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# Properties of Polynomially Riesz Operators on Some Banach Spaces

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**Abstract**—The purpose of this paper is to investigate some properties of polynomially Riesz operators acting on some Banach spaces, moreover a partially positive answer to the M.R.F. Smyth's [29] conjecture is given.

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## 1. INTRODUCTION AND NOTATIONS

Let $X$ and $Y$ be two Banach spaces. We denote by $\mathcal{L}(X,Y)$ the set of bounded linear operators from $X$ into $Y$. The subset of all compact operators of $\mathcal{L}(X,Y)$ is designated by $\mathcal{K}(X,Y)$. If $A \in \mathcal{L}(X,Y)$, we write $N(A) \subseteq X$ and $R(A) \subseteq Y$ for the null space and range of $A$. We set $\alpha(A) := \dim N(A)$, $\beta(A) := \mathrm{codim} R(A)$. The set of upper semi-Fredholm operators is defined by

$$\Phi_+(X,Y) = \{A \in \mathcal{L}(X,Y) : \alpha(A) < \infty \text{ and } R(A) \text{ is closed in } Y\},$$

and the set of lower semi-Fredholm operators is defined by

$$\Phi_-(X,Y) = \{A \in \mathcal{L}(X,Y) : \beta(A) < \infty \quad (\text{then } R(A) \text{ is closed in } Y)\}.$$

Operators in $\Phi_\pm(X,Y) := \Phi_-(X,Y) \bigcup \Phi_+(X,Y)$ are called semi-Fredholm operators from $X$ into $Y$ while $\Phi_-(X,Y) \bigcap \Phi_+(X,Y)$ denotes the set of Fredholm operators from $X$ into $Y$. If $A \in \Phi(X,Y)$, the number $i(A) = \alpha(A) - \beta(A)$ is called the index of $A$. If $X = Y$, then $\mathcal{L}(X,Y)$, $\mathcal{K}(X,Y)$, $\Phi_+(X,Y)$, $\Phi_\pm(X,Y)$, and $\Phi(X,Y)$ are replaced, respectively, by $\mathcal{L}(X)$, $\mathcal{K}(X)$, $\Phi_+(X)$, $\Phi_\pm(X)$, and $\Phi(X)$. For the properties of these sets we refer to [12, 18].

Let $A \in \mathcal{L}(X)$. The spectrum of $A$ will be denoted by $\sigma(A)$. The resolvent set of $A$, $\rho(A)$, is the complement of $\sigma(A)$ in the complex plane. An operator $A \in \mathcal{L}(X)$ is called a Weyl operator if $A$ is a Fredholm operator with index zero. The following definitions are well known: the Wolf essential spectrum is defined by

$$\sigma_e(A) = \{\lambda \in \mathbb{C} : \lambda I - A \text{ is not Fredholm}\},$$

and the Weyl spectrum $\omega(A)$ is

$$\omega(A) = \{\lambda \in \mathbb{C} : \lambda I - A \text{ is not Weyl}\}.$$

(Obviously $\sigma_e(T) \subseteq \omega(T)$).

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**Definition 1.1.** Let $F \in \mathcal{L}(X, Y)$. $F$ is called a Fredholm perturbation if $U + F \in \Phi(X, Y)$ whenever $U \in \Phi(X, Y)$. $F$ is called a upper (resp. lower) semi-Fredholm perturbation if $F + U \in \Phi_+(X, Y)$ (resp. $\Phi_-(X, Y)$) whenever $U \in \Phi_+(X, Y)$ (resp. $\Phi_-(X, Y)$).

The sets of Fredholm, upper semi-Fredholm and lower semi-Fredholm perturbations are denoted by $\mathcal{F}(X, Y), \mathcal{F}_+(X, Y)$ and $\mathcal{F}_-(X, Y)$, respectively. In particular, it is shown that $\mathcal{F}(X, Y)$, $\mathcal{F}_+(X, Y)$ and $\mathcal{F}_-(X, Y)$ are closed subsets of $\mathcal{L}(X, Y)$, and if $X = Y$, then $\mathcal{F}(X) = \mathcal{F}(X, X)$, $\mathcal{F}_+(X) = \mathcal{F}_+(X, X)$ are closed two-sided ideals of $\mathcal{L}(X)$. Recently, in the second edition of his book [26], Schechter proved that $\mathcal{F}_-(X) = \mathcal{F}_-(X, X)$ is also a closed two-sided ideal of $\mathcal{L}(X)$.

It is worth noticing that, in general, the structure ideal of $\mathcal{L}(X)$ is extremely complicated. Most of the results on ideal structure deal with the well known closed ideals which have arisen from applied work with operators. We can quote, for example, compact operators, strictly singular operators; strictly cosingular operators, upper and lower semi-Fredholm perturbations and Fredholm perturbations. In general, we have $\mathcal{K}(X) \subset \mathcal{S}(X) \subset \mathcal{F}_+(X) \subseteq \mathcal{F}(X)$ and $\mathcal{K}(X) \subset \mathcal{CS}(X) \subset \mathcal{F}_-(X) \subseteq \mathcal{F}(X)$ where $\mathcal{S}(X)$ (resp. $\mathcal{CS}(X)$) stands for the ideal of strictly singular (resp. strictly cosingular) operators of $\mathcal{L}(X)$ (see [7]). The inclusion $\mathcal{S}(X) \subset \mathcal{F}_+(X)$ is due to Kato (cf. [18]) while $\mathcal{CS}(X) \subset \mathcal{F}_-(X)$ was proved by Vladimirskii [31].

An operator $R \in \mathcal{L}(X)$ is called a Riesz operator if $\lambda - R \in \Phi(X)$ for all scalars $\lambda \neq 0$. Let $\mathcal{R}(X)$ denote the class of all Riesz operators. For our purpose we recall that Riesz operators satisfy the Riesz-Schauder theory of compact operators, $\mathcal{R}(X)$ is not an ideal of $\mathcal{L}(X)$ [7] and $\mathcal{F}(X)$ is the largest proper two-sided ideal of $\mathcal{L}(X)$ contained in $\mathcal{R}(X)$ [25]. Hence $\mathcal{K}(X), \mathcal{S}(X), \mathcal{CS}(X), \mathcal{F}_+(X)$ and $\mathcal{F}_-(X)$ are also contained in $\mathcal{R}(X)$. For more information concerning this family of operators we refer to [12, 25].

**Definition 1.2.** An operator $A \in \mathcal{L}(X)$ is said to be polynomially Riesz (resp. polynomially Fredholm perturbation) if there exists a nonzero complex polynomial $P$ such that $P(A)$ is a Riesz operator (resp. $P(A)$ is a Fredholm perturbation).

By virtue of the famous theorem of Lomonosov [20], it follows that every polynomially compact operator has a non-trivial invariant subspace. This result does not apply to the case of polynomially Riesz operators. Indeed, C. Read [24] gave one example of a strictly singular operator which has not a non-trivial invariant subspace, this can be seen as an answer to the fact that Fredholm perturbations, in general, are not polynomially compact. One of the major questions in operator theory is the characterization of Riesz operators (resp. Polynomially Riesz operators) in Banach spaces. This subject is related directly to the geometry of Banach spaces. Progress in this area has been made with the appearance of hereditarily indecomposable Banach spaces and the results established in this direction by T. Gowers, B. Maurey, T. Schlumprecht and other mathematicians ([1, 3–5, 15, 16, 28]). This attractive discovered made it possible to divide the structure of Banach spaces into two categories, which was interpreted by the dichotomy theorem of T. Gowers [15].

A bounded operator $A$ on $X$ is called quasi-nilpotent if $\lim\limits_{n \longrightarrow +\infty} ||A^n||^{\frac{1}{n}} = 0$ (which is equivalent to $\sigma(A) = \{0\}$).

It is a long standing question whether every Riesz operator splits as the quasinilpotent operator and a compact operator. Such decomposition is called a West decomposition. This last was proved in the case of Hilbert spaces by T.T. West [34]. The extension of the result to the case of $l_p$-spaces $(1 \leq p < \infty)$ and more generally to the spaces having F.D.P.B.D (Finite Dimensional P-block Decomposition) property was obtained by K. Davidson and D. Herrero [8]. In (1988), using the concept of the $B$-convexity of $L_p[0,1]$ $(1 < p < \infty)$ spaces, H.J. Zhong [35] showed this decomposition for Riesz operators acting on these spaces but the problem is always open in the case of an arbitrary Banach space. However M.R.F. Smyth [29, p. 149] conjectured that every Riesz operator can be decomposed into the sum of a Fredholm perturbation and a quasinilpotent operator, this bring us to give the following definition of the Smyth decomposition.

**Definition 1.3.** Let $X$ be a Banach space and let $R$ be a Riesz operator in $\mathcal{L}(X)$. We say that $R$ satisfies the Smyth decomposition if $R = F + Q$ where $F$ is a Fredholm perturbation and $Q$ is a quasinilpotent operator.

It is easy to observe that every Riesz operator which has the West decomposition satisfies the Smyth decomposition. The reverse is an open problem, owing to the fact that there is not yet a very good knowledge and description of the class of Fredholm perturbations via compact and quasinilpotent operators. We recall just that in $l_p$-spaces $(1 \leq p < \infty)$, these two decompositions are equivalent, this is an immediate consequence of the uniqueness of the nontrivial closed two-sided ideal in $\mathcal{L}(l_p)$ [12].

The paper is summarized as follows:
First of all, we establish a principal result concerning the class of polynomially Fredholm perturbations in Banach spaces. Motivated by the conjecture due to M.R.F. Smyth [29], a characterization of the set of polynomially Riesz operators on various Banach spaces is given. We close this paper by discussing some interesting questions.

## 2. MAIN RESULTS

We start our analysis by the following crucial result which will be used in the sequel.

**Theorem 2.1.** *Let $X$ be a Banach space and let $T \in \mathcal{L}(X)$ such that there exists a non-constant complex polynomial $P$ for which the set $\sigma_e(P(T))$ has an empty interior, then the interior of $\sigma_e(T)$ is empty. Moreover if $T$ is a polynomially Riesz operator, then $\sigma_e(T) = \omega(T)$ is finite. Also if $T$ is a Riesz operator then $T$ is polynomially Fredholm perturbation if and only if $T^n$ is a Fredholm perturbation for some $n \in \mathbb{N}$.*

**Proof.** If the set $\sigma_e(P(T))$ has an empty interior, the spectral mapping theorem gives that $p(Int(\sigma_e(T))) = Int(\sigma_e(p(T))) = \emptyset$, the fact that $P$ is a non-constant analytic function, then it is an open mapping, so if the set $Int(\sigma_e(T))$ is non empty, thus $Int(p(\sigma_e(T)))$ must be nonempty, which is a contradiction and the result for the first assertion is concluded. Now, if $p(T)$ is Riesz for some nonzero complex polynomial $P$, then it follows that $p(\sigma_e(T)) = \sigma_e(p(T)) = \{0\}$, which implies that $\sigma_e(T)$ is finite. To prove the equality $\sigma_e(T) = \omega(T)$, it suffices to show that $\mathbb{C}\backslash\sigma_e(T) \subseteq \mathbb{C}\backslash\omega(T)$. Let $\lambda \in \mathbb{C}\backslash\sigma_e(T)$, the set $\mathbb{C}\backslash\sigma_e(T)$ is connected and contains in particular the resolvent set of the operator $T$. The stability of the index on the connected components shows that $\lambda \in \mathbb{C}\backslash\omega(T)$ and gives the result of the second assertion. Now, let $p$ be a nonzero polynomial such that $p(T)$ is a Fredholm perturbation. Writing $p(\lambda) = a_0(\lambda - \lambda_1)\ldots(\lambda - \lambda_m)$, we have $a_0(T - \lambda_1 I)\ldots(T - \lambda_m I) = F$, where $F$ is a Fredholm perturbation. Since $T$ is a Riesz operator, we get $T - \lambda_i I$ is a Fredholm operator for each nonzero $\lambda_i$, then there exists at least one $\lambda_i$ such that $\lambda_i = 0$ $(1 \leq i \leq m)$: if it was not the case, then $F$ would be a Fredholm perturbation and a Fredholm operator, which contradicts the infinite-dimensionality of $X$. We deduce that, for some $1 \leq n \leq m$, $a_0 T^n S = F$ where $S \in \Phi(X)$. Therefore $T^n$ is a Fredholm perturbation. Indeed, since $S \in \Phi(X)$, there exist $K_1 \in \mathcal{K}(X), A_0 \in \mathcal{L}(X)$ such that $SA_0 = I - K_1$ (see [27, Theorem 3.7]), thus $a_0 T^n S A_0 = F A_0$, this fact implies that $a_0 T^n (I - K_1) = F A_0$ and consequently $a_0 T^n = F A_0 + a_0 T^n K_1$, but $F \in \mathcal{F}(X)$ and $K_1 \in \mathcal{K}(X)$, this gives that $F A_0 + a_0 T^n K_1 \in \mathcal{F}(X)$ and finally $T^n \in \mathcal{F}(X)$ which achieves the proof. The converse is trivial. □

**Remark 2.1.** It is obvious to see the following implications: $T$ is polynomially compact $\Longrightarrow T$ is polynomially Fredholm perturbation $\Longrightarrow T$ is polynomially Riesz $\Longrightarrow \omega(T)$ is finite.

We note that in general, the reverse of the two first implications is not true, indeed, for the first implication, it suffices to take the example established by C. Read [24] concerning a strictly singular operator (thus is polynomially Fredholm perturbation and consequently is polynomially Riesz) but it is not polynomially compact because it does not have any nontrivial invariant subspace.

Concerning the second implication, we can take the following celebrated example due to C. Foias and C. Pearcy [11, 17, 22]:

$T : l_2 \longrightarrow l_2$ defined by $T(e_1) = 0$ and $T(e_{n+1}) = \tau_n e_n, n \geq 1$ where $\{e_n\}_1^{+\infty}$ is the canonical basis of $l_2$ and the sequence $\{\tau_n\}_1^{+\infty}$ is given by $\{\frac{1}{2}, \frac{1}{2^4}, \frac{1}{2}, \frac{1}{2^{16}}, \frac{1}{2}, \frac{1}{2^4}, \frac{1}{2}, \frac{1}{2^{64}}, \frac{1}{2}, \frac{1}{2^4}, \cdots\}$, this sequence can be written in the form $\tau_n = \frac{1}{2^{4^k}}$ if $n = 2^k \pmod{2^{k+1}}$. For all $n \geq 1, T^n$ is not a compact operator, indeed, one knows that if $S$ is a nonzero bounded operator commuting with $T$, then $S$ can not be compact (see [11, Theorem 5, p. 403]) and since that every power of $T$ commutes with $T$, we obtain that $T^n$ is not a compact operator, but $T$ is quasinilpotent, then $T$ is a Riesz operator and in particular it is polynomially Riesz, the use of Theorem 2.1 together with the uniqueness of the closed two-sided ideal in $\mathcal{L}(l_2)$ implies that $T$ is not polynomially compact.

A Banach space is said to be decomposable if it is the topological direct sum of two closed infinite dimensional subspaces. A Banach space is said to be hereditarily indecomposable (in short H.I space) if it does not contain any decomposable subspace. The class of hereditarily indecomposable Banach spaces was first introduced and investigated by T. Gowers and B. Maurey [16].

A Banach space $X$ is said to be quotient hereditarily indecomposable (in short Q.H.I) if no infinite dimensional quotient of a subspace of $X$ is decomposable.

One of the main facts relating to the classes of H.I and Q.H.I Banach spaces is the following result:

**Lemma 2.1.** Let $X$ be a Banach space.

(a) If $X$ is a complex H.I Banach space, then every operator in $\mathcal{L}(X)$ can be written in the form $\lambda I + S$ where $\lambda \in \mathbb{C}$ and $S \in \mathcal{S}(X)$.

(b) If $X$ is a complex Q.H.I Banach space, then every operator in $\mathcal{L}(X)$ can be written in the form $\lambda I + S$ where $\lambda \in \mathbb{C}$ and $S \in \mathcal{CS}(X)$.

For more details we refer to [9, 13, 16].

**Remark 2.2.** It is worth noticing that the reverse of assertions a) and b) of Lemma 2.1 are in general false, indeed V. Ferenczi [10] gave a rather complex construction of an indecomposable Banach space $X$ having copies of $l_p$ (then it is not hereditarily indecomposable) with $\mathcal{L}(X) = \mathbb{C}I \bigoplus \mathcal{S}(X)$. Another characterization of H.I Banach spaces was established by the same author and asserts that:

$$X \quad \text{is} \quad \text{H.I} \Longleftrightarrow \mathcal{L}(Y,X) = \mathbb{C}I_{(Y,X)} \bigoplus \mathcal{S}(Y,X)$$

for every closed subspace $Y$ of $X$ (here $I_{(Y,X)}$ is the canonical injection from $Y$ into $X$).

**Definition 2.1.** Let $X$ be a Banach space, $X$ is said to be a Smyth Banach space if for every $R \in \mathcal{R}(X)$, $R$ satisfies the Smyth decomposition.

In the following proposition we will prove that many well known spaces in the literature are a Smyth Banach spaces.

**Proposition 2.1.** *The following spaces are Smyth Banach spaces*:

1. $l_p (1 \le p < \infty)$;
2. $L_p (1 < p < \infty)$;
3. $l_p \times l_r, (r \neq p)$;
4. H.I and Q.H.I Banach spaces;
5. $Z = X \times Y$ where $X$ is a reflexive H.I Banach space and $Y$ is a closed subspace of $X$ such that $\dim X/Y = \infty$.

**Proof.** For assertions **(1)** and **(2)**, this follows immediately from the fact that every Riesz operator defined on these two spaces satisfies the West decomposition (see [8, 35]).

**(3)** Assume that $p < r$, let $A \in \mathcal{L}(l_p \times l_r)$, then $A$ can be written in the form $A = \begin{pmatrix} B & C \\ D & E \end{pmatrix}$ where $B \in \mathcal{L}(l_p)$, $C \in \mathcal{L}(l_r, l_p)$, $D \in \mathcal{L}(l_p, l_r)$ and $E \in \mathcal{L}(l_r)$. On the other hand, we have $\mathcal{L}(l_r, l_p) = \mathcal{K}(l_r, l_p)$, $\mathcal{L}(l_p, l_r) = \mathcal{S}(l_p, l_r)$ (see [19, Definition 2, c. 1]), which shows that $\begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix} \in \mathcal{F}(l_p \times l_r)$, moreover, it can be observed that $B \in \mathcal{R}(l_p)$ and $E \in \mathcal{R}(l_r)$, it follows that $B = K + Q$ and $E = K' + Q'$ where $K \in \mathcal{K}(l_p)$, $K' \in \mathcal{K}(l_r)$ and $Q, Q'$ are quasinilpotents, thus $\begin{pmatrix} B & C \\ D & E \end{pmatrix} = \begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix} + \begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & K' \end{pmatrix} + \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & Q' \end{pmatrix}$. The sum of the three first operators is a Fredholm perturbation in $\mathcal{L}(l_p \times l_r)$ while the sum of the two last is quasinilpotent (since they are quasinilpotents operators which commutate), this ends the proof of this assertion.

Let us note that on this space P. Volkmann [32] showed the existence of two maximal ideals such that their intersection is the class of strictly singular operators.

**(4)** In the case of H.I and Q.H.I Banach spaces the Smyth decomposition is trivial, indeed on these spaces, we deduce directly that $\mathcal{R}(X) = \mathcal{S}(X) = \mathcal{F}(X)$ if $X$ is H.I and $\mathcal{R}(X) = \mathcal{CS}(X) = \mathcal{F}(X)$ if $X$ is Q.H.I.

**(5)** Here, we have $\mathcal{L}(X,Y) = \mathcal{S}(X,Y) = \mathcal{F}(X,Y)$, which implies that $\mathcal{L}(Y,X) = \mathcal{F}(Y,X)$ (see [13]), moreover $\mathcal{R}(X) = \mathcal{S}(X) = \mathcal{F}(X)$ and $\mathcal{R}(Y) = \mathcal{S}(Y) = \mathcal{F}(Y)$. The remainder of the proof is similar to the case of assertion **3)**. On this space, M. Gonzalez [13] gave a negative answer concerning the relationship between semi-Fredholm perturbations and the sets of strictly singular and strictly cosingular operators. □

The Proposition 2.2 gives the structure of polynomially Riesz operators on Smyth Banach spaces.

**Proposition 2.2.** *Let $X$ be a Smyth Banach space, then $T \in \mathcal{L}(X)$ is polynomially Riesz if and only if $\omega(T)$ is finite. In this case, $T$ is decomposed into a finite direct sum*

$$T = \bigoplus_{i=1}^{n} (F_i + Q_i + \lambda_i I),$$

*where*

*(i) the $F_i$ are Fredholm perturbations*;

*(ii) the $Q_i$ are quasinilpotents*;

*(iii) $\omega(T) = \{\lambda_1, \cdots, \lambda_n\}$.*

**Proof.** According to Theorem 2.1, if $T$ is polynomially Riesz, then $\omega(T)$ is finite, now we examine the reverse, if $\omega(T) = \{\lambda_1, \ldots, \lambda_n\}$ and by taking the polynomial $p(\lambda) = \prod_{i=1}^{n} (\lambda - \lambda_i)$ we can establish that $p(T)$ is a Riesz operator. Indeed we have $\sigma_e(p(T)) = p(\sigma_e(T)) = \{0\}$ which gives the result. On the other hand we can find a collection $\{\Delta_1, \cdots, \Delta_n\}$ of closed subsets of $\sigma(T)$ satisfying

(i) $\bigcup_{i=1}^{n} \Delta_i = \sigma(T)$;

(ii) $\Delta_i \bigcap \Delta_j = \emptyset$ if $i \neq j$;

(iii) $\lambda_i \in \Delta_i$ for $i = 1, \ldots, n$.

If for every $i = 1, \ldots, n, N_i$ is a neighborhood of $\Delta_i$ which contains no other points of $\sigma(T)$, by using the spectral projections $P = \frac{1}{2\pi i} \int_{\partial N_i} (\lambda I - T)^{-1} d\lambda$ corresponding to $\Delta_i$ $(i = 1, \ldots, n)$, we can decompose $T$ as

$$T = \bigoplus_{i=1}^{n} (T_i + \lambda_i I),$$

where $\sigma_e(T_i) = \{0\}$, i.e., $T_i$ is a Riesz operator for every $i = 1, \ldots, n$. Next, making use of the Smyth decomposition of Riesz operators, each $T_i$ is the sum of a Fredholm perturbation and a quasinilpotent operator $Q_i$. Also, we can argue as follows: Writing $p(\lambda) = (\lambda - \lambda_1) \ldots (\lambda - \lambda_n)$, it follows that

$$p(T) = \bigoplus_{j=1}^{n} \left[ \prod_{i=1}^{n} (K_j + Q_i + (\lambda_j - \lambda_i) I) \right].$$

As already observed, for every $j = 1, \ldots, n$, the $j$-th direct sum in can be written into the sum of Fredholm perturbation and an operator of the form $V_j = Q_j^n + \alpha_{n-1} Q_j^{n-1} + \ldots \alpha_1 Q_j$ for some $\alpha_i \in \mathbb{C}$ $(i = 1, \ldots, n-1)$.

Now, it is known that if $a$ and $b$ are quasinilpotents elements which commutate in a normed algebra, then $a + b$ and $ab$ are both quasinilpotents, thus we can easily check that $V_j$ is quasinilpotent for every $j = 1, \ldots, n$ and consequently

$$T = \bigoplus_{i=1}^{n} (F_i + Q_i + \lambda_i I),$$

where

(i) the $F_i$ are Fredholm perturbations;

(ii) the $Q_i$ are quasinilpotents;

(iii) $\omega(T) = \{\lambda_1, \cdots, \lambda_n\}$.

which achieves the proof. □

**Remark 2.3.** On $L_p[0,1](1 \leq p < \infty)$ spaces, by taking account of the coincidence of all classes of Fredholm and semi-Fredholm perturbations [33, p. 287], we conclude that $T$ polynomially strictly singular operator $\Longleftrightarrow T$ polynomially strictly cosingular operator $\Longleftrightarrow T$ polynomially Fredholm perturbations $\Longrightarrow T$ polynomially Riesz.

Moreover, on these spaces we have the following Lemma:

**Lemma 2.2.** *Let $X = L_p[0,1]$ $(1 \leq p < \infty)$ and let $T \in \mathcal{L}(X)$, then $T$ polynomially compact $\Longleftrightarrow$ $T$ polynomially Fredholm perturbation.*

**Proof.** It suffices to prove the second implication. The use of [21, Theorem 1b)] gives the result. □

Let us give now the following decomposition of strictly singular operators on $L_p[0,1]$ spaces $(1 < p < \infty)$.

**Proposition 2.3.** *Let $X = L_p[0,1](1 < p < \infty)$, and $S$ be a strictly singular operator on $X$, then $S$ can be written in the form $S = K + Q + R$, where $K$ is a compact operator, $Q$ is nilpotent $(Q^2 = 0)$ and $PR = RP = 0$ for a suitably selected projection $P$ on $X$.*

**Proof.** It suffices to prove the result for $p > 2$, the case $1 < p < 2$ is established by duality.

Let $S$ be a strictly singular operator $X$, then for every infinite dimensional subspace $M$ in $X$, we can find an infinite dimensional subspace $H$ such that $H \subseteq M$ and the restriction of $S$ to $H$ is compact. In addition, according to the Kadec-Pelczynski Theorem [6] which asserts that an infinite dimensional closed subspace $M$ of $X$, is either isomorphic to $l_2$ and complemented in $X$, or it contains a subspace which is isomorphic to $l_p$ and complemented in $X$, we can find a non-trivial projection $P$ on $X$ such that $SP$ is a compact operator, which shows that $S$ can be written as the following $2 \times 2$ operator matrix relative to the direct sum $P(X) \bigoplus (I-P)(X)$:

$$S = \begin{pmatrix} PSP & PS(I-P) \\ (I-P)SP & (I-P)S(I-P) \end{pmatrix}.$$

Note that $PSP$ and $(I-P)SP$ are both compact. Therefore if we put

$$K = \begin{pmatrix} PSP & 0 \\ (I-P)SP & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & PS(I-P) \\ 0 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 0 \\ 0 & (I-P)S(I-P) \end{pmatrix}.$$

Then $K$ is compact, $Q$ is nilpotent $(Q^2 = 0)$ and $PR = RP = 0$. This proves the result. □

**Remark 2.4.** It is clear that if the operator $(I-P)S$ is compact, we obtain that $S = K_1 + Q$ where $K_1$ is compact and $Q$ is nilpotent.

## 3. CONCLUDING REMARKS AND OPEN QUESTIONS

In this last section, we present some remarks on the two famous examples of strictly singular not compact operators, the first operator is due to I. Gohberg, A. Markus and I.A. Feldmann [12] and is constructed on $L_p[-1,1]$ $(1 \leq p < \infty)$ spaces while the second one, it can be constructed at the same time on an H.I Banach space of T. Gowers and B. Maurey or on the complementably minimal Banach space $S$ due to T. Schlumprecht [28].

**Example 3.1.** Writing $L_p[-1,1]$ under the form

$$L_p[-1,1] = L_p[0,1] \bigoplus L_p[-1,0].$$

Let $N \subseteq L_p[0,1]; p \geq 1$ the closed linear hull of the system of functions

$$y_k(t) = \begin{cases} 2^{\frac{k}{p}}, & \text{if } t \in [2^{-k}, 2^{1-k}] \quad (k = 1, 2, \dots), \\ 0 & \text{if } t \notin [2^{-k}, 2^{1-k}] \quad (k = 1, 2, \dots). \end{cases}$$

Let $(e_k)_{k=1}^{\infty}$ be the canonical base of $l_p$. The operator $S \in \mathcal{L}(N, l_p)$ defined by the equalities $S(y_k) = e_k$ $(k = 1, 2, \dots)$ maps $N$ isometrically onto $l_p$, and the subspace $N$ has a direct complement in $L_p[0,1]$, this follows from the fact that the bounded operator $Q$ defined by

$$Q(x) = \sum_{k=1}^{\infty} f_k(x) y_k (x \in L_p[0,1])$$

where $f_k(x) = \int_0^1 x(t)[y_k(t)]^{p-1} dt (x \in L_p[0,1], k = 1, 2, \dots)$, is a projection which maps the whole space $L_p[0,1]$ onto $N$. Now, let $R \subseteq L_2[-1,0]$ be the closed linear hull of the orthonormal system of Rademacher functions

$$r_k(t) = \operatorname{sign} \sin(2^k \pi t) \quad (k = 1, 2, \dots).$$

Then $R$ is a closed subspace in every $L_p[-1,0]$ $(1 \leq p < \infty)$ and in $R$ the norms of all the spaces $L_p (1 \leq p < \infty)$ are topologically equivalent, moreover if $p \geq 2$ there exists a direct complement to $R$ in $L_p[-1,0]$. The projection here is given by

$$\overline{Q}(x) = \sum_{k=1}^{\infty} \left( \int_{-1}^{0} x(s) r_k(s) ds \right) r_k \quad (x \in L_p[-1,0]).$$

Thus $L_p[-1,1] = L_p[0,1] \bigoplus L_p[-1,0] = L_p[-1,0] \bigoplus N \bigoplus F$ (where $F$ is the direct complement to $N$ in $L_p[0,1]$). We define the operator $T$ by

$$T\left(x + z + \sum_k \alpha_k y_k\right) = \sum_k \alpha_k r_k \quad (x \in L_p[-1,0],\ z \in F) \quad (\text{if } 1 \leq p < 2),$$

and

$$T\left(x + y + \sum_k \alpha_k r_k\right) = \sum_k \alpha_k y_k \quad (x \in L_p[0,1],\ y \in G) \quad (\text{if } p > 2),$$

where $G$ is the direct complement to $R$ in $L_p[-1,0]$).

**Remark 3.1.** $T$ is a bounded strictly singular operator on $L_p[-1,1] (1 \leq p < \infty)$ but it is not compact, moreover $T$ is nilpotent, indeed $(T^2 = 0)$ and for the case $p = 1$, this gives an example of a weakly compact (not compact) operator.

**Example 3.2.** Let $c_{00}$ be the vector space of sequences in $\mathbb{R}$ for which only finitely coordinates are not zero and $S$ is the complementably minimal Banach space constructed by T. Schlumprecht in [28]. On the other hand, in [1] the authors showed the existence of a seminormalized block sequences $(x_i{}^*)_{i=1}^{\infty}$ in $S^*$ and an increasing sequence $C(l)_{l \in \mathbb{N}} \subseteq \mathbb{R}^+$ such that $C(l) \longrightarrow +\infty$ if $l \longrightarrow +\infty$, for which the following condition holds: $\mathcal{H})$ If $(z_i)_{i=1}^{\infty}$ is a block sequence in $S$, so that for each $i \in \mathbb{N}, x_i^*(z_i) = 1$ and $x_{i-1}^* < z_i < x_{i+1}^*$ $(x_0^* = 0)$, then for any $2 \leq l \in \mathbb{N}$ and $(\lambda_i)_{i=1}^{\infty} \in c_{00}$, we have

$$\left\| \sum_{i=1}^{\infty} \lambda_i e_i \right\|_l \leq \frac{1}{C(l)} \left\| \sum_{i=1}^{\infty} \lambda_i z_i \right\|_S .$$

(where $||.||_l$ is an equivalent norm to that of the space $S$). Then the operator $\widetilde{T} = \sum_{i=1}^{\infty} x_i^* \bigotimes e_i$, with $\widetilde{T}(x) = \sum_{i=1}^{\infty} x_i^*(x) e_i$, for $x \in S$, is bounded, strictly singular, but not compact.

**Remark 3.2.** This operator shows that hereditarily indecomposable Banach space of T. Gowers and B. Maurey is not simple, i.e. in this case, the set of compact operators is not the unique nontrivial closed two-sided ideal, we will prove that $\widetilde{T}$ is quasinilpotent. Indeed, Since $\widetilde{T}$ is strictly singular, then its spectrum $sp(\widetilde{T})$ must check the Riesz-Schauder theory, an easy calculation shows that the equation $\widetilde{T}(x) = \lambda x$ $(x \neq 0)$ does not have a nonzero solution $\lambda$ which proves that the spectrum of $\widetilde{T}$ is reduced to the set $\{0\}$.

**Remark 3.3.** Also let us note that the null space of strictly singular operators in Banach spaces can be of finite dimensions. In the preceding example, the space $N(\widetilde{T})$ is of infinite dimensional, since the set $\mathbb{N}\Big\backslash \bigcup_{i=1}^{+\infty} \mathrm{Supp}(x_i^*)$ is infinite (where $\mathrm{Supp}(x_i^*)$ is the support of $x_i^*$ in $S^*$) (for more details, see [1]).

We complete this study by stating some interesting questions.

*Question 1.* Let $X$ be a Banach space and let $\mathcal{I}(X)$ be a family of closed two sided ideals in $\mathcal{L}(X)$ defined by:

$$\mathcal{I}(X) = \{J \text{ such that for all } T \in J, T \text{ satisfies the West decomposition}\}.$$

Observing that $\mathcal{I}(X)$ is not empty, since it contains at least the ideal of compact operators.

Can we to say that $\mathcal{I}(X)$ must contain a maximal element (for the inclusion) represented by the ideal of Fredholm perturbations?

*Question 2.* Is it true that on every not simple H.I. Banach space, we can construct a strictly singular operator such that all its powers are not compacts?

*Question 3.* Is it true that all strictly singular operators acting on H.I Banach spaces satisfy the West decomposition?

*Question 4.* Is it true that all Riesz operators on $L_1[0, 1]$ satisfy the Smyth decomposition or the West decomposition? Here the class of Fredholm perturbations is nothing else but the ideal of weakly compact operators (see [23]).

*Question 5.* Let $X$ denote a Banach space, we say that $T \in \mathcal{L}(X)$ satisfies the problem of Salinas if there exists a compact operator $K$ such that $\sigma_\omega(T) = \sigma(T + K)$ [2]. It is further remarked that if this property is satisfied for every bounded operator on $X$, then every Riesz operator has the West decomposition, the problem whether the second property implies the first one is open, we recall that they are satisfied at the same time on $l_p$ $(1 \leq p < \infty)$ spaces. Also, it should be observed that in the case of the famous Argyros-Haydon Banach space $X$ for which the Calkin algebra $\mathcal{L}(X)/\mathcal{K}(X)$ is isomorphic to the complex field $\mathbb{C}$ [4], then these two properties hold.

*Question 6.* By which criterion, the richness of the algebra of bounded operators on Banach spaces is measured. By misuse of language, a rich algebra of bounded operators on a certain Banach space is connected to the fact that for example the space is decomposed and an infinite number of its subspaces have an infinite number of projections, this can be seen of another angle, indeed let $\mathcal{K}(\mathbb{C})$ be the family of all nonempty compact sets of the complex plane, a rich Banach space is a space for which the mappings spectra and essential spectra defined by

$$\begin{array}{ll} \sigma : \mathcal{L}(X) \longrightarrow \mathcal{K}(\mathbb{C}) & \quad \sigma_e : \mathcal{L}(X) \longrightarrow \mathcal{K}(\mathbb{C}) \\ \qquad A \longrightarrow \sigma(A) & \qquad\quad A \longrightarrow \sigma_e(A) \end{array}$$

are surjective. Let us note that this property is satisfied if $X$ is a Hilbert space. Unfortunately, it is not the case of hereditarily indecomposable Banach spaces (or a finite product of H.I Banach spaces), since here $\mathcal{K}(\mathbb{C})$ is replaced by a subset of the family of all convergent sequences in $\mathbb{C}$ as the range of the first mapping and as the range of the second one, we have the family of all the sets which are made up of one element (or finite number of elements).

Now the question is of knowing, which is the structure of Banach spaces $X$ such that these two mappings are surjective at the same time?

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