Research Article

Some Remarks on Perturbation Classes of Semi-Fredholm and Fredholm Operators

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We show the existence of Banach spaces X, Y such that the set of strictly singular operators $\mathcal{S}(X,Y)$ (resp., the set of strictly cosingular operators $\mathcal{CS}(X,Y)$) would be strictly included in $\mathcal{F}_+(X,Y)$ (resp., $\mathcal{F}_-(X,Y)$) for the nonempty class of closed densely defined upper semi-Fredholm operators $\Phi_+(X,Y)$ (resp., for the nonempty class of closed densely defined lower semi-Fredholm operators $\Phi_-(X,Y)$).

1. Introduction

It is well known that the work of T. Kato [1] on strictly singular operators has been the starting point of an interesting and complex domain in the operator theory, that is, Fredholm and semi-Fredholm perturbations between two Banach spaces X and Y denoted by $\mathcal{F}(X,Y)$, $(\mathcal{F}_+(X,Y), \mathcal{F}_-(X,Y))$, it has been the object of many works studying and analysing these operators, especially the inclusion between all these classes and the stability problem by passing to the dual (see [2–10]). The difficulty to study these questions comes from the fact that their properties are related directly to the geometry of Banach spaces. The new thing in this paper consists in studying all these classes for closed densely defined perturbed semi-Fredholm and Fredholm operators which are not necessarily bounded. For $X = Y$, Latrach and Dehici [11, Lemma 2.3] have shown that $\mathcal{F}(X) = \mathcal{F}^b(X)$, where $\mathcal{F}^b(X)$ refers to the class of Fredholm perturbations acting on $\Phi(X) \cap \mathcal{L}(X)$, it forms the largest closed two-sided ideal in the set of Riesz operators $\mathcal{R}(X)$. The framework of the sets $\mathcal{F}_+(X)$, $\mathcal{F}_+^b(X)$, $\mathcal{F}_-(X)$, and $\mathcal{F}_-^b(X)$ is quite different, we just notice the trivial inclusions $(\mathcal{F}_+(X) \subseteq \mathcal{F}_+^b(X)$ and $\mathcal{F}_-(X) \subseteq \mathcal{F}_-^b(X))$. These comments are also applicable if $X \neq Y$. Here, by means of Heriditarily indecomposable Banach spaces of

Gowers-Maurey denoted by X_{GM}, we will show that $\mathcal{S}(X_{GM} \times X_{GM}, X_{GM}) \subsetneq \mathcal{F}_+(X_{GM} \times X_{GM}, X_{GM})$ (resp., $\mathcal{CS}(X_{GM}^*, X_{GM}^* \times X_{GM}^*) \subsetneq \mathcal{F}_-(X_{GM}^*, X_{GM}^* \times X_{GM}^*)$), moreover, we will prove that the inclusions $\mathcal{S}(Z) \subsetneq \mathcal{F}_+^b(Z)$ (resp., $\mathcal{CS}(Z^*) \subsetneq \mathcal{F}_-^b(Z^*)$) are strict for an infinity of Banach spaces Z.

2. Preliminaries and notations

First of all, let us start with recalling some definitions and results about Fredholm theory.

Let X and Y be two Banach spaces, we denote by $\mathcal{C}(X, Y)$ the space of closed densely defined operators from X into Y, and $\mathcal{L}(X, Y)$ denote the space of bounded linear operators from X into Y. If $A \in \mathcal{C}(X, Y)$, $N(A)$ (resp., $R(A)$) denote the kernel (resp., the range) of A. Setting

$$\alpha(A) := \dim N(A), \qquad \beta(A) := \operatorname{codim} R(A). \tag{2.1}$$

The set of upper semi-Fredholm operators is defined by

$$\Phi_+(X, Y) := \{A \in \mathcal{C}(X, Y) : \alpha(A) < \infty \, (\text{and } R(A) \text{ is closed in } Y)\}, \tag{2.2}$$

while the set of lower semi-Fredholm operators is given by

$$\Phi_-(X, Y) := \{A \in \mathcal{C}(X, Y) : \beta(A) < \infty \, (\text{then } R(A) \text{ is closed in } Y)\}. \tag{2.3}$$

We denote by $\Phi(X, Y)$ the set $\Phi_+(X, Y) \bigcap \Phi_-(X, Y)$. If $A \in \Phi(X, Y)$, the index of A is the number $i(A) := \alpha(A) - \beta(A)$. When $X = Y$, the sets $\mathcal{L}(X, Y)$, $\mathcal{C}(X, Y)$, $\Phi_+(X, Y)$, $\Phi_-(X, Y)$, and $\Phi(X, Y)$ are replaced, respectively, by $\mathcal{L}(X)$, $\mathcal{C}(X)$, $\Phi_+(X)$, $\Phi_-(X)$, and $\Phi(X)$.

Definition 2.1. Let X and Y be two Banach spaces and $T \in \mathcal{L}(X, Y)$. T is said to be strictly singular, if its restriction to every closed infinite-dimensional subspace of X is not an isomorphism.

Let $\mathcal{S}(X, Y)$ denote the set of strictly singular operators from X into Y. In general, strictly singular operators are not compact (see [12, 4]). If $X = Y$, $\mathcal{S}(X) := \mathcal{S}(X, X)$ is a closed two-sided ideal of $\mathcal{L}(X)$ containing $\mathcal{K}(X)$ and if X is a separable Hilbert space, then $\mathcal{S}(X) := \mathcal{K}(X)$. For basic properties of strictly singular operators, we refer to [5, 8, 10, 12].

Let N be a closed subspace of a Banach space X. we denote by π_N the quotient map $X \to X/N$. The codimension of N, $\operatorname{codim}(N)$, is defined to be the dimension of the vector space X/N.

Definition 2.2. Let X, Y be two Banach spaces and $T \in \mathcal{L}(X, Y)$. T is said to be strictly cosingular if there exists no closed subspace N of Y with $codim(N) = \infty$ such that $\pi_N T : X \to Y/N$ is surjective.

Let $\mathcal{CS}(X, Y)$ denote the set of strictly cosingular operators on X. This class of operators was introduced by Pelczynski [13]. The set $\mathcal{CS}(X, Y)$ is a closed subspace of $\mathcal{L}(X, Y)$. If $X = Y$, $\mathcal{CS}(X) := \mathcal{CS}(X, X)$ forms a closed two-sided ideal of $\mathcal{L}(X)$ (see [14]).

Definition 2.3. Let X and Y be two Banach spaces and let $F \in \mathcal{L}(X,Y)$. F is called a Fredholm perturbation if $U + F \in \Phi(X,Y)$ whenever $U \in \Phi(X,Y)$. F is called an upper (resp., lower) semi-Fredholm perturbation if $U + F \in \Phi_+(X,Y)$ (resp., $U + F \in \Phi_-(X,Y)$) whenever $U \in \Phi_+(X,Y)$ (resp., $U \in \Phi_-(X,Y)$).

The sets of Fredholm, upper semi-Fredholm, and lower semi-Fredholm perturbations are denoted by $\mathcal{F}(X,Y)$, $\mathcal{F}_+(X,Y)$, and $\mathcal{F}_-(X,Y)$, respectively. If, in Definition 2.3, we replace $\Phi(X,Y)$, $\Phi_+(X,Y)$, and $\Phi_-(X,Y)$ by $\Phi^b(X,Y)$, $\Phi^b_+(X,Y)$, and $\Phi^b_-(X,Y)$, we obtain the sets $\mathcal{F}^b(X,Y)$, $\mathcal{F}^b_+(X,Y)$, and $\mathcal{F}^b_-(X,Y)$. These classes of operators were introduced and investigated in [12]. In particular, it is shown that $\mathcal{F}^b(X,Y)$ is a closed subset of $\mathcal{L}(X,Y)$, and $\mathcal{F}^b(X) := \mathcal{F}^b(X,X)$ is a closed two-sided ideal of $\mathcal{L}(X)$. In general, we have

$$\mathcal{K}(X,Y) \subseteq \mathcal{S}(X,Y) \subseteq \mathcal{F}^b_+(X,Y) \subseteq \mathcal{F}^b(X,Y), \tag{2.4}$$

$$\mathcal{K}(X,Y) \subseteq \mathcal{SC}(X,Y) \subseteq \mathcal{F}^b_-(X,Y) \subseteq \mathcal{F}^b(X,Y). \tag{2.5}$$

The inclusion $\mathcal{S}(X,Y) \subseteq \mathcal{F}^b_+(X,Y)$ is due to Kato [1], whereas the inclusion $\mathcal{CS}(X,Y) \subseteq \mathcal{F}^b_-(X,Y)$ was proved by Vladimirski [14].

Let X be a Banach space and $R \in \mathcal{L}(X)$. R is said to be a Riesz operator if R satisfies Riesz-Schauder theory of compact operators. The set of all Riesz operators will be denoted by $\mathcal{R}(X)$. However, we point out that, in general, $\mathcal{R}(X)$ is not an ideal of $\mathcal{L}(X)$. Moreover, M. Schechter [7] has proved that $\mathcal{F}^b(X)$ is the largest closed two-sided ideal of $\mathcal{R}(X)$. By using (2.4) and (2.5), we deduce that the classes $\mathcal{K}(X), \mathcal{S}(X), \mathcal{CS}(X), \mathcal{F}^b_+(X) := \mathcal{F}^b_+(X,X)$, and $\mathcal{F}^b_-(X) := \mathcal{F}^b_-(X,X)$ are included in $\mathcal{R}(X)$, therefore, if S belongs to one of these sets, 0 is an accumulation point of its spectrum (see [15, 7]).

Let X and Y be two Banach spaces and $A \in \mathcal{C}(X,Y)$. For every $x \in D(A)$ (the domain of A), we write

$$\|x\|_A := \|x\| + \|Ax\| \quad \text{(graph norm)}. \tag{2.6}$$

As already observed, $D(A)$ endowed with the norm $\|\cdot\|_A$ is a Banach space denoted by X_A, and A, as operator from X_A into Y, is bounded. If $D(A) \subseteq D(J)$, then J is A-defined. Furthermore, we have

$$\alpha(\hat{A}) = \alpha(A), \qquad \beta(\hat{A}) = \beta(A), \qquad R(\hat{A}) = R(A), \qquad \alpha(\hat{A}+\hat{J}) = \alpha(A+J), \tag{2.7}$$

$$\beta(\hat{A}+\hat{J}) = \beta(A+J), R(\hat{A}+\hat{J}) = R(A+J). \tag{2.8}$$

It is clear that the relations (2.7) and (2.8) lead to

$$A \in \Phi_+(X,Y) \Longleftrightarrow \hat{A} \in \Phi_+(X_A,Y), \tag{2.9}$$

$$A \in \Phi_-(X,Y) \Longleftrightarrow \hat{A} \in \Phi_-(X_A,Y), \tag{2.10}$$

$$A \in \Phi(X,Y) \Longleftrightarrow \hat{A} \in \Phi(X_A,Y). \tag{2.11}$$

3. Main results

We start this study by stating the following result which was established in [16].

PROPOSITION 3.1. *Let X and Y be two Banach spaces, then*

$$\mathcal{F}^b(X,Y) = \mathcal{F}(X,Y). \tag{3.1}$$

Before completing our analysis, let us give some elements which will be useful afterwards. First, let us recall the definition of totally incomparable Banach spaces that has been introduced for the first time by H. Rosenthal [17].

Definition 3.2. Two infinite-dimensional Banach spaces X and Y are called totally incomparable if there exists no infinite-dimensional Banach space Z which is isomorphic to a subspace of X and to a subspace of Y.

It should be observed that every two different spaces from the set $\{c_0\} \bigcup \{l_p\}$ are totally incomparable ([4]). More precisely, we have the following.

Let $p \in [1, \infty[$, if $p < r$ (resp. $r < p$), then

$$\mathcal{L}(l_r,l_p) = \mathcal{K}(l_r,l_p) = \mathcal{S}(l_r,l_p) = \mathcal{F}^b(l_r,l_p) = \mathcal{F}(l_r,l_p)$$
$$(\text{resp.}, \mathcal{L}(l_r,l_p) = \mathcal{S}(l_r,l_p) = \mathcal{F}^b(l_r,l_p) = \mathcal{F}(l_r,l_p) \neq \mathcal{K}(l_r,l_p)). \tag{3.2}$$

For more examples satisfying the previous identities, we can quote, for example, [18].

On the other hand, it is easy to observe that if X and Y are totally incomparable, then any bounded operator from X into Y is strictly singular. Moreover, the definition of Fredholm perturbations allows us to establish the following result.

LEMMA 3.3. *Let X and Y be two Banach spaces such that $\mathcal{L}(X,Y) = \mathcal{F}^b(X,Y) = \mathcal{F}(X,Y)$, then $\Phi^b(X,Y) = \varnothing$.*

We give now the definition of hereditarily indecomposable Banach spaces which will be used afterwards.

Definition 3.4. Let X be a Banach space. X is said to be indecomposable if it can not be written as a direct sum of two closed infinite-dimensional subspaces.

Definition 3.5. Let X be a Banach space. X is said to be hereditarily indecomposable (H.I) if all of its closed infinite-dimensional subspaces are indecomposables.

For a detailed study on these spaces, we refer to the famous results established by Gowers and Maurey [19, 20] in which we can find an example of a separable reflexive hereditarily indecomposable Banach space denoted by X_{GM} whose dual quotients inherit this property.

THEOREM 3.6 (see [21, Theorem 2.1]). *Let X be an X_{GM} Banach space, then*
(a)

$$\mathcal{L}(X_{GM} \times X_{GM}, X_{GM}) = \mathcal{F}^b_+(X_{GM} \times X_{GM}, X_{GM}) \neq \mathcal{S}(X_{GM} \times X_{GM}, X_{GM}), \tag{3.3}$$

(b)

$$\mathcal{L}(X^*{}_{GM}, X^*{}_{GM} \times X^*{}_{GM}) = \mathcal{F}^b_-(X^*{}_{GM}, X^*{}_{GM} \times X^*{}_{GM})$$
$$\neq \mathcal{CS}(X^*{}_{GM}, X^*{}_{GM} \times X^*{}_{GM}). \tag{3.4}$$

Remark 3.7. As an immediate consequence of this theorem, we deduce that

(a)

$$\mathcal{L}(X_{GM} \times X_{GM}, X_{GM}) = \mathcal{F}^b(X_{GM} \times X_{GM}, X_{GM}) \neq \mathcal{S}(X_{GM} \times X_{GM}, X_{GM}), \tag{3.5}$$

(b)

$$\mathcal{L}(X^*{}_{GM}, X^*{}_{GM} \times X^*{}_{GM}) = \mathcal{F}^b(X^*{}_{GM}, X^*{}_{GM} \times X^*{}_{GM})$$
$$\neq \mathcal{CS}(X^*{}_{GM}, X^*{}_{GM} \times X^*{}_{GM}). \tag{3.6}$$

We will prove that Theorem 3.6 remains true, respectively, for the perturbation classes $\mathcal{F}_+(X_{GM} \times X_{GM}, X_{GM})$ and $\mathcal{F}_-(X^*{}_{GM}, X^*{}_{GM} \times X^*{}_{GM})$; however, the proofs are more complicated.

The following lemma is essential in proving Theorem 3.10 which is regarded as an extension of Theorem 3.6 to the closed densely defined (unbounded) semi-Fredholm perturbed operators.

LEMMA 3.8. *Let X be an X_{GM} Banach space, then*
(a) $\Phi_+(X_{GM} \times X_{GM}, X_{GM}) \neq \varnothing$ *and* (b) $\Phi_-(X^*{}_{GM}, X^*{}_{GM} \times X^*{}_{GM}) \neq \varnothing$.

Proof. (a) For the class $\Phi_+(X_{GM} \times X_{GM}, X_{GM})$, the proof is based on the separability of the space $X_{GM} \times X_{GM}$ (with respect to the topology of its norm) and that of the space $X^*{}_{GM}$ endowed with the $*$-weak topology. In fact, [3] ensures the existence of a compact injective operator with a dense range from X_{GM} to $X_{GM} \times X_{GM}$, this implies that the operator $K^{-1} : R(K) \subseteq X_{GM} \times X_{GM} \to X_{GM}$ is a closed densely defined Fredholm operator and therefore, $\Phi(X_{GM} \times X_{GM}, X_{GM}) \neq \varnothing$, one sees that $\Phi_+(X_{GM} \times X_{GM}, X_{GM}) \neq \varnothing$.

(b) A similar approach by duality allows us to establish the result for the class of lower semi-Fredholm operators $\Phi_-(X^*{}_{GM}, X^*{}_{GM} \times X^*{}_{GM})$. $\square$

The next proposition, owing to Weis [9], will play a fundamental role in the proof of Theorem 3.10.

PROPOSITION 3.9. *Let Y be a Banach space, then*
(a) $\mathcal{L}(Y, Z) = \mathcal{S}(Y, Z) \bigcup \Phi_+(Y, Z)$ *for every Banach space Z if and only if Y is an hereditarily indecomposable Banach space,*
(b) $\mathcal{L}(X, Y) = \mathcal{CS}(X, Y) \bigcup \Phi_-(X, Y)$ *for every Banach space X if and only if the quotients of Y are hereditarily indecomposable Banach spaces.*

We now prove the following theorem.

THEOREM 3.10. *Let X be an X_{GM} Banach space, then*
(a) $\mathcal{L}(X_{GM} \times X_{GM}, X_{GM}) = \mathcal{F}_+(X_{GM} \times X_{GM}, X_{GM}) \neq \mathcal{S}(X_{GM} \times X_{GM}, X_{GM})$,
(b) $\mathcal{L}(X^*_{GM}, X^*_{GM} \times X^*_{GM}) = \mathcal{F}_-(X^*_{GM}, X^*_{GM} \times X^*_{GM}) \neq \mathcal{CS}(X^*_{GM}, X^*_{GM} \times X^*_{GM})$.

Proof. (a) It suffices to establish the inclusion $\mathcal{L}(X_{GM} \times X_{GM}, X_{GM}) \subseteq \mathcal{F}_+(X_{GM} \times X_{GM}, X_{GM})$. □

Let $S \in \Phi_+(X_{GM} \times X_{GM}, X_{GM})$, and let j be the embedding operator from X_S to $X_{GM} \times X_{GM}$ defined by $j : (D(S), \|\cdot\|_S) = X_S \rightarrow X_{GM} \times X_{GM}$, $j(x) = x$, one sees that j is strictly singular. In fact, since $S \in \Phi_+(X_{GM} \times X_{GM}, X_{GM})$, the relation (2.9) shows that $\widehat{S} \in \Phi_+(X_S, X_{GM})$, this implies the existence of finite codimensional subspace H in X_S, which is isomorphic to $R(\widehat{S}) = R(S)$. Furthermore, as $R(S)$ is a closed hereditarily indecomposable subspace of X_{GM}, then H will inherit this property in the Banach space X_S, which allows us to conclude that X_S is a hereditarily indecomposable Banach space. Moreover, $j \notin \Phi_+(X_S, X_{GM} \times X_{GM})$ because if $j \in \Phi_+(X_S, X_{GM} \times X_{GM})$, we would have $X_S \cong X_{GM} \times X_{GM}$, this contradicts the fact that $X_{GM} \times X_{GM}$ is not a hereditarily indecomposable Banach space. Next, by applying the Proposition 3.9(a), we deduce that j is a strictly singular operator from X_S into X_{GM}.

Let us take now some bounded operator $T \in \mathcal{L}(X_{GM} \times X_{GM}, X_{GM})$, we should show first that the spaces $X_{S+T} = (D(S+T), \|\cdot\|_{S+T})$ and $X_S = (D(S), \|\cdot\|_S)$ are isomorphic.

Indeed, let $x \in X_{S+T} = (D(S+T), \|\cdot\|_{S+T})$, then

$$
\begin{aligned}
\|x\|_{S+T} &= \|x\| + \|(S+T)x\| \\
&\leq \|x\| + \|S(x)\| + \|T(x)\| \\
&\leq \|x\| + \|S(x)\| + M\|x\| \\
&\leq (1+M)(\|x\| + \|S(x)\|) \\
&\leq (1+M)\|x\|_S.
\end{aligned}
\tag{3.7}
$$

Moreover, if $x \in X_S$, we can establish the following estimates:

$$
\begin{aligned}
\|x\|_S &= \|x\| + \|S(x)\| \\
&\leq \|x\| + \|(S+T)(x) - T(x)\| \\
&\leq \|x\| + \|(S+T)(x)\| + \|T(x)\| \\
&\leq \|x\| + \|(S+T)(x)\| + M\|x\| \\
&\leq (1+M)\|x\|_{S+T},
\end{aligned}
\tag{3.8}
$$

and finally,

$$
\frac{\|x\|_S}{(1+M)} \leq \|x\|_{S+T} \leq (1+M)\|x\|_S,
\tag{3.9}
$$

this ensures that the spaces X_{S+T} and X_S are isomorphic.

This isomorphism will be denoted by h, which is defined by $h(x) = x$. On the other hand, the operator $\widehat{T+S}$ defined by $\widehat{T+S} : X_{S+T} \rightarrow X_{GM}, (T+S)(x) = (Tjh)(x) + (Sjh)(x)$ $\forall x \in X_{S+T}$ is an element of $\Phi_+(X_{S+T}, X_{GM})$, this follows immediately from the fact that Tjh and Sjh belong, respectively, to the classes $\mathcal{S}(X_{S+T}, X_{GM})$ and $\Phi_+(X_{S+T}, X_{GM})$; next, by the use of the relation (2.9), we obtain $T+S \in \Phi_+(X_{GM} \times X_{GM}, X_{GM})$ and, therefore,

$T \in \mathcal{F}_+(X_{GM} \times X_{GM}, X_{GM})$. Let us now consider the projection operator $\mathrm{Pr} : X_{GM} \times X_{GM} \to X_{GM}$ defined by $\mathrm{Pr}(x, y) = x$. Obviously, $\mathrm{Pr} \in \mathcal{L}(X_{GM} \times X_{GM}, X_{GM})$, but this operator is not strictly singular because its restriction to the subspace $X \times \{0\}$ is an isomorphism; consequently, $\mathcal{L}(X_{GM} \times X_{GM}, X_{GM}) = \mathcal{F}_+(X_{GM} \times X_{GM}, X_{GM}) \neq \mathcal{S}(X_{GM} \times X_{GM}, X_{GM})$, which achieves the proof.

(b) Let $S \in \Phi_-(X_{GM}^*, X_{GM}^* \times X_{GM}^*)$ and let $X_S = (D(S), \|.\|_S)$, then the operator j^* defined from X_S to X_{GM}^* by $j^*(x) = x$ is not an element of $\Phi_-(X_S, X_{GM}^*)$ because if we suppose that this is not the case, we obtain $X_S \simeq X_{GM}^*$ and, therefore, $X_{GM}^*/N(S) \simeq X_S/N(S) \simeq R(S)$, which is a finite codimensional decomposable subspace of $X_{GM}^* \times X_{GM}^*$, this contradicts the fact that the quotients of X_{GM}^* are indecomposable Banach spaces. Moreover, the Proposition 3.9(b) ensures that j^* is a strictly cosingular operator.

Let us take $T \in \mathcal{L}(X_{GM}^*, X_{GM}^* \times X_{GM}^*)$; as in the proof of (a), the operator $\widehat{T+S}$ defined from X_{S+T} to $X_{GM}^* \times X_{GM}^*$ can be written under the form $\widehat{T+S}(x) = (Tj^*h)(x) + (Sj^*h)(x) \ \forall x \in X_{S+T}$, which gives that this operator is an element of the set of $\Phi_-(X_{S+T}, X_{GM}^* \times X_{GM}^*)$, this follows immediately from the fact that the operators Tj^*h and Sj^*h belong, respectively, to the classes $\mathcal{CS}(X_{S+T}, X_{GM}^* \times X_{GM}^*)$ and $\Phi_-(X_{S+T}, X_{GM}^* \times X_{GM}^*)$; next, by the use of the relation (2.10), we infer that $T + S \in \Phi_-(X_{GM}^*, X_{GM}^* \times X_{GM}^*)$ and, therefore, $T \in \mathcal{F}_-(X_{GM}^*, X_{GM}^* \times X_{GM}^*)$, this gives that $\mathcal{L}(X_{GM}^*, X_{GM}^* \times X_{GM}^*) = \mathcal{F}_-(X_{GM}^*, X_{GM}^* \times X_{GM}^*)$. Now consider the operator i defined from X_{GM}^* to $X_{GM}^* \times X_{GM}^*$ by $i : X_{GM}^* \to X_{GM}^* \times X_{GM}^*$, $i(x) = (x, 0)$, this operator is not strictly cosingular. In fact, since $i \in \mathcal{L}(X_{GM}^*, X_{GM}^* \times X_{GM}^*)$, then $i \in \mathcal{F}_-(X_{GM}^*, X_{GM}^* \times X_{GM}^*)$. Let H be the closed subspace $H = \{(0, y), y \in X_{GM}^*\}$ and denote, by π_H, the quotient map $\pi_H : X_{GM}^* \times X_{GM}^* \to (X_{GM}^* \times X_{GM}^*)/H$. Clearly, the operator $\pi_H o i : X_{GM}^* \to (X_{GM}^* \times X_{GM}^*)/H$ is surjective. Since $codim(H) = \infty$, we infer that i is not strictly cosingular from X_{GM}^* to $X_{GM}^* \times X_{GM}^*$. Consequently, $\mathcal{F}_-(X_{GM}^*, X_{GM}^* \times X_{GM}^*) \neq \mathcal{CS}(X_{GM}^*, X_{GM}^* \times X_{GM}^*)$, which ends the proof.

Given a complex Banach space X and an operator $T \in \mathcal{L}(X)$, we define

$$
\begin{aligned}
\sigma_e(T) &= \{\lambda \in \mathbb{C} : \lambda I - T \notin \Phi^b(X)\}, \\
\sigma_+(T) &= \{\lambda \in \mathbb{C} : \lambda I - T \notin \Phi_+^b(X)\}, \\
\sigma_-(T) &= \{\lambda \in \mathbb{C} : \lambda I - T \notin \Phi_-^b(X)\}.
\end{aligned}
\tag{3.10}
$$

It is well known that $\sigma_e(T)$ is a nonempty compact set of the field $\mathbb{C}$ because it coincides with the spectrum of the image of T in the Calkin algebra $\mathcal{L}(X)/\mathcal{K}(X)$ (see [22]). On the other hand, it is clear that

$$
\sigma_+(T) \bigcup \sigma_-(T) \subseteq \sigma_e(T).
\tag{3.11}
$$

Moreover, the stability of the index of a semi-Fredholm operator under small perturbations [4, Proposition 2.c.9] provides the inclusions

$$
\begin{aligned}
Fr(\sigma_e(T)) &\subseteq \sigma_+(T), \\
Fr(\sigma_e(T)) &\subseteq \sigma_-(T),
\end{aligned}
\tag{3.12}
$$

where $Fr(\sigma_e(T))$ denotes the boundary of the set $\sigma_e(T)$.

The following result shows the fact that the sets $\sigma_e(T)$ possess empty interiors in the field $\mathbb{C}$ enables us to derive some nice properties of the classes of semi-Fredholm perturbations. More precisely, we have the following proposition.

PROPOSITION 3.11. *Let X be a Banach space such that $\overset{0}{\sigma_e}(T) = \varnothing$ for every $T \in \mathcal{L}(X)$, then*

$$\mathcal{F}_+^b(X) = \mathcal{F}_-^b(X) = \mathcal{F}^b(X). \tag{3.13}$$

Proof. In this case, we obtain that $\Phi^b(X) = \Phi_+^b(X) = \Phi_-^b(X)$. In fact, let $T \in \mathcal{L}(X)$, from the inclusions (3.11) and (3.12), we conclude that $F_r(\sigma_e(T)) = \sigma_e(T)\backslash\overset{0}{\sigma_e}(T) = \sigma_e(T) = \sigma_+(T) = \sigma_-(T)$. To prove (3.13), it suffices to establish the identities $\Phi^b(X) = \Phi_+^b(X) = \Phi_-^b(X)$. We will restrict our proof to the inclusion $\Phi_+^b(X) \subseteq \Phi^b(X)$ (the inclusion $\Phi_-^b(X) \subseteq \Phi^b(X)$ may be checked in the same way). This is equivalent to show that $C(\Phi^b(X)) \subseteq C(\Phi_+^b(X))$, where $C(\Phi^b(X))$ and $C(\Phi_+^b(X))$ denote, respectively, the sets $\mathcal{L}(X)\backslash\Phi^b(X)$ and $\mathcal{L}(X)\backslash\Phi_+^b(X)$. Let us consider $A \in C\Phi^b(X)$, then $A \notin \Phi^b(X)$, which implies that $0 \in \sigma_e(A) = \sigma_+(A)$ and, consequently, $A \notin \Phi_+^b(X)$, this allows us to get $A \in C\Phi_+^b(X)$ and it ends the proof. $\square$

Finally, our last result in this work is stated by the following theorem.

THEOREM 3.12. *Let Z be an X_{GM} Banach space and let $X = X_{GM} \times X_{GM} \cdots \times X_{GM}$ (n times, $n \geq 2$). Denote by $Y = X \times Z = X_{GM} \times \cdots \times X_{GM}$ (n + 1 times), then*
    (a)

$$\mathcal{F}_+^b(Y) = \mathcal{F}^b(Y) \neq \mathcal{S}(Y), \tag{3.14}$$

    (b)

$$\mathcal{F}_-^b(Y^*) = \mathcal{F}^b(Y^*) \neq \mathcal{CS}(Y^*). \tag{3.15}$$

Proof. First, we observe that every operator $A \in \mathcal{L}(Y)$ can be written under the form

$$A = \left(\begin{array}{cc} \begin{pmatrix} A_{11} & . & . & . & A_{1n} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ A_{n1}. & . & . & . & A_{nn} \end{pmatrix} & B \\ C & D \end{array}\right), \tag{3.16}$$

where $A_{ij} \in \mathcal{L}(X_{GM})$ $\forall i, j = 1,\ldots,n$, $B \in \mathcal{L}(X_{GM},X) = \mathcal{F}^b(X_{GM},X)$, $C \in \mathcal{L}(X,X_{GM}) = \mathcal{F}^b(X,X_{GM})$ (because $\mathcal{F}^b(H,M) = \mathcal{L}(H,M)$ if and only if $\mathcal{F}^b(M,H) = \mathcal{L}(M,H)$, see [22]), and $D \in \mathcal{L}(X_{GM})$.

Let us denote

$$A_0 = \left(\begin{pmatrix} A_{11} & . & . & . & A_{1n} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ A_{n1}. & . & . & . & A_{nn} \end{pmatrix} \quad \begin{matrix} 0 \\ \\ \\ \\ \\ \end{matrix} \\ \qquad 0 \qquad\qquad\qquad\qquad D \right). \tag{3.17}$$

We have $\sigma_e(A) = \sigma_e(A_0)$. Since $\operatorname{card}\sigma_e(A_{ij}) = \operatorname{card}\sigma_e(D) = 1$, $\forall i,j = 1,\dots,n$ (see [21, Proposition 3.1]), we infer that the set $\sigma_e(A)$ consists of an isolated point with finite number in $\mathbb{C}$, thus its interior is empty. Moreover, according to Proposition 3.3, we get $\mathscr{F}^b(Y) = \mathscr{F}^b_+(Y)$. Thus $P_j = \begin{pmatrix} 0 & j_Z \\ 0 & 0 \end{pmatrix}$, where $j_Z : X_{GM} \to X, j_Z(x) = (x,0,\dots,0)$ gives us an operator in $\mathscr{F}^b(Y) = \mathscr{F}^b_+(Y)$, which is not strictly singular.

Second, we show that if H is a reflexive Banach space, we obtain that $[\mathscr{F}^b_+(H)]^* = \mathscr{F}^b_-(H^*)$. Thus $P_j{}^* \in \mathscr{F}^b_-(Y^*)$. However, $P_j{}^*$ is not strictly singular because $j_Z{}^*$ is surjective. $\qquad\square$

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