 2.2. *A Banach space X is said to be quotient hereditarily indecomposable (in short Q.H.I) if no infinite dimensional quotient of a subspace of X is decomposable.*

The class of hereditarily indecomposable Banach spaces was first introduced and investigated by T. Gowers and B. Maurey [14]. Noting also that the hereditarily indecomposable Banach space X_{GM} constructed by the last authors also gave the first example of an indecomposable Banach space.

Definition 2.3. *A Banach space X is hereditarily finitely decomposable if the maximal number of (infinite dimensional) subspaces of X forming a direct sum in X is finite. For $n \geq 1$, X is HD_n if this number is equal to n.*

Definition 2.4. *The space X is n-quotient decomposable and we write $X \in QD_n$, if n is the maximal number of the integers k such that X has a quotient which is the direct sum of k closed infinite dimensional subspaces.*

Let X be a Banach space and M be a closed infinite dimensional subspace of X, M is said to be complemented in X if there exists an infinite dimensional subspace M of X such that $X = M \bigoplus N$. Through this paper, N will be denoted by $X \ominus M$.

We start our study by the definition of exotic Banach spaces which will be adopted in this paper.

Definition 2.5. *Let X be a complex Banach space. We say that X is an exotic Banach space if the Wolf essential spectrum of each bounded operator on X has an empty interior.*

Theorem 2.1. *If X is an exotic reflexive Banach space then X^* is exotic .*

Proof. Let $A \in \mathcal{L}(X^*)$, then $A^* \in \mathcal{L}(X^{**}) = \mathcal{L}(X)$. The result follows immediately from the fact that $\lambda I - A \in \Phi(X)$ if and only if $\lambda I^* - A^* \in \Phi(X^*)$.

Remark 2.1. *Noting that the property that $A \in \Phi(X)$ if and only if $A^* \in \Phi(X^*)$ is also valid for the case of closed densely defined operators on reflexive Banach spaces since $D(A)$ is dense in X if and only if $D(A^*)$ is dense in X^*, better than that, we have $A^{**} = A$ (see for example [2]). On the other hand, the fact that A^* is closed and the relations $\beta(A^*) = \alpha(A), \alpha(A^*) = \beta(A)$ [11] complete the proof.*

Theorem 2.2. *If X is an exotic Banach space and Y is an infinite dimensional complemented closed subspace of X, then Y is an exotic Banach space.*

Proof. If this is not the case, so there exists $\widetilde{A} \in \mathcal{L}(Y)$ for which $Int\sigma_e(\widetilde{A}) \neq \emptyset$, hence $A = \begin{pmatrix} \widetilde{A} & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{L}(X)$ and $Int\sigma_e(A) = Int\sigma_e(\widetilde{A}) = \emptyset$ (since $\sigma_e(\widetilde{A}) = \sigma_e(A)$) which is a contradiction.

Remark 2.2. *H.I, Q.H.I, HD_n and QD_n are exotic Banach space since the essential spectra of bounded operators on these spaces are finite sets.*

One of the major questions which can come to the spirit is the following one: If we have a Banach space X such that the essential spectrum of every bounded operator on it, has an empty interior, is it true that this space is one of the spaces HD_n, QD_n?

The answer of this question is negative and is given by the following proposition.

Proposition 2.1. Let X_s be the Shift Banach space constructed in [15], then for all $A \in \mathcal{L}(X_s)$, the set $\sigma_e(A)$ is a compact connected set with empty interior of the complex plane $\mathbb{C}$.

Proof. Let T be the unit circle in the complex plane $\mathbb{C}$ and $\mathbf{C}(\mathrm{T})$ the Banach space of the scalar continuous functions on T. We denote by ϕ the algebra homomorphism and projection $\phi : \mathcal{L}(X) \longrightarrow \mathcal{L}(X)$ onto the subspace consisting of Toeplitz operators with absolutely summable coefficients and define the mapping $\Psi : \mathcal{L}(X) \longrightarrow \mathbf{C}(\mathrm{T})$ as the composition of ϕ with the Fourier transform. Then Ψ is a continuous (better than that it is piecewise regular) algebra homomorphism. Moreover, for all $A \in \mathcal{L}(X_s)$, we have that $\sigma_e(A)$ is the set $[\Psi(A)](\mathrm{T})$ [15], this shows that this set is a connected compact set with empty interior of the complex plane $\mathbb{C}$.

Remark 2.3. In the case of this Banach space, the Fredholm domain of the right shift $R \in \mathcal{L}(X_s)$ has exactly two connected components. Indeed, the corresponding function to this operator is $\Psi(R)$ defined by $\Psi(R)(\lambda) = \lambda$, then $\Psi(R)(\mathrm{T}) = \sigma_e(R) = \mathrm{T}$, thus $\phi_R = \{\lambda \in \mathbb{C}/|\lambda| > 1\} \bigcup \{\lambda \in \mathbb{C}/|\lambda| < 1\}$.

Our first result here is given by the following theorem given in the general context.

Theorem 2.3. *Let X be a complex Banach space and let F be a closed subspace of X, then $X\backslash F + F \subseteq X\backslash F$ and the set $X\backslash F$ is connected in X.*

Proof. The fact that $X\backslash F + F \subseteq X\backslash F$ is trivial. Moreover, if $x \in X\backslash F$, then x and $-x$ can be connected by the path $\varphi(t) := e^{it}x$ $(0 \leq t \leq \pi)$. Now, let $x, y \in X\backslash F$. Assume that the line segment connecting x and y meets F, in other words, there exists $t \in]0, 1[$ such that

$$(1 - t)x + ty \in F. \tag{$\star$}$$

Assume also that the line segment connecting x and $-y$ meets F, i.e., there exists $s \in]0, 1[$ such that

$$(1 - s)x - sy \in F.$$

Then, multiplying by $\frac{t}{s}$, we obtain that

$$\frac{t}{s}(1 - s)x - ty \in F. \tag{$\star\star$}$$

Addicting $(\star)$ and $(\star\star)$, we get $\left((1 - t) + \frac{t}{s}(1 - s)\right) x \in F$ which is a contradiction since the real number $\left((1 - t) + \frac{t}{s}(1 - s)\right)$ is strictly positive. This shows that one of the line segments mentioned above lies in $X\backslash F$ and therefore x and y can be connected by a path □

Let us give now the following fundamental lemma du to L. Weis [30].

Lemma 2.1. *Let X be a complex Banach space. Then*

$(\imath)$ X is a H.I Banach space if and only if for every Banach space Y, we have $\mathcal{L}(X, Y) = \mathcal{S}(X, Y) \bigcup \Phi_+(X, Y)$;

$(\imath\imath)$ X is a Q.H.I Banach space if and only if for every Banach space Y, we have $\mathcal{L}(Y, X) = \mathcal{CS}(Y, X) \bigcup \Phi_-(Y, X)$;

As a consequence of Theorem 2.3 and Lemma 2.1, we have

Corollary 2.1. *Let X be a complex Banach space. Then*

$(\imath)$ If *X is a H.I Banach space then $\mathcal{L}(X) = \mathbb{C}I_X \bigoplus \mathcal{S}(X)$;*

$(\imath\imath)$ If *X is a Q.H.I Banach space then $\mathcal{L}(X) = \mathbb{C}I_X \bigoplus \mathcal{CS}(X)$.*

Proof. According to Lemma 2.1, we have that $\mathcal{L}(X) = \Phi_+(X) \bigcup \mathcal{S}(X)$ for H.I Banach spaces and $\mathcal{L}(X) = \Phi_-(X) \bigcup \mathcal{CS}(X)$ for Q.H.I Banach spaces. On the other hand, it is well known that the sets $\Phi_+(X)$ and $\Phi_-(X)$ are open [21], Theorem 2.3 applied to the sets $\mathcal{S}(X)$, $\Phi_+(X)$ and $\mathcal{CS}(X)$, $\Phi_-(X)$ shows that $\Phi_+(X)$ and $\Phi_-(X)$ are connected in $\mathcal{L}(X)$, now by taking account to the fact that $I_d \in \Phi_+(X) \bigcap \Phi_-(X)$ and the stability of the index on the connected sets, we obtain that $\Phi_+(X) = \Phi_0(X)$ for H.I Banach spaces and $\Phi_-(X) = \Phi_0(X)$ for Q.H.I Banach spaces. Now the result follows immediately from the Gelfand-Mazur theorem (since Fredholm operators in $\mathcal{L}(X)$ are nothing but else the representatives of invertible elements in the algebra $\mathcal{L}(X)/\mathcal{S}(X)$).

Proposition 2.2. *Let X be a complex Banach space.*

$(\imath)$ If *X is a H.I Banach space then $\partial(\Phi(X)) = \mathcal{S}(X)$;*

$(\imath\imath)$ If X is a Q.H.I Banach space then $\partial(\Phi(X)) = \mathcal{CS}(X)$.

Proof. We restrict our proof to H.I Banach spaces, the case of Q.H.I spaces can be established by the same arguments.
Let $A \in \partial(\Phi(X))$ then $A \notin \Phi_+(X)$ [21], this proves that $A \in \mathcal{S}(X)$ and consequently $\partial(\Phi(X)) \subseteq \mathcal{S}(X)$. On the other hand, let $S \in \mathcal{S}(X)$ and given $\{\lambda_n\}_{n=0}^{+\infty}$ a nonzero sequence of complex scalars converging to 0, then the sequence of operators $\lambda_n I + S$ are in $\Phi(X)$ and converging in norm to S, this implies that $S \in \partial(\Phi(X))$ and yields $\mathcal{S}(X) \subseteq \partial(\Phi(X))$ which ends the proof. **Second proof:** It suffices to prove that $\Phi(X)$ is dense in $\mathcal{L}(X)$. Indeed, let S be a strictly singular operator on a H.I Banach space, then S is the limit in norm topology of the sequence of Fredholm operators given by $S_n = \frac{1}{n}I + S$, this

gives the result for H.I Banach spaces, the case of Q.H.I Banach spaces can be established by the same argument.

Remark 2.4. [25] Nothing that in general Banach spaces, the boundary of Fredholm operators $\partial\Phi(X)$ does not coincide necessarily with the set of Riesz operators. Indeed, it suffices to show that this last set is not closed. For this, giving the celebrated example due to Kakutani [25]. Let $\mathcal{H}$ be a separable Hilbert space with orthonormal basis $(h_m)_{m=1}^{\infty}$. Let $(\alpha_n)_{n=1}^{\infty}$ the sequence of positive real numbers given by $\alpha_n = e^{-k}$ for $n = 2^k(2l+1), k, l, ...$ and the operator A given by $Ah_n = \alpha_n h_{n+1}, n \in \mathbb{N}$. Then $||A|| = \sup_{n\in\mathbb{N}} |\alpha_n|$, $A^p h_n = \alpha_n\alpha_{n+1}.....\alpha_{n+p-1}h_{n+p}$ and so $||A^p|| = \sup_{n\in\mathbb{N}}(\alpha_n\alpha_{n+1}.....\alpha_{n+p-1})$. We can observe that $\alpha_1.....\alpha_{2^s-1} = \prod_{j=1}^{s-1} e^{-j2^{s-j-1}}$, hence $(\alpha_1\alpha_2......\alpha_{2^s-1})^{\frac{1}{2^s-1}} > (\prod_{j=1}^{s-1} e^{\frac{-j}{2^{j+1}}})^2$ and if $\lambda = \sum_{j=1}^{\infty} \frac{j}{2^{j+1}}$, then $e^{-2\lambda} \leq r_\sigma(A) = \lim_{n\longrightarrow\infty} ||A^n||^{\frac{1}{n}}$, this implies that A is not quasinilpotent.

Now define the operator A_p by

$$A_p h_n = \begin{cases} 0 \quad n = 2^p(2l+1), \quad l = 0, 1, \dots, \\ \alpha_n h_{n+1} \quad \text{otherwise} \end{cases}$$

Then A_p is nilpotent and

$$(A - A_p)h_n = \begin{cases} e^{-p}h_{n+1}, \quad n = 2^p(2l+1), \quad l = 0, 1, \dots \\ 0 \quad \text{otherwise} \end{cases}$$

Thus $||A_p - A|| = e^{-p}$ which tends to 0 if $p \longrightarrow \infty$ and consequently $A_p \longrightarrow A$ in the norm topology in $\mathcal{L}(\mathcal{H})$. Moreover, it is easy to show that the punctual spectrum of A is empty, on the other hand the fact that $r_\sigma(A) \geq e^{-2\lambda}$ implies the existence of a nonzero scalar β such that $\beta \in \sigma_\omega(A)$, thus $\sigma_\omega(A) \neq \{0\}$ and consequently A is not a Riesz operator. For more details concerning the boundary of Fredholm and semi-Fredholm operators we refer to [28].

Also, it was given in [4, p. 45] an example of an operator which asserts that the hereditarily indecomposable Banach space of T. Gowers and B. Maurey X_{GM} is not simple, in other words, the class of compact operators is not the unique nontrivial closed two-sided ideal in $\mathcal{L}(X_{GM})$.

Theorem 2.4. *Let X be a complex Banach space. Then*

$(\imath)$ If *X is a H.I Banach space and $A : X \longrightarrow X$ a closed densely defined upper semi-Fredholm operator on X, then the Banach space $X_A = (D(A), ||.||_A)$ is a H.I Banach space.*

$(\imath\imath)$ *If X is a Q.H.I Banach space and $A : X \longrightarrow X$ a closed densely defined lower semi-Fredholm operator on X, then the Banach space $(D(A), ||.||_A)$ is a Q.H.I Banach space.*

Proof. We prove just the first assertion, the second one can be established in the same way. Let $A \in \Phi_+(X)$, then $\widehat{A} \in \Phi_+^b(X_A, X)$, hence $X_A = X_1 \bigoplus X_2$ such that X_1 is finite dimensional and X_2 is isomorphic to $R(A)$, on the other hand, since $R(A) = R(\widehat{A})$ is a closed infinite dimensional Banach space of a hereditarily indecomposable Banach space, so $R(A)$ is a H.I Banach space and consequently X_A is H.I.

We present here a surprising result derived on the exotic Banach spaces and established by [5].

Theorem 2.5. (see [5]) *Let X be an exotic Banach space, then*

$$\mathcal{F}_+^b(X) = \mathcal{F}_-^b(X) = \mathcal{F}^b(X) = \mathcal{F}(X).$$

Remark 2.5. As it was already mentioned in [5], the sets $\mathcal{F}_+(X)$ and $\mathcal{F}_-(X)$ are closed sets, moreover, we have just the inclusions $\mathcal{F}_+(X) \subseteq \mathcal{F}_+^b(X)$ and $\mathcal{F}_-(X) \subseteq \mathcal{F}_-^b(X)$.

Here, we give a general construction of Banach subspaces Z, Z' for which $\mathcal{F}_+^b(Z) \neq \mathcal{S}(Z)$ and $\mathcal{F}_-^b(Z') \neq \mathcal{S}(Z')$.

Proposition 2.3. Let Z be an exotic Banach space for which there exists a closed complemented subspace Y such that the subspaces Y and $Z \ominus Y$ are essentially incomparable and Y isomorphic to a closed subspace of $Z \ominus Y$, then $\mathcal{F}_+^b(Z) \neq \mathcal{S}(Z)$, moreover if Z is reflexive, so $\mathcal{F}_-^b(Z^*) \neq \mathcal{CS}(Z^*)$.

Proof. Assume that Y and $Z \ominus Y$ are essentially incomparable, then $\mathcal{L}(Y, Z \ominus Y) = \mathcal{F}^b(Y, Z \ominus Y)$. On the other hand, there exists an isomorphism $J : Y \longrightarrow Z \ominus Y$, thus, it is clear that $A = \begin{pmatrix} 0 & J \\ 0 & 0 \end{pmatrix} \in \mathcal{F}^b(Z) = \mathcal{F}^b_+(Z)$ but $A \notin \mathcal{S}(Z)$.

Let us given Z reflexive, it is easy to observe that if Y and $Z \ominus Y$ are essentially incomparable, then Y^* and $(Z \ominus Y)^*$ are essentially incomparable. On the other hand, $A \in \mathcal{F}^b(Y, Z \ominus Y)$ if and only if $A^* \in \mathcal{F}^b((Z \ominus Y)^*, Y^*)$. By Theorem 2.1, Z^* is exotic, we can also see that $J^* : (Z \ominus Y)^* \longrightarrow Y^*$ is not strictly cosingular, moreover, we have $\mathcal{F}^b_-(Z^*) = [\mathcal{F}^b_+(Z)]^* = \mathcal{F}^b(Z^*) = [\mathcal{F}^b(Z)]^*$, this shows that $B = \begin{pmatrix} 0 & 0 \\ J^* & 0 \end{pmatrix} \in \mathcal{F}^b_-(Z^*)$ but $B \notin \mathcal{CS}(Z^*)$.

Definition 2.6. *A Banach space X is said to be weakly compactly generated whenever there exists a weakly compact subset K of X such that the closed linear span of K, $[K]$, is all of X.*

Proposition 2.4. Reflexive Banach spaces and separable Banach spaces are weakly compactly generated.

Proof. In reflexive Banach spaces we take $K = B_X$ (the closed unit ball of X) while in the case of separable Banach spaces we can take $K = \left\{ \frac{x_n}{n\|x_n\|} \right\}_{n=1}^{+\infty} \bigcup \{0\}$ where $\{x_n\}_{n=1}^{+\infty}$ is a dense set in X. □

Remark 2.6. Recall that weakly compactly generated hereditarily indecomposable Banach spaces are necessarily separable. Indeed, if this is not the case, thus X must be decomposable [6], which is a contradiction. A good example for a compactly generated hereditarily indecomposable Banach space is X_{GM} which is reflexive and so separable.

Lemma 2.2. *Let X be a Banach space such that for every semi-Fredholm operator T in $\mathcal{L}(X)$, the set $\mathbb{C} \backslash \sigma_e(T)$ is connected, then T is Fredholm with index 0.*

Proof. The result follows from the fact that the set $\mathbb{C} \backslash \sigma_e(T)$ contains the resolvent of T and the stability of the index on the connected components. We denote by $\mathcal{A}$ the class of Banach spaces for which every Fredholm operator is of index 0.

Remark 2.7. It is not difficult to see that $\mathcal{A}$ contains all the $H.I, Q.H.I, HD_n$ and QD_n Banach spaces.

Proposition 2.5. Let X and Y be two essentially incomparable Banach spaces in $\mathcal{A}$, then $X \bigoplus Y \in \mathcal{A}$.

Proof. Assume that $T \in \mathcal{L}(X \bigoplus Y)$, so, T can be written in the matrix form $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $A \in \mathcal{L}(X), D \in \mathcal{L}(Y), B \in \mathcal{L}(Y, X), B \in \mathcal{L}(X, Y)$, on the other hand, it is shown that $T \in \Phi^b(X \bigoplus Y)$ if and only if $A \in \Phi^b(X)$ and $D \in \Phi^b(Y)$ and $ind(T) = ind(A) + ind(D)$ (see [13]), this completes the proof. □.

Proposition 2.6. Let X be a weakly compactly generated hereditarily indecomposable Banach space for which X^* (the dual of X) is w^*-separable and let A be a closed densely defined (not bounded) upper semi-Fredholm (resp. lower semi-Fredholm operator) on X, then the index of A is equal to 0.

Proof. Let $A \in \Phi_+(X)$, then $\widehat{A} \in \Phi^b_+(X_A, X)$, by applying Theorem 2.4, we obtain that X_A is hereditarily indecomposable Banach space, clearly, the fact that X is weakly compactly generated implies that X is a separable Banach space and consequently X_A is separable. On the other hand, X^* is w^*-separable (by hypothesis), this implies that the set $\Phi^b(X_A, X)$ is non empty. Indeed, we can construct a one-to-one compact operator $\widetilde{K}$ between X and X_A having a dense range [12], it follows that $\widetilde{K}^{-1}$ is a closed densely defined Fredholm operator with index 0. Now by Theorem 2.3 and Lemma 2.1, the set $\Phi^b_+(X_A, X)$ is connected, thus the index is constant on this set, since $\widehat{A}, \widetilde{K}^{-1} \in \Phi_+(X_A, X)$, we conclude that $ind(A) = ind(\widehat{A}) = ind(\widetilde{K}^{-1}) = 0$ which ends the proof.

Proposition 2.7. Let $X \in \mathcal{A}$ be an exotic Banach space, then no proper subspaces or proper quotient subspaces of X are isomorphic to X.

Proof. Assume that there exists a closed subspace Y of X and an isomorphism $S : X \longrightarrow Y$, thus $S \in \Phi^b_+(X) = \Phi^b_-(X) = \Phi^b(X)$ and consequently $S \in \Phi_0(X)$ which is a contradiction since Y is proper of X.

Proposition 2.8. Let X be a separable Hilbert space and given a compact subset K of the complex plane $\mathbb{C}$, then there exists a bounded operator A on X for which $\sigma(A) = K$.

Proof. Without loss of generality, we may assume that $X = l^2$, let $(e_n)_{n=1}^{\infty}$ be the orthonormal basis for X and $(\lambda_n)_{n=1}^{\infty}$ a dense sequence in K. Define A as follows: $A(\sum_{n=1}^{+\infty} \alpha_n e_n) = \sum_{n=1}^{+\infty} \lambda_n \alpha_n e_n$ where $(\alpha_n)_{n=1}^{\infty} \in l^2$. It is clear that $\sigma(A)$ contains K. Now, we prove the opposite inclusion, if $\lambda \notin K$, then $\inf\{|\lambda - \beta|; \beta \in K\} > 0$ and so $S(\sum_{n=1}^{+\infty} \alpha_n e_n) = \sum_{n=1}^{+\infty} (\lambda - \lambda_n)^{-1} \alpha_n e_n$ is a bounded operator on X and $(\lambda I_d - A)S = S(\lambda I_d - A) = I_d$, this gives that $\lambda \notin \sigma(A)$ which completes the proof.

Proposition 2.9. Let X be a Banach space such that the Wolf essential spectrum of each bounded operator on X does not contain any boundary of a nonempty open set in the complex plane, then X is an exotic Banach space.

Proof. Indeed, in this case, the essential spectrum of every bounded operator has necessarily an empty interior. Nothing that the converse of Proposition 2.9 is in general false (see the example of Shift space X_s and Remark 2.3).

Proposition 2.10. Let X be a Banach space such that the Wolf essential spectrum of every bounded operator on X does not contain any boundary of a nonempty open set in the complex plane, then $\sigma : \mathcal{L}(X) \longrightarrow \mathcal{K}(\mathbb{C})$ is not surjective, more precisely the range of this mapping is included in the set of all compact sets with empty interiors.

Proof. Let us given $A \in \mathcal{L}(X)$. First of all, we prove that the set $Int(\sigma(A))\backslash\sigma_e(A)$ is empty. To do it, assume that $Int(\sigma(A))\backslash\sigma_e(A) \neq \emptyset$, let Ω be a connected component of this set and let $\lambda \in \overline{\Omega}$ such that $\lambda \notin \Omega$; this gives that $\lambda \in \partial\sigma(A)$. Next, we put $S_\lambda = A - \lambda I$; since $\lambda \notin \sigma_e(A)$, the operator S_λ is Fredholm with index 0. Indeed, the fact that $0 \in \partial\sigma(S_\lambda)$ asserts the existence of invertible operators arbitrary close to S_λ, the assertion follows from the continuity of the index. Hence, 0 is an isolated point of $\sigma(S_\lambda)$. Indeed, since $S_\lambda \in \Phi_0(X)$, then $Ker(S_\lambda) \neq \{0\}$. Furthermore, there exists $\epsilon > 0$ such that $\alpha(S_\lambda)$ is constant on the set $\{\lambda \in \mathbb{C} \text{ such that } 0 < |\lambda| < \epsilon\}$. Since $0 \in \partial\sigma(S_\lambda)$, $\alpha(S_\lambda)$ must be 0; now, we may assume that $\epsilon > 0$ is sufficiently small for which S_λ is still Fredholm with index 0 when $0 < |\lambda| < \epsilon$. Thus, S_λ is invertible when $0 < |\lambda| < \epsilon$. On the other hand, the fact that 0 is isolated in $\sigma(S_\lambda)$ shows that λ is isolated in $\sigma(A)$ and contradicts the fact that $\lambda \in \overline{\Omega}$. The set $Int(\sigma(A))\backslash\sigma_e(A)$ must be empty and consequently $Int(\sigma(A)) \subseteq \sigma_e(A)$. The use of Proposition 2.9 gives the result.

Proposition 2.11. Let X be a Banach space such that the Wolf essential spectrum of every bounded operator on X does not contain any boundary of a nonempty open set in the complex plane, then for every $A \in \mathcal{L}(X)$, we have $\sigma(A) = \sigma_e(A) \bigcup \mathcal{H}$ where $\mathcal{H}$ is a set of isolated eigenvalues with finite multiplicities.

Proof. By Proposition 2.9, the spectrum of A has an empty interior. Next, If $\lambda \in \partial\sigma(A) = \sigma(A)\backslash Int\sigma(A) = \sigma(A)$, then $S = \lambda I - A \in \Phi_0(X)$. As in the proof of Proposition 2.10, we get that 0 is an isolated eigenvalue with finite multiplicity of S and so, λ is an isolated eigenvalue with finite multiplicity for A.

Proposition 2.11. Let X be a Banach space such that the Wolf essential spectrum of every bounded operator on X does not contain any boundary of a nonempty open set in the complex plane and if $A \in \mathcal{C}(X)$ (not bounded) such that $\rho(A) \neq \emptyset$, then $0 \in \sigma_e[(\lambda - A)^{-1}]$ for all $\lambda \in \rho(A)$.

Proof. By Proposition 2.12, if $\lambda \in \rho(A)$, we have $\sigma(\lambda - A)^{-1} = \sigma_e(\lambda - A)^{-1} \bigcup \mathcal{H}$ where $\mathcal{H}$ is a set of isolated eigenvalues of finite multiplicities, on the other hand, the fact that A is not bounded shows that $D(A) \neq X$, now let $\mu \in \rho(A)$. It is clear that $0 \in \sigma(\mu - A)^{-1}$, assume now that 0 is an eigenvalue of finite multiplicity of the invertible operator $(\mu - A)^{-1}$, this is impossible, thus necessarily $0 \in \sigma_e(\mu - A)^{-1}$ which gives the result.

Proposition 2.13. Let X be a H.I (resp. Q.H.I) Banach space and $A \in \mathcal{C}(X)$ (not bounded) such that A generates a C_0 semigroup on X, then we have necessarily $(\lambda - A)^{-1}$ is a bounded strictly singular operator on X (resp. $(\lambda - A)^{-1}$ is a bounded strictly cosingular operator on X) for all $\lambda \in \rho(A)$.

Proof. This is an immediate consequence of Proposition 2.12 since here the essential spectrum of every bounded operator is a singleton.

Proposition 2.14. Under the hypothesis of Proposition 2.13, we have $\sigma_w(A) = \emptyset$, in particular, we have A is a Fredholm operator with index 0.

Proof. Let $\lambda_0 \in \rho(A)$, for all $\lambda \in \mathbb{C}$, we write

$$\lambda I - A = (I + (\lambda - \lambda_0)(\lambda_0 I - A)^{-1})(\lambda_0 I - A),$$

Proposition 2.13 asserts that $(\lambda_0 I - A)^{-1}$ is strictly singular if X is H.I and strictly cosingular if X is Q.H.I, it follows that

$$\mathrm{ind}(I + (\lambda - \lambda_0)(\lambda_0 I - A)^{-1}) = \mathrm{ind}(I) = 0.$$

Furthermore

$$\mathrm{ind}(\lambda_0 I - A) = 0.$$

By using the Theorem of Atkinson, we obtain that $\mathrm{ind}(\lambda I - A) = 0$, in particular if $\lambda = 0$ we deduce that A is Fredholm with index 0.

Proposition 2.15. Let X be a H.I or Q.H.I Banach space and $A \in \mathcal{C}(X)$, then we have the following three alternatives.

(i) A is bounded.

(ii) $\sigma(A) = \mathbb{C}$.

(iii) $\sigma_\omega(A) = \emptyset$.

Proof. If A is not bounded and $\sigma(A) \neq \mathbb{C}$, thus there exists $\lambda_0 \in \rho(A)$, it is easy to show that necessarily we have $(\lambda_0 I - A)^{-1}$ is strictly singular in the case of H.I Banach spaces and strictly cosingular in the case of Q.H.I Banach spaces now the result is an immediate consequence of Proposition 2.14.

Example (see [24]) Let X be a H.I Banach space with a Schauder basis, for example the space X_{GM} constructed by T. Gowers and B. Maurey (see [15]) which is reflexive and having a Schauder (not unconditional) basis. For $x \in X$, we write $x = (x_1, x_2, \ldots)$ with respect to this basis. Define linear subspace

$$D(S) = \{x \in X : (0, x_1, 0, x_3, \ldots) \in X\}$$

and $S : D(S) \longrightarrow X$ by $S(x) = (0, x_1, 0, x_3, \ldots)$. Since every $x \in X$ with finite support belongs to $D(S)$, then $D(S)$ is dense in X. On the other hand, by using the continuity of the coordinate functionals in X, it is easy to deduce that S is closed (not bounded), thus $S \in \mathcal{C}(X)$. To prove that its spectrum is equal $\mathbb{C}$, it suffices to show that S is not Fredholm, indeed it is easy to see that the equation $S(x) = 0$ has an infinity of solutions.

Corollary 2.2. *Let X be a H.I or Q.H.I Banach space. The spectra mapping $\sigma : \mathcal{L}(X) \longrightarrow \mathcal{K}(\mathbb{C})$ is not surjective, more precisely the range of this mapping is included in the set of all converging sequences of the complex plane.*

Proof. This follows from the fact that the spectrum of every strictly singular or strictly cosingular operator satisfies Riesz-Schauder theory.

Proposition 2.16. Let X be a Banach space and $A \in \mathcal{L}(X)$ for which $\sigma_c(A) = \sigma(A)$, then $\sigma_e(A) = \sigma_\omega(A) = \sigma(A)$.

Proof. Since $\sigma_c(A) \subseteq \sigma_\omega(A) \subseteq \sigma(A)$, it follows that $\sigma_c(A) = \sigma_\omega(A) = \sigma(A)$. Next, if $\lambda \in \sigma_\omega(A)$ such that $\lambda \notin \sigma_e(A)$, thus $\lambda I - A \in \Phi(X)$, hence $R(\lambda I - A)$ is closed in X which is a contradiction, the inclusion $\sigma_e(A) \subseteq \sigma_\omega(A)$ completes the proof.

Proposition 2.17. Let $\mathcal{H}$ be a separable Hilbert space space and K is a compact set, then there exists $A \in \mathcal{L}(\mathcal{H})$ such that $\sigma_e(A) = K$.

Proof. This result was announced without proof in [22]. Giving for example the proof of the case of the unit disc and the unit circle in $\mathbb{C}$. Taking $K = \overline{D}(0,1)$ equipped with the Lebesgue measure ν on $\mathbb{R}^2$ and $\mathcal{H} = L^2(\overline{D}(0,1))$. Define A on $\mathcal{H}$ by the following:

$$(Af)(\lambda) = \lambda f(\lambda); \lambda \in \overline{D}(0,1), f \in \mathcal{H}.$$

If $\lambda \notin \overline{D}(0,1)$, then $\sup\{|\lambda-\alpha|; \alpha \in \overline{D}(0,1)\} < \infty$ and then we can define an operator B on $\mathcal{H}$ by $(Bf)(\alpha) = (\lambda-\alpha)^{-1}f(\lambda); f \in \mathcal{H}$. Hence $B(\lambda I - A) = (\lambda I - A)B = I_d$ which gives that $\lambda \notin \sigma(A)$. Now, if $\lambda \in \overline{D}(0,1)$ and $\lambda \notin \sigma(A)$, then $\lambda \in \rho(A)$ and $(\lambda I - A)^{-1} \in \mathcal{L}(\mathcal{H})$. Let f_ϵ denote the characteristic functions of the sets $\{\alpha; |\lambda-\alpha| < \epsilon\}$ multiplied by $(\nu\{\alpha; |\lambda-\alpha| < \epsilon\})^{-\frac{1}{2}}$, then

$$1 = ||f_\epsilon||_2 \leq ||(\lambda I - A)^{-1}||||(\lambda I - A)f_\epsilon||_2$$

$$= ||(\lambda I - A)^{-1}|| \left(\int\limits_{\overline{D}(0,1)} |(\lambda-\alpha)f_\epsilon(\alpha)|^2 d\nu \right)^{\frac{1}{2}} \leq ||(\lambda I - A)^{-1}||\epsilon.$$

If $\epsilon \longrightarrow 0$, we obtain a contradiction, hence λ must be in $\sigma(A)$ and so, it follows that $\sigma(A) = K$. Moreover, if $Af = \theta f$ for some $\theta \in \mathbb{C}$, then for all $\lambda \in \overline{D}(0,1), \lambda f(\lambda) = \theta f(\lambda)$. So f is the null function almost every where, thus, A has no eigenvalues. On the other hand, for every $\lambda \in \sigma(A)$, then $\lambda I - A$ is not onto since $\beta(\alpha-\lambda)^{-1} \notin L^2(\overline{D}(0,1))$ for $\beta \neq 0$, so the nonzero constant functions β do not belong to the range of $\lambda I - A$ which is a dense set. Indeed, For any $f \in L^2(\overline{D}(0,1))$, let

$$f_n(z) = \begin{cases} f(z) & \text{if } |z-\lambda| \geq \frac{1}{n}, \\ 0 & \text{if } |z-\lambda| < \frac{1}{n}. \end{cases}$$

Thus $f_n \longrightarrow f$ in $L^2(\overline{D}(0,1))$, this implies that $R(\lambda I - A)$ is dense and consequently $\overline{D}(0,1) = \sigma(A) = \sigma_c(A)$. Finally, the result follows from Proposition 2.16.

If $K = \{z \in \mathbb{C}; |z| = 1\}$, taking H a separable infinite dimensional Hilbert space with an orthogonal basis $(\xi_n)_{n\in\mathbb{Z}}$, we define $U(\xi_n) = \xi_{n+1}$. Then U is isometric and surjective, so it is a unitary and it is easy to check that $\sigma(U) = \sigma_c(U) = \sigma_e(A) = K$.

REFERENCES

1. G. Androulakis and Th. Schlumprecht, J. London. Math., Soc.(2) **64** (3), 655 (2001).
2. H. Brezis, *Analyse Fonctionnelle, Théorie et Applications* (Masson, 1983).
3. S. R. Caradus, W. E. Pfaffenberger, and B. Yood, *Calkin Algebras and Algebras of Operators on Banach Spaces* (Marcel Dekker, New York, 1974).
4. A. Dehici and N. Boussetila, Lobachevskii Journal of Mathematics **32**, 39 (2011).
5. A. Dehici and K. Saoudi, Int. J. Math. Math. Sci, ID 26254 (2007).
6. J. Diestel, *Geometry of Banach spaces* (Selected Topics. Lecture notes in Mathematics 485 Berlin-Heidelberg-New York: Springer 1975).
7. V. Ferenczi, Can. J. Math. **51** (3), 566 (1999).
8. V. Ferenczi, Stud. Math. **123** (2), 135 (1997).
9. I. C. Gohberg, A. Markus, and I. A. Feldman, Amer. Math. Soc. Tran. Ser. 2. **61**, 63 (1967).
10. I. C. Gohberg and M. G. Krein, Amer. Math. Soc. Tran. Ser. 2. **13**, 185 (1960).
11. S. Goldberg, *Unbounded linear operators* (McGraw-Hill, New York, 1966).
12. S. Goldberg and A. Kruse, Proc. Amer. Math. Soc. **13**, 808 (1962).
13. M. Gonzalez, Math. Z. **215**, 621 (1994).
14. T. W. Gowers, Geom. Func. Anal. **6** (6), 1083 (1996).
15. T. W. Gowers and B. Maurey, J. Amer. Math. Soc. **6**, 851 (1993).
16. T. W. Gowers and B. Maurey, Math. Ann. **307**, 543 (1997).
17. T. Kato, J. Anal. Math. **6**, 261 (1958).
18. J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces, vol I* (Springer-Verlag, 1977).
19. B. Maurey, *Operators theory and exotic Banach spaces* (Series of lectures, Université Paris 6, 1995).
20. V. Milman, J. Func. Anal. Appl. **3**, 77 (1969).
21. V. Müller, *Spectral theory of linear operators and spectral systems in Banach algebra* (Birkhäuser, Basel, Switzerland, 2007).
22. G. J. Murphy, Glasgow Math. J. **23**, 83 (1982).
23. A. Pelczynski, Bull. Acad. Polon. Sci. **13**, 31–41 (1965).
24. F. Räbiger and W. J. Ricker, Aspects of operator theory in heriditarily indecomposable Banach spaces, Universität Tübingen-University of New South Wales, Conferenza tenuta il 16 settembre, 1997.
25. C. E. Rickart, *General theory of Banach algebra* (Princeton, Van Nostrand, 1960).

26. M. Schechter, Bull. Amer. Math. Soc. **74**, 1139 (1968).
27. Th. Schlumprecht, Israel J. Math. **76** (1-2), 81 (1991).
28. H. Skhiri, Proc. Amer. Math. Soc. **126** (5), 1381 (1998).
29. J.I. Vladimirskii, Sov. Math. Dokl. **8**, 739 (1967).
30. L. Weis, Math. Z. **178**, 429 (1981).