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# Spectral Analysis of a Transport Operator Arising in Growing Cell Populations

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**Abstract** The present paper is concerned with the spectral analysis of a transport-like operator derived from a model introduced by Rotenberg describing the growth of a cell population. Each cell of this population is distinguished by its degree of maturity $\mu$ and its maturation velocity $v$. The biological boundaries of $\mu = 0$ and $\mu = a$ $(a > 0)$ are fixed and tightly coupled through mitosis. At mitosis daughter cells and mother cells are related by a general reproduction rule which covers all known biological ones. We first discuss in detail the spectrum of the streaming operator for smooth and partly smooth boundary conditions. Next, we discuss the existence and nonexistence of eigenvalues of the transport operator in the half plane $\{\lambda \in \mathbb{C} : \ Re\lambda > -\underline{\sigma}\}$ where $-\underline{\sigma}$ denotes the spectral bound of the streaming operator. In particular, the strict monotonicity of the leading eigenvalue (when it exists) of the transport operator with respect to different parameters of the equation is also considered. We close the paper by describing in detail the various essential spectra of the transport operator for wide classes of collision and boundary operators.

**Key words** transport equation · boundary conditions · positivity in the lattice sense · spectral analysis.

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## 1 Introduction

In [26] M. Rotenberg proposed the following partial differential equation

$$\frac{\partial \psi}{\partial t}(\mu, \upsilon, t) = -\upsilon \frac{\partial \psi}{\partial \mu}(\mu, \upsilon, t) - \sigma(\mu, \upsilon)\psi(\mu, \upsilon, t) + \int_0^c r(\mu, \upsilon, \upsilon')\psi(\mu, \upsilon', t)d\upsilon'$$
$$= A_K\psi = S_K\psi + B\psi \qquad (1.1)$$

to describe the growth of cells of a population where $S_K$ denotes the streaming operator and $B$ stands for the collision operator (the integral part of $A_K$). In this model cells are distinguished by their degree of maturity $\mu \in [0, a]$, $a > 0$, and their maturation velocity $\upsilon \in [0, c]$, $c > 0$. The degree of maturation $\mu$ is then defined so that $\mu = 0$ at the birth and $\mu = a$ at the death. Equation (1.1) describes the number density of cell population as a function of the degree of maturation $\mu$, the maturation velocity $\upsilon$ and time $t$. The function $r(., ., .)$ denotes the transition rate at which cells change their maturation velocities from $\upsilon$ to $\upsilon'$.

This model is one of the models of structured population dynamics with inherited properties. Inherited property models allow memory of generation time and among such models are the age-time and maturity-time models of proliferating cells population with inherited cycle length of Lebowitz and Rubinow [20]. These models are based on the assumption that the duration of the cycle from cell birth to mitosis is determined at birth.

Rotenberg discussed essentially the Fokker-Plank approximation of (1.1) for which he obtained numerical solutions. Using eigenfunction expansion technique Van der Mee and Zweifel obtained analytical solutions for a variety of boundary conditions [29]. The first theoretical approach to establish the well-posedness of (1.1) supplemented with Lebowitz and Rubinow boundary conditions can be found in [8, 28]. We quote also the works [18] and [19] where a stationary nonlinear version of Rotenberg was considered. Here the transition rate and the total transition cross section were allowed to depend on the density of population while the boundary conditions are modeled by a nonlinear reproduction law. Despite these works, to our knowledge, the spectral analysis of the operators $S_K$ and $A_K$ even for simple reproduction laws has not yet been investigated. The main purpose of this work is to fill this gap and to discuss various aspects of the spectral theory of the operators $S_K$ and $A_K$. The boundary conditions will be modeled by a general linear boundary operator, i.e., at the mitosis the daughter cells and parent cells are related by a general reproduction rule containing in particular all those considered in the papers [8, 20, 26, 28, 29]. The paper is organized as follows:

- Introduction,
- Notations and preliminaries,
- Spectral properties of $S_K$,
- Compactness results,
- Existence of the leading eigenvalues of $A_K$,
- The strict monotonicity of the leading eigenvalue of $A_K$,
- Essential spectra of $A_K$.

In Section 2 we make precise the functional setting of the problem and establish some preparation results required in the rest of the paper. The aim of Section 3 is to deal with the spectral theory of the streaming operator $S_K$ involving both smooth (compact) and partly smooth transition operators (cf. assumptions $(A1)$ and $(A2)$). Very precise results are given, in particular, the spectrum of the transition operator $K$ enters in play and it behaves like a collision operator at the boundary. Section 5 deals with the existence of eigenvalues of $A_K$ in $\{\lambda \in \mathbb{C}$ such that $Re\lambda > -\underline{\sigma}\}$ where $\underline{\sigma}$ denotes the spectral bound of $S_K$. Existence and nonexistence results of eigenvalues are given. The problem concerning the strict monotonicity of the leading eigenvalue of the operator $A_K$ with respect to the parameters of the equation is the main purpose of Section 6. We use the comparison results of the spectral radius of positive operators obtained in [21]. We show, in particular, that the leading eigenvalue (when it exists) increases strictly with respect to $K$ and $B$. Finally, in Section 7, we will describe the various essential spectra of the operator $A_K$ for general transition operators. Our analysis is based on the compactness results of Section 4 (Theorem 4.1), Proposition 7.1 and the knowledge of the precise picture of essential spectra of the operator $S_0$ (i.e. $K = 0$). We show, in particular, that for collision operators $B$ satisfying the assumption $(\mathcal{A}3)$ (cf. Section 4) and a sizable class of transition operators $K$ the essential spectra of $A_K$ and $S_0$ coincide.

## 2 Notations and Preliminaries

In this section we introduce the different notions and notations which we shall need in sequel. Let us first make precise the functional setting of the problem. Let

$$X_p := L_p([0, a] \times [0, c]; d\mu dv)$$

where $a > 0$, $c > 0$ and $1 \leq p < \infty$. We denote by $X_p^0$ and $X_p^1$ the following boundary spaces

$$X_p^0 := L_p(\{0\} \times [0, c]; v\mathrm{d}v),$$
$$X_p^1 := L_p(\{a\} \times [0, c]; v\mathrm{d}v)$$

endowed with their natural norms. In the sequel $X_p^0$ and $X_p^1$ will often be identified with $L_p([0, c]; vdv)$.

We define the partial Sobolev space $W_p$ by

$$W_p = \left\{\psi \in X_p \text{ such that } v\frac{\partial\psi}{\partial\mu} \in X_p\right\}.$$

It is well known (see [2] or [8]) that any $\psi$ in $W_p$ has traces on the spatial boundary $\{0\}$ and $\{a\}$ which belong to the spaces $X_p^0$ and $X_p^1$, respectively. They are denoted, respectively, by $\psi^0$ and $\psi^1$.

Let $K$ be the following boundary operator

$$\begin{cases} K \,:\, X_p^1 \to X_p^0 \\ \qquad u \to Ku \end{cases}$$

We define the free streaming operator $S_K$ by

$$\begin{cases} S_K\ :\ D(S_K)\subset X_p \longrightarrow X_p \\ \quad \psi\ \longrightarrow S_K\psi(\mu,v) = -v\dfrac{\partial\psi}{\partial\mu}(\mu,v) - \sigma(\mu,v)\psi(\mu,v) \\ D(S_K) = \{\psi\in W_p \text{ such that } \psi^0 = K\psi^1\}, \end{cases}$$

where the function $\sigma(.,.)$ is bounded below and belongs to $L^1_{loc}[(0,a)\times(0,c)]$.

Consider now the resolvent equation for the operator $S_K$,

$$(\lambda - S_K)\psi = \varphi, \tag{2.1}$$

where $\varphi$ is a given function of $X_p$, $\lambda\in\mathbb{C}$ and the unknown $\psi$ must be sought in $D(S_K)$. Let $\underline{\sigma}$ be the real defined by

$$\underline{\sigma} = \text{ess-}\inf\{\sigma(\mu,v),\ (\mu,v)\in[0,a]\times[0,c]\}.$$

For $Re\lambda > -\underline{\sigma}$ the solution is formally given by

$$\psi(\mu,v) = \psi(0,v)\, e^{-\frac{1}{v}\int_0^\mu(\lambda+\sigma(\mu',v))d\mu'} + \frac{1}{v}\int_0^\mu e^{-\frac{1}{v}\int_{\mu'}^\mu(\lambda+\sigma(\tau,v))d\tau}\ \varphi(\mu',v)\,d\mu'. \tag{2.2}$$

Accordingly, for $\mu = a$, we get

$$\psi(a,v) = \psi(0,v)\, e^{-\frac{1}{v}\int_0^a(\lambda+\sigma(\mu',v))d\mu'} + \frac{1}{v}\int_0^a e^{-\frac{1}{v}\int_{\mu'}^a(\lambda+\sigma(\tau,v))d\tau}\ \varphi(\mu',v)\,d\mu'. \tag{2.3}$$

In the sequel we shall need the following operators

$$P_\lambda\ :\ X_p^0 \longrightarrow X_p^1,\, u \longrightarrow (P_\lambda u)(0,v) := u(0,v)\, e^{-\frac{1}{v}\int_0^a(\lambda+\sigma(\mu',v))d\mu'};$$

$$Q_\lambda : X_p^0 \longrightarrow X_p,\, u \longrightarrow (Q_\lambda u)(0,v) := u(0,v)\, e^{-\frac{1}{v}\int_0^\mu(\lambda+\sigma(\mu',v))d\mu'};$$

$$\begin{cases} \Pi_\lambda : X_p \longrightarrow X_p^1, \\ \qquad \varphi \longrightarrow (\Pi_\lambda\varphi)(\mu,v) := \dfrac{1}{v}\displaystyle\int_0^a e^{-\frac{1}{v}\int_{\mu'}^a(\lambda+\sigma(\tau,v))d\tau}\ \varphi(\mu',v)\,d\mu'; \end{cases}$$

and

$$\begin{cases} \Xi_\lambda : X_p \longrightarrow X_p, \\ \qquad \varphi \longrightarrow (\Xi_\lambda\varphi)(\mu,v) := \dfrac{1}{v}\displaystyle\int_0^\mu e^{-\frac{1}{v}\int_{\mu'}^\mu(\lambda+\sigma(\tau,v))d\tau}\ \varphi(\mu',v)\,d\mu'. \end{cases}$$

Clearly, for $\lambda$ satisfying $Re\lambda > -\underline{\sigma}$, the operators $P_\lambda$, $Q_\lambda$, $\Pi_\lambda$ and $\Xi_\lambda$ are bounded. One readily checks that the norms of $P_\lambda$ and $Q_\lambda$ satisfy

$$\|P_\lambda\| \le e^{-\frac{a}{c}(Re\lambda+\underline{\sigma})} \ \text{ and } \ \|Q_\lambda\| \le (p\,(Re\lambda+\underline{\sigma}))^{-\frac{1}{p}}.$$

Moreover, a simple calculation using the Hölder inequality shows that

$$\|\Pi_\lambda\| \le (Re\lambda+\underline{\sigma})^{-\frac{1}{q}} \ \text{ and } \ \|\Xi_\lambda\| \le\ (Re\lambda+\underline{\sigma})^{-1}$$

where $q$ is the conjugate exponent of $p$, i.e. $q = \frac{p}{p-1}$. Using the operators above and the fact that $\psi$ must satisfy the boundary conditions, (2.3) may be written abstractly in the form

$$\psi^1 = P_\lambda K \psi^1 + \Pi_\lambda \varphi. \tag{2.4}$$

Similarly, Eq. (2.2) becomes

$$\psi = Q_\lambda K \psi^1 + \Xi_\lambda \varphi. \tag{2.5}$$

Throughout this paper we denote by $\lambda_K$ the real

$$\lambda_K := \begin{cases} -\underline{\sigma} & \text{if } r_\sigma(K) \leq 1 \\ -\underline{\sigma} + \frac{c}{a}\log(r_\sigma(K)) & \text{if } r_\sigma(K) > 1. \end{cases}$$

Clearly, the solution of Eq. (2.4) reduces to the invertibility of the operator $\mathcal{U}(\lambda) := I - P_\lambda K$ (which is the case if $Re\lambda > \lambda_K$). This amounts to

$$\psi^1 = \{\mathcal{U}(\lambda)\}^{-1}\Pi_\lambda \varphi$$

where $\{\mathcal{U}(\lambda)\}^{-1} = \sum_{n\geq 0}(P_\lambda K)^n$. This together with (2.5) gives

$$\psi = Q_\lambda K\{\mathcal{U}(\lambda)\}^{-1}\Pi_\lambda\varphi + \Xi_\lambda\varphi.$$

Accordingly, for $Re\lambda > \lambda_K$, the resolvent of the operator $S_K$ may be written in the form

$$(\lambda - S_K)^{-1} = \sum_{n\geq 0} Q_\lambda K (P_\lambda K)^n \Pi_\lambda + \Xi_\lambda. \tag{2.6}$$

Let $X$ be a Banach space and $T$ a linear operator on $X$. As usually we denote by $\sigma(T)$, $\sigma_c(T)$, $\sigma_r(T)$, $\sigma_p(T)$ and $\rho(T)$ the spectrum, the continuous spectrum, the residual spectrum, the point spectrum and the resolvent set of $T$, respectively. We say that $\lambda_0 \in \sigma_p(T)$ is the leading eigenvalue of $T$ if $\lambda_0 \in \mathbb{R}$ and, for every $\lambda \in \sigma(T)$, $\mathrm{Re}\lambda < \lambda_0$. The set of all bounded linear operators on $X$ will be denoted by $\mathcal{L}(X)$. If $T \in \mathcal{L}(X)$, we denote by $r_\sigma(T)$ the spectral radius of $T$.

We close this section by recalling some facts about positive operators on $L_p$ spaces. Let $\Omega$ be an open subset of $\mathbb{R}^m$, $m \geq 1$, and let $E_p := L_p(\Omega)$, $1 \leq p < \infty$, be the Banach space of equivalence classes of measurable functions on $\Omega$ whose $p$'th power is integrable. It's dual space is $E_q$ where $q = \frac{p}{p-1}$. The positive cone $E^+_{p,0}$ of $E_p$ is given by

$$E^+_{p,0} := \{f \in E_p \ : \ f(x) \geq 0 \ \ \mu \ a.e.\ x \in \Omega\}.$$

The set of strictly positive elements in $E_p$ is denoted by

$$E^+_p := \{f \in E_p \ : \ f(x) > 0 \ \ \mu \ a.e.\ x \in \Omega\}.$$

Note that $E^+_p$ coincides with the set of quasi-interior points of $E_p$, i.e.

$$E^+_p := \{f \in E^+_{p,0} \ : < f, f' >> 0 \ \ \forall f' \in E^+_{q,0}\backslash\{0\}\}$$

where $< .,. >$ is the duality pairing.

**Definition 2.1** We say that $T \in \mathcal{L}(E_p)$ is positive on $E_p$ if $T(E^+_{p,0}) \subseteq E^+_{p,0}$. $T$ is called strictly positive if $T(E^+_{p,0} \backslash \{0\}) \subseteq E^+_p$.

**Definition 2.2** An operator $T \in \mathcal{L}(E_p)$ is called irreducible if, for all $f \in E^+_{p,0} \backslash \{0\}$, there exists $n \in \mathbb{N}\backslash\{0\}$ such that $T^n f \in E^+_p$.

Consider two positive operators $A$ and $B$ in $\mathcal{L}(E_p)$. It is well known that if $A$ and $B$ satisfy $A \leq B$ (i.e. $A - B$ is positive), then $r_\sigma(A) \leq r_\sigma(B)$. The next result owing to I. Marek [21, Theorem 4.4] provides sufficient conditions under which the latter inequality is strict. More precisely:

**Theorem 2.1** *Let $A$ and $B$ be two positive operators in $\mathcal{L}(E_p)$ satisfying $A \leq B$ and $A \neq B$. If $A$ is not quasinilpotent, $B$ is irreducible and power compact (i.e. $B^n$ is compact for some integer $n \geq 1$), then $r_\sigma(A) < r_\sigma(B)$.*

The next two results are also required below. The following one is established in [14, p. 67].

**Theorem 2.2** *Let $T \in \mathcal{L}(E_p)$ be a positive compact operator satisfying*

$$\exists \varphi \geq 0, \quad \varphi \neq 0 \quad \text{and } \alpha > 0 \text{ such that } T\varphi \geq \alpha\varphi.$$

*Then $T$ has an eigenvalue $\lambda_0 \geq \alpha$ with a corresponding nonnegative eigenfunction.*

**Corollary 2.1** Let $T \in \mathcal{L}(E_p)$ be a positive compact non quasinilpotent operator. Then $r_\sigma(T)$ is an eigenvalue of $T$ with a corresponding nonnegative eigenfunction.

*Proof* Let $\lambda \in \mathbb{C}$ be an eigenvalue of $T$ such that $|\lambda| = r_\sigma(T)$. We have $T(\varphi) = \lambda\varphi$ with $\varphi \neq 0$. This implies that $|\lambda|\,|\varphi| \leq T(|\varphi|)$. It follows from Theorem 2.2 that there exists $\lambda_0 \geq |\lambda| = r_\sigma(T)$ which completes the proof. □

For the theory of positive operators on general Banach lattices (resp. $L_p$-spaces) we refer to [14] or [22] (resp. [31]).

*Remark 2.1* Note that for $\lambda > -\underline{\sigma}$, the operators $P_\lambda$, $Q_\lambda$, $\Pi_\lambda$ and $\Xi$ are positive in the lattice sense. Hence, it follows from (2.6) that, if $K \geq 0$, $(\lambda - S_K)^{-1}$ is also positive on $X_p$ for all $\lambda > \lambda_0$.

## 3 Spectral Properties of $S_K$

The purpose of this section is to derive, under reasonable hypotheses on the transition operator $K$, a precise description of the spectrum of the streaming operator $S_K$. We shall also discuss the influence of the transition operators on the leading eigenvalue (when it exists). To do so, we will first consider the case of smooth transition operators, i.e., $K$ satisfies the assumption:

$$(\mathcal{A}1) \quad \begin{cases} K \text{ is a positive operator (in the lattice sense)} \\ \text{and some power of } K \text{ is compact.} \end{cases}$$

We define the sets

$$\mathbf{U} = \{\lambda \in \mathbb{C} \text{ such that } Re\lambda > -\underline{\sigma}\} \quad \text{and} \quad P(S_K) = \sigma(S_K) \cap \mathbf{U}.$$

Our first result is the following.

**Theorem 3.1** *Let $p \in [1, +\infty)$ and assume that the transition operator $K$ satisfies the hypothesis $(\mathcal{A}1)$. Then:*

(i) *$P(S_K)$ consists of, at most, isolated eigenvalues with finite algebraic multiplicity.*
(ii) *If $P(S_K) \neq \emptyset$, then $S_K$ has a leading eigenvalue $\lambda(a)$.*
(iii) *$P(S_K) \neq \emptyset$ if and only if $\lim_{\lambda \to -\underline{\sigma}} r_\sigma(P_\lambda K) > 1$. Furthermore, if $\lambda(a)$ exists, then*

$$-\underline{\sigma} \leq \lambda(a) \leq -\underline{\sigma} + \frac{c}{a}\log(r_\sigma(K)). \tag{3.1}$$

*In particular, if $\sigma(\mu, v) = \sigma$, then $P(S_K) \neq \emptyset$ if and only if $r_\sigma(K) > 1$ (regardless of $a$).*
(iv) *If $r_\sigma(K) \leq 1$, then $P(S_K) = \emptyset$ for all $a$.*
(v) *If $r_\sigma(K) > 1$, then $P(S_K) \neq \emptyset$, at least, for small $a$ and $\lambda(a) \to +\infty$ as $a \to 0$.*

*Proof* Let us first observe that if $r_\sigma(P_\lambda K) < 1$ for all $\lambda \in \mathbf{U}$, then $I - P_\lambda K$ is boundedly invertible. Hence, the solution of (2.4) can be written as

$$\psi^1 = (I - P_\lambda K)^{-1}\Pi_\lambda \varphi, \quad \forall \lambda \in \mathbf{U}.$$

This shows that $\mathbf{U} \subseteq \rho(S_K)$ and then $P(S_K) = \emptyset$.

Now we suppose that $r_\sigma(P_\lambda K) > 1$ for some $\lambda \in \mathbf{U}$. Clearly, for all $\lambda > -\underline{\sigma}$, we have $P_\lambda \leq e^{-\frac{a}{c}(\lambda+\underline{\sigma})} I$, where $I$ denotes the identity operator on $L_p([0, c]), vdv)$. (Here we make the identification $X_p^1 \sim X_p^0 \sim L_p([0, c]), vdv)$). Consequently,

$$P_\lambda K \leq e^{-\frac{a}{c}(\lambda+\underline{\sigma})} K, \qquad \forall \lambda \geq -\underline{\sigma} \tag{3.2}$$

On the other hand, by $(\mathcal{A}1)$, there exists $N \in \mathbb{N}^*$ such that $(K)^N$ is compact. Moreover, (3.2) implies $(P_\lambda K)^N \leq (K)^N$ $\forall \lambda \geq -\underline{\sigma}$. So, applying the Dodds-Fremlin comparison theorem for compact operators [3], we find that $(P_\lambda K)^N$ is compact for $\lambda \geq -\underline{\sigma}$. Next, using the analyticity of the operator valued function $\mathbf{U} \ni \lambda \to (P_\lambda K)^N$ [13, p. 365 ], we infer the compactness of $(P_\lambda K)^N$ for all $\lambda$ in $\mathbf{U}$. On the other hand, the inequality $(P_\lambda K)^{N+1} \leq P_\lambda K K^N$ implies that

$$\|(P_\lambda K)^{N+1}\| \leq \|P_\lambda K(K)^N\|.$$

Since $P_\lambda K \to 0$ strongly as $\lambda \to +\infty$, the use of Lemma 3.7 in [13, p. 151] together with the compactness of $K^N$ implies that $P_\lambda K\,(K)^N \to 0$ in the operator norm as $\lambda \to +\infty$. This shows that $\|(P_\lambda K)^{N+1}\| \to 0$ as $\lambda \to +\infty$ and therefore

$$r_\sigma((P_\lambda K)^{N+1}) \to 0 \text{ as } \lambda \to \infty. \tag{3.3}$$

It follows from (3.3) together with Gohberg-Shmul'yan's theorem (see [11, Theorem 11.4, p. 258]), that $(I - (P_\lambda K)^{N+1})^{-1}$ is a degenerate-meromorphic operator function on $\mathbf{U}$ (i.e. $(I-(P_\lambda K)^{N+1})^{-1}$ is holomorphic on $\mathbf{U}$ except for a set $S$ of isolated points where $(I-(P_\lambda K)^{N+1})^{-1}$ has poles and the coefficients of the principal part have finite rank). From

$$I-(P_\lambda K)^{N+1}=(I-P_\lambda K)\,(I+P_\lambda K+....+(P_\lambda K)^N)$$

we conclude that

$$(I-P_\lambda K)^{-1}=(I+P_\lambda K+....+(P_\lambda K)^N)\,(I-(P_\lambda K)^{N+1})^{-1}$$

is degenerate-meromorphic on $\mathbf{U}$. So, if $\lambda\notin S$, Eq. (2.4) becomes $\psi^1=(I-P_\lambda K)^{-1}\Pi_\lambda\varphi$. By inserting $\psi^1$ into (2.5) we get $\psi=(\lambda-S_K)^{-1}\varphi$ where $(\lambda-S_K)^{-1}=Q_\lambda K(I-P_\lambda K)^{-1}\Pi_\lambda+\Xi_\lambda$. Thus $(\lambda-S_K)^{-1}$ is degenerate-meromorphic on $\mathbf{U}$ which ends the proof of (i).

(ii) If $\lambda_0\in P(S_K)$, then there exists $\varphi\neq 0$ such that $P_{\lambda_0}K\varphi=\varphi$. Thus, $(P_{\lambda_0}K)^N\varphi=\varphi$ and therefore $|\varphi|\leq|(P_{\beta_0}K)^N\varphi|\leq(P_{\beta_0}K)^N|\varphi|$ where $\beta_0=Re\lambda_0$. This implies,

$$r_\sigma((P_{\beta_0}K)^N)\geq 1. \tag{3.4}$$

On the other hand, according to Theorem 0.4 in [23], $r_\sigma((P_\beta K)^N)$ is a continuous strictly decreasing function of $\beta$ in $]-\underline{\sigma},+\infty[$. Moreover, by the spectral mapping theorem [4, p. 569], there exists $\alpha(\beta_0)\in\sigma(P_{\beta_0}K)$ such that $(\alpha(\beta_0))^N=r_\sigma((P_{\beta_0}K)^N)$, i.e. $\alpha(\beta_0)=\sqrt[N]{r_\sigma((P_{\beta_0}K)^N)}$. Thus $\alpha(\beta)$ is also a continuous strictly decreasing function of $\beta$ in $]-\underline{\sigma},+\infty[$. On the other hand, (3.4) (resp. (3.3)) shows that $\alpha(\beta_0)\geq 1$ (resp. $\lim_{\beta\to+\infty}\alpha(\beta)=0$). Accordingly, there exists (a unique) $\lambda\geq\beta_0$ such that $\alpha(\lambda)=1$, i.e. $\lambda=\lambda(a)$ which is the leading eigenvalue of $S_K$.

(iii) In order to prove this statement we restrict ourselves to $\sigma(S_K)\cap(-\underline{\sigma},+\infty)$. Hence, proceeding as in the proof of the second assertion we find that the leading eigenvalue $\lambda(a)$ is characterized by

$$r_\sigma(P_{\lambda(a)}K)=1. \tag{3.5}$$

Hence, $\lambda(a)$ exists if and only if $\lim_{\lambda\to-\underline{\sigma}}r_\sigma(M_\lambda H)>1$. If $\lambda(a)$ exists, using (3.2) and (3.5) we get $1\leq e^{\frac{a}{c}(\lambda(a)-\underline{\sigma})}r_\sigma(K)$. Hence,

$$-\underline{\sigma}\leq\lambda(a)\leq-\underline{\sigma}+\frac{c}{a}\log(r_\sigma(K)).$$

Assume now $\sigma(\mu,v)=\sigma$, then $P_{-\underline{\sigma}}\leq I$ and consequently $P_{-\underline{\sigma}}K\leq K$ which completes the proof of (iii).

(iv) Note that as in (iii), $P_{(-\sigma)}K\leq K$. Hence, if $\lim_{\lambda\to-\sigma}r_\sigma(K)\leq 1$, then $\lim_{\lambda\to-\sigma}r_\sigma(P_\lambda K)\leq 1$. The assertion is then an immediate consequence of (iii).

(v) Let $\overline{\lambda}$ be an arbitrary real satisfying $\overline{\lambda}>-\underline{\sigma}$. Clearly $P_{\overline{\lambda}}\to I$ strongly as $a\to 0$. Now using the compactness of $(K)^N$ we see that $\lim_{a\to 0}\|(P_{\overline{\lambda}}K)^{N+1}-(K)^{N+1}\|=0$ and consequently $\lim_{a\to 0}r_\sigma(P_{\overline{\lambda}}K)=r_\sigma(K)>1$. This shows that, for $a$ small enough, $r_\sigma(P_{\overline{\lambda}}K)>1$ and therefore $\lambda(a)$ exists and $\lambda(a)>\overline{\lambda}$. Next using the fact that $\overline{\lambda}$ is an arbitrary real in $]-\underline{\sigma},+\infty[$ we infer that $\lambda(a)\to\infty$ as $a\to 0$. □

**Theorem 3.2** *Let $p \in [1, \infty)$ and assume that $K$ is a nonnegative compact transition operator. Then the following statements hold:*

(i) *$P(S_K)$ is bounded and, for every $\eta > 0$, $\sigma(S_K) \cap \{\lambda \in \mathbb{C} : Re\lambda > -\underline{\sigma} + \eta\}$ is finite.*
(ii) *Assume that $\sigma \in L^\infty[(0,a) \times (0,c)]$. If $r_\sigma(K) > 1$, then there exists a positive constant $\nu$ such that $\lambda(a) \geq -\|\sigma\|_{L^\infty} + \dfrac{\nu}{a}$.*
(iii) *Let $\sigma(\mu, v) = \sigma$. If $r_\sigma(K) = 1$, then $-\sigma \in \sigma_p(S_K)$.*

Let $X$ be a Banach space and denote by $\mathbf{B}$ its closed unit ball. A set $\mathcal{J} \subseteq \mathcal{L}(X)$ is collectively compact if and only if the set $\mathcal{J}\mathbf{B} = \{Kx : K \in \mathcal{J}, \ x \in \mathbf{B}\}$ has compact closure.

Before proving Theorem 3.2 we first establish the following lemma.

**Lemma 3.1** Let $K$ be an arbitrary compact transition operator. Then $(I - P_\lambda K)^{-1}$ exists for $\lambda$ in the half plane $\{\lambda \in \mathbb{C} : Re\lambda > -\underline{\sigma}\}$ with $|Im\lambda|$ sufficiently large.

*Proof* Notice that if the transition operator $K$ is compact, then there exists a sequence of finite rank operators which converges, in the operator norm, to $K$. Hence, it suffices to establish the result for a finite rank operator, that is, $K = \sum_{k=1}^n K_k$, $K_k = \langle ., \vartheta_k \rangle \zeta_k$ where $n \in \mathbb{N}$, $\vartheta_k \in X_1^q$, $\zeta_k \in X_p^0$ and $q$ denotes the conjugate exponent of $p$. Thus we may restrict ourselves to a transition operator of rank one which we denote also by $K$, namely, $K. := \langle ., \vartheta \rangle \zeta$ where $\zeta \in X_p^0$ and $\vartheta \in X_q^1$.

Let $\lambda$ be a complex number such that $Re\lambda > -\underline{\sigma}$. The dual of the operator $P_\lambda K$ is given by $(P_\lambda K)^* = K^* \widetilde{P}_\lambda$ where

$$\widetilde{P}_\lambda : X_q^1 \longrightarrow X_q^0, u \longrightarrow (\widetilde{P}_\lambda u)(0, v) := u(a, v)\, e^{-\frac{1}{v}\int_0^a (\lambda + \sigma(\mu', v))d\mu'} \tag{3.6}$$

and

$$K^* : X_q^0 \longrightarrow X_q^1, u \longrightarrow (K^* u)(0, v) := \langle \zeta, u \rangle \vartheta \tag{3.7}$$

where $\zeta$ and $\vartheta$ are the functions appearing in the expression of $K$.

Let $\lambda_0$ be the real defined by

$$\lambda_0 := -\underline{\sigma} + \frac{c}{a}\log(r_\sigma(K)).$$

Clearly, if $Re\lambda > \lambda_0$, then $\|P_\lambda K\| < 1$ and consequently, the half plane $Re\lambda > \lambda_0$ is contained in $\rho(S_K)$. So, it suffices to establish Lemma 3.1 in the strip $\{\lambda \in \mathbb{C}$ such that $-\underline{\sigma} < Re\lambda \leq \lambda_0\}$.

*Claim 1* If $\lambda$ belongs to the strip $-\underline{\sigma} < Re\lambda \leq \lambda_0$, then $(K^* \widetilde{P}_\lambda)$ converges to 0, for the strong operator topology, as $|Im\lambda| \to +\infty$.

Let $\varphi \in X_q^1$. It follows from (3.6) and (3.7) that

$$K^* \widetilde{P}_\lambda \varphi := \langle \zeta, P_\lambda \varphi \rangle \vartheta = \int_0^c \vartheta(v)\zeta(v') e^{-\frac{1}{v'}\int_0^a (\lambda + \sigma(\mu', v'))d\mu'} \varphi(a, v') v' dv'$$

Let $(\lambda_n)_n$ be a sequence of complex number such that $\lambda_n = \eta + it_n$ where $\eta \in ]-\underline{\sigma}, \lambda_0]$ an $t_n \to +\infty$ as $n \to \infty$. Thus

$$|(K^* \widetilde{P}_{\lambda_n}\varphi)(a, v)| = \left| \int_0^c \vartheta(v)\zeta(v')e^{-\frac{1}{v'}\int_0^a(\eta+\sigma(\mu',v'))d\mu'} e^{\frac{ai}{v'}t_n}\varphi(a, v')v'dv' \right|.$$

Applying the Riemann-Lebesgue lemma we find

$$\lim_{n\to\infty} \left| \int_0^c \vartheta(v)\zeta(v')e^{-\frac{1}{v'}\int_0^a(\eta+\sigma(\mu',v'))d\mu'} e^{\frac{ai}{v'}t_n}\varphi(a, v')v'dv' \right| = 0 \text{ a.e. on } \{a\} \times (0, c).$$

Accordingly,

$$\lim_{n\to+\infty} |(K^* \widetilde{P}_{\lambda_n}\varphi)(a, v)| = 0 \text{ a.e. on } \{a\} \times (0, c).$$

Furthermore, for every integer $n$, we have:

$$|(K^* \widetilde{P}_{\lambda_n}\varphi)(a, v)| \le \int_0^c |\vartheta(v)|\, |\zeta(v')|\, |\varphi(a, v')|v'dv' \in X_q^1.$$

Then according to the dominated convergence theorem of Lebesgue, we have

$$\lim_{n\to+\infty} \|K^* \widetilde{P}_{\lambda_n}\varphi\|_{X_q^1} = 0.$$

This proves the first claim.

*Claim 2* The family $\{K^* \widetilde{P_\lambda}, -\underline{\sigma} < Re\lambda \le \lambda_0\}$ is collectively compact.

Let $\mathbf{B}_q$ denote the unit ball of the space $X_q^1$ and let $(\psi_n)_{n\in\mathbb{N}}$ be a sequence in $\cup_\lambda(K^* \widetilde{P}_\lambda \mathbf{B}_q), \lambda \in \{\lambda \in \mathbb{C} : -\underline{\sigma} < Re\lambda \le \lambda_0\}$. Then there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $\mathbf{B}_q$ such that $\psi_n = K^* \widetilde{P}_{\lambda_n}\varphi_n,\ n = 1, 2, ....$ It is clear that the sequence $(y_n = \widetilde{P}_{\lambda_n}\varphi_n)_{n\in\mathbb{N}}$ is bounded in $X_q^0$. So, it follows from the compactness of $K^*$ that $(\psi_n = K^* y_n)_{n\in\mathbb{N}}$ has a converging subsequence in $\overline{\cup_\lambda(K^* \widetilde{P}_\lambda \mathbf{B}_q)}$. This ends the proof of the claim.

*Claim 3* Let $\lambda$ be in the strip $-\underline{\sigma} < Re\lambda \le \lambda_0$. Then $\lim_{|Im\lambda|\to+\infty} r_\sigma\ (P_\lambda K) = 0$.

In view of the Claims 1, 2 and Proposition 3.1 in [1] we have

$$\lim_{|Im\lambda|\to+\infty} \|(K^* \widetilde{P}_\lambda)^2\| = 0 \quad \text{uniformly on } \{\lambda \in \mathbb{C} : -\underline{\sigma} < Re\lambda \le \lambda_0\}.$$

Therefore, since $r_\sigma(K^* \widetilde{P}_\lambda) \le \|(K^* \widetilde{P}_\lambda)^n\|^{\frac{1}{n}}$ with $n = 1, 2, ...$, we conclude that

$$\lim_{|Im\lambda|\to+\infty} r_\sigma(K^* \widetilde{P}_\lambda) = 0 \quad \text{uniformly on} \quad \{\lambda \in \mathbb{C} : -\underline{\sigma} < Re\lambda \le \lambda_0\}.$$

Next, the use of the equality $r_\sigma(K^* \widetilde{P}_\lambda) = r_\sigma(P_\lambda K)$ proves the claim.

Now according to Claim 3, there exists $M > 0$ such that for any $\lambda$ in the strip $-\underline{\sigma} < Re\lambda \leq \lambda_0$ satisfying $|Im\lambda| > M$, we have $r_\sigma(P_\lambda K) < 1$. This completes the proof of Lemma 1. □

*Proof of Theorem 3.2*

(i) As mentioned above, if $Re\lambda > \lambda_0$, then $r_\sigma(P_\lambda K) < 1$ and therefore $\sigma(S_K) \cap \{\lambda \in \mathbb{C} : Re\lambda > \lambda_0\} = \emptyset$. Next, using Lemma 3.1 we conclude that there exists $M > 0$ such that

$$P(S_K) \subseteq \{\lambda \in \mathbb{C} : -\underline{\sigma} < Re\lambda \leq \lambda_0 \text{ and } |Im\lambda| \leq M\}.$$

This proves the boundedness of $P(S_K)$. Moreover, for any $\eta > 0$ such that $-\underline{\sigma} + \eta < \lambda_0$, $P(S_K) \cap \{\lambda \in \mathbb{C} : -\underline{\sigma} + \eta < Re\lambda \leq \lambda_0\}$ is confined in a compact subset of the complex plane and, then, it is necessarily finite since it is discrete.

(ii) Let $\varepsilon \in (0, c)$ and define the operator $K_\varepsilon$ by $K_\varepsilon : u \to I_\varepsilon Ku$ where $I_\varepsilon$ denotes the operator $I_\varepsilon : u \to \chi_{(\varepsilon,c)}u$ and $\chi_{(\varepsilon,c)}(.)$ stands for the characteristic function of $(\varepsilon, c)$. Obviously, $K_\varepsilon \leq K$ and $\|K_\varepsilon - K\| \to 0$ as $\varepsilon \to 0$ (use the compactness of $K$). Let $\varphi_\varepsilon$ be a positive eigenfunction of $K_\varepsilon$ associated with the eigenvalue $r_\sigma(K_\varepsilon)$. Let $\lambda > -\underline{\sigma}$. It is clear that $P_\lambda K\varphi_\varepsilon \geq P_\lambda K_\varepsilon\varphi_\varepsilon$. On the other hand, the fact that $\varphi_\varepsilon(v) = 0$ if $v \in [0, \varepsilon[$ implies that

$$P_\lambda\varphi_\varepsilon \geq e^{-a\left(\frac{\lambda+\|\sigma\|_{L^\infty}}{\varepsilon}\right)}\varphi_\varepsilon.$$

Similarly,

$$P_\lambda K_\varepsilon\varphi_\varepsilon \geq e^{-a\left(\frac{\lambda+\|\sigma\|_{L^\infty}}{\varepsilon}\right)}K_\varepsilon\varphi_\varepsilon.$$

Hence, $P_\lambda K \geq e^{-a(\frac{\lambda+\|\sigma\|_{L^\infty}}{\varepsilon})}K_\varepsilon$ and consequently,

$$r_\sigma(P_\lambda K) \geq e^{-a\left(\frac{\lambda+\|\sigma\|_{L^\infty}}{\varepsilon}\right)}\, r_\sigma(K_\varepsilon). \tag{3.8}$$

Owing to the fact that $r_\sigma(P_{\lambda(a)}K) = 1$, thus for $\lambda = \lambda(a)$, (3.8) becomes

$$1 \geq e^{-a\left(\frac{\lambda+\|\sigma\|_{L^\infty}}{\varepsilon}\right)}r_\sigma(K_\varepsilon).$$

Let $\varepsilon$ be small enough so that $r_\sigma(K_\varepsilon) > 1$ (note that by Corollary 0.2 in [23], $r_\sigma(K_\varepsilon) \to r_\sigma(K) > 1$ as $\varepsilon \to 0$). Then

$$\lambda(a) \geq -\|\sigma\|_{L^\infty} + \frac{\varepsilon}{a}\, \log(r_\sigma(K_\varepsilon)).$$

This ends the proof. □

In the following we denote by $\lambda(K)$ the leading eigenvalue of the operator $S_K$ (when it exists). We will now discuss the monotonicity properties of $\lambda(K)$. To do so, we consider two transition operators $K_1$ and $K_2$ satisfying $K_1 \leq K_2$ and $K_1 \neq K_2$.

**Theorem 3.3** *Let $K_1$ and $K_2$ be two transition operators satisfying ($\mathcal{A}1$). If $\lambda(K_1)$ exists, then $\lambda(K_2)$ exists and $\lambda(K_1) \leq \lambda(K_2)$. Moreover, if there exists an integer $m$ such that $(P_{\lambda(K_1)}K_2)^m$ is strictly positive, then $\lambda(K_1) < \lambda(K_2)$.*

*Proof* By hypothesis, there exist two integers $n_1$ and $n_2$ such that $(K_1)^{n_1}$ and $(K_2)^{n_2}$ are compact. Let $n_3 = \max(n_1, n_2)$. It follows from (3.2) together with the Dodds-Fremlin theorem [3] that $(P_\lambda K_1)^{n_3}$ and $(P_\lambda K_2)^{n_3}$ are compact for all $\lambda$ belonging to $]-\underline{\sigma}, \infty[$. In particular, $(P_{\lambda(K_1)}K_1)^{n_3}$ and $(P_{\lambda(K_1)}K_2)^{n_3}$ are positive compact operators on $X_p^1$. As already seen in the proof of Theorem 3.1, $\lambda$ is an eigenvalue of $S_K$ if and only if 1 is an eigenvalue of $P_\lambda K$. So we conclude that

$$r_\sigma(P_{\lambda(K_1)}K_1) \geq 1. \tag{3.9}$$

On the other hand, since $K_1 \leq K_2$ and $K_1 \neq K_2$, then $P_{\lambda(K_1)}K_1 \leq P_{\lambda(K_1)}K_2$ and $P_{\lambda(K_1)}K_1 \neq P_{\lambda(K_1)}K_2$. This implies that $r_\sigma(P_{\lambda(K_1)}K_1) \geq r_\sigma(P_{\lambda(K_1)}K_2)$. But $P_{\lambda(K_1)}K_2$ is irreducible and power compact, then using (3.9) and Theorem 2.1 we infer that

$$r_\sigma[P_{\lambda(K_1)}K_2]^{n_3} > 1. \tag{3.10}$$

Clearly, $[P_\lambda K_2]^{n_3}$ is an analytic operator-valued function whose values are compact for all $\lambda > -\underline{\sigma}$. Moreover, we have $\lim_{\lambda\to\infty} \|[P_\lambda K_2]^{n_3}\| = 0$ (see the proof of Theorem 3.1), thus the use of Theorem 0.4 in [23] implies that the function $]-\underline{\sigma}, +\infty) \ni \lambda \to r_\sigma([P_\lambda K_2]^{n_3})$ is strictly decreasing. This together with (3.10) implies that there exists a unique $\overline{\lambda} > \lambda(K_1)$ such that $r_\sigma([P_{\overline{\lambda}}K_2]^{n_3}) = 1$. Now the spectral mapping theorem yields $\overline{\lambda} = \lambda(K_2)$ and the proof is complete. □

Let us now consider the case of partly smooth transition operators:

$$(\mathcal{A}2) \qquad \begin{cases} K = K_1 + K_2 \text{ with } K_i \geq 0\ i = 1, 2, \quad K_2 \text{ is compact} \\ \text{if } 1 < p < \infty \text{ or weakly compact if p=1.} \end{cases}$$

**Theorem 3.4** *Let $p \in [1, \infty)$ and suppose that the hypothesis ($\mathcal{A}2$) is satisfied. Then the following assertions hold:*

(i) *$\sigma(S_K) \cap \{\lambda \in \mathbb{C} : Re\lambda > \lambda_{K_1}\}$ consists of, at most, isolated eigenvalues with finite algebraic multiplicity.*
(ii) *If $\sigma(S_K) \cap \{\lambda \in \mathbb{C} : Re\lambda > \lambda_{K_1}\} \neq \emptyset$, then $S_K$ has a leading eigenvalue $\lambda(a)$.*
(iii) *If $\lim_{\lambda\to\lambda_{K_1}} r_\sigma(P_\lambda K_2) > 1$, then $\sigma(S_K) \cap \{\lambda \in \mathbb{C} : Re\lambda > \lambda_{K_1}\} \neq \emptyset$.*

*Proof*

(i) Consider again the problem (2.1) which is now equivalent to solving in $X_p^1$ the following one

$$\psi^1 = P_\lambda K_1\psi^1 + P_\lambda K_2\psi^1 + \Pi_\lambda\varphi. \tag{3.11}$$

Clearly, if $\lambda > \lambda_{K_1}$, then the operator $I - P_\lambda K_1$ is boundedly invertible and (3.11) becomes $\psi^1 = F_\lambda\psi^1 + L_\lambda\varphi$ where $F_\lambda := (I - P_\lambda K_1)^{-1}P_\lambda K_2$ and $L_\lambda := (I - P_\lambda K_1)^{-1}\Pi_\lambda$. As already mentioned, $P_\lambda \to 0$ strongly as $\lambda \to \infty$ for all $p$ in $[1, \infty)$. For $p \in (1, \infty)$, $K_2$ is compact and therefore $\|P_\lambda K_2\| \to 0$ as

$\lambda \to \infty$ in the operator topology (use Lemma 3.7 in [13, p. 151]). Now let $p = 1$, and $\lambda_2 > \lambda_{K_1}$. It follows from the estimate $(I - P_\lambda K_1)^{-1} \le (I - P_{\lambda_2} K_1)^{-1}$ (valid for $\lambda > \lambda_2$) that $(F_\lambda)^3 \le (I - P_{\lambda_2} K_2)^{-1} P_\lambda K_2 (F_{\lambda_2})^2 \forall \lambda > \lambda_2$. Since $K_2$ is weakly compact, applying Corollary 13 in [4, p. 510] we infer that $(F_{\lambda_2})^2$ is compact. Using again Lemma 3.7 in [13, p. 151] we get

$$\|(F_\lambda)^3\| \le \|(I - P_{\lambda_2} K_1)^{-1}\| \, \|P_\lambda K_2 (F_{\lambda_1})^2\| \to 0 \text{ as } \lambda \to \infty.$$

Since $r_\sigma(F_\lambda) \le \|F_\lambda^n\|^{\frac{1}{n}}$, $n = 1, 2, 3, \ldots$, we have $r_\sigma(F_\lambda) \to 0$ as $\lambda \to +\infty$ for all $p \in [1, \infty)$. Now applying the Gohberg-Shmul'yan theorem [11] we get the desired result.

(ii) This assertion follows from the fact that $(\lambda - S_K)^{-1}$ is positive for large $\lambda$ (see [30]).

(iii) Let $\lambda > \lambda_{K1}$. The estimate $F_\lambda \ge P_\lambda K_2$ implies $r_\sigma(F_\lambda)) \ge r_\sigma(P_\lambda K_2)$. Hence, if $\lim_{\lambda \to \lambda_{K_1}} r_\sigma(P_\lambda K_2) > 1$, then

$$\lim_{\lambda \to \lambda_{K_1}} r_\sigma(F_\lambda) \ge \lim_{\lambda \to \lambda_{K_1}} r_\sigma(P_\lambda K_2) > 1.$$

Moreover, since $F_\lambda^3$ is compact on $X_p^0$, $1 \le p < \infty$ (see the proof of (1)) and satisfies $\lim_{\lambda \to \infty} \|(F_\lambda)^3\| \to 0$, the use of Theorem 0.4 in [23] and the spectral mapping theorem shows that $r_\sigma(F_\lambda)$ is a continuous strictly decreasing function of $\lambda$ satisfying $\lim_{\lambda \to +\infty} r_\sigma(F_\lambda) = 0$. Therefore there exists $\bar{\lambda} > \lambda_{K_1}$ such that $r_\sigma(F_{\bar{\lambda}}) = 1$ which is the leading eigenvalue. □

## 4 Compactness Results

We now consider the transport operator $A_K = S_K + B$ where $B$ is the bounded operator given by

$$\begin{cases} B : X_p \longrightarrow X_p \\ \quad \psi \longrightarrow \displaystyle\int_0^c r(\mu, v, v')\psi(\mu, v')dv' \end{cases} \tag{4.1}$$

with $r(., ., .)$ is a measurable function from $[0, a] \times [0, c] \times [0, c]$ to $\mathbb{R}^+$.

The purpose of this section is to give some compactness results which play a crucial role in our subsequent analysis (cf. Sections 5, 6 and 7). Note that, in the classical neutron transport theory, similar results are already present in the literature (see, for example, [15, 16]).

Observe that the operator $B$ acts only on the maturation velocity $v'$, so $\mu$ may be viewed merely as a parameter in $[0, a]$. Hence, we may consider $B$ as a function $B(.) : \mu \in [0, a] \longrightarrow B(\mu) \in \mathcal{Z}$ where $\mathcal{Z} := \mathcal{L}(L_p([0, c], dv))$.

In the following we will make the assumptions:

$$(\mathcal{A}3)\quad \begin{cases} \text{the function } B(.) \text{is strongly measurable,} & (4.2)\\ \text{there exists a compact subset } \mathcal{C} \subseteq \mathcal{Z} \text{ such that} & \\ B(\mu) \in \mathcal{C} \text{ a.e. on } [0, a], & (4.3)\\ \text{and } B(\mu) \in \mathcal{K}(L_1([0, c], dv)) \text{ a.e. on } [0, a] & (4.4) \end{cases}$$

where $\mathcal{K}(L_p([0, c], dv))$ denotes the set of all compact operators on $L_p([0, c], dv)$. Obviously, (4.3) implies that

$$B(.) \in L^\infty(]0, a[, \mathcal{Z}). \tag{4.5}$$

Let $\psi \in X_p$. It is easy to see that $(B\psi)(\mu, v) = B(\mu)\psi(\mu, v)$ and then, by (4.5), we have

$$\int_0^c |(B\psi)(\mu, v)|^p dv \leq \|B(.)\|^p_{L^\infty(]0,a[,\mathcal{Z})} \int_0^c |\psi(\mu, v)|^p dv$$

and therefore

$$\int_0^a \int_0^c |(B\psi)(\mu, v)|^p dv d\mu \leq \|B(.)\|^p_{L^\infty(]0,a[,\mathcal{Z})} \int_0^a \int_0^c |\psi(\mu, v)|^p dv d\mu.$$

This leads to the estimate

$$\|B\|_{\mathcal{L}(X_p)} \leq \|B(.)\|_{L^\infty(]0,a[,\mathcal{Z})}. \tag{4.6}$$

The interest of collision operators in the form (4.1) which satisfy $(\mathcal{A}3)$ lies in the following lemma.

**Lemma 4.1** Assume that $(\mathcal{A}3)$ holds true. Then $B$ can by approximated, in the uniform topology, by a sequence $(B_n)_n$ of operators of the form

$$\kappa_n(\mu, v, v') = \sum_{j=1}^{n} \eta_j(\mu)\theta_j(v)\beta_j(v')$$

where $\eta_j(.) \in L^\infty([0, a], d\mu)$, $\theta_j(.) \in L_p([0, c], dv)$ and $\beta_j(.) \in L_q([0, c], dv)$ ($q$ denotes the conjugate of $p$).

*Proof* Let $\varepsilon > 0$. By the assumption (4.3) there exist $B_1, .., B_m$ such that $(B_i)_i \subset K$ and $K \subset \bigcup_{1\leq i\leq m} B(B_i, \varepsilon)$ where $B(B_i, \varepsilon)$ is the open ball, in $\mathcal{K}(L_p([0, c], dv))$, centered at $B_i$ with radius $\varepsilon$.

Let $A_1 = B(B_1, \varepsilon)$, $A_2 = B(B_2, \varepsilon) - A_1$,.., $A_m = B(B_m, \varepsilon) - A_{m-1}$. Clearly, $A_i \cap A_j = \emptyset$ if $i \neq j$ and $K \subset \bigcup_{1\leq i\leq m} A_i$. Let $1 \leq i \leq m$ and denote by $I_i$ the set

$$I_i = B^{-1}(A_i) = \{\mu \in ]0, a[ \text{ such that } B(\mu) \in A_i\}.$$

Hence we have $I_i \cap I_j = \emptyset$ if $i \neq j$ and $]0, 1[= \cup_{i=1}^m I_i$.

Consider now the following step function from $]0, a[$ to $\mathcal{Z}$ defined by

$$S(\mu) = \sum_{i=1}^{m} \chi_{I_i}(\mu) B_i$$

where $\chi_{I_i}(.)$ denotes the characteristic function of $I_i$. Obviously, $S(.)$ satisfies (4.2), (4.3) and (4.4). Then using (4.5) we get $B - S \in L^\infty(]0, a[, \mathcal{Z})$. Moreover, an easy calculation leads to

$$\|B - S\|_{L^\infty(]0,a[,\mathcal{Z})} \le \varepsilon.$$

Now, using (4.6) we obtain

$$\|B - S\|_{\mathcal{L}(X_p)} \le \|B - S\|_{L^\infty(]0,a[,\mathcal{Z})} \le \varepsilon.$$

Hence, we infer that the operator $B$ may be approximated (in the uniform topology) by operators of the form

$$U(\mu) = \sum_{i=1}^{m} \eta_i(\mu) B_i$$

where $\eta_j(.) \in L^\infty([0, a], d\mu)$ and $B_i \in \mathcal{K}(L_p([0, c], dv))$. On the other hand, each compact operator $B_i$ on $L_p([0, c], dv)$ is a limit (for the norm topology) of a sequence of finite rank operators because $L_p([0, c], dv)$ $(1 \le p < \infty)$ admits a Schauder basis. This ends the proof. □

**Theorem 4.1** *Assume that* $(\mathcal{A}3)$ *holds true. Then, for any* $\lambda \in \mathbb{C}$ *such that* $Re\lambda > \lambda_K$, *the operator* $(\lambda - S_K)^{-1}B$ *is compact on* $X_p$, $1 < p < \infty$, *and weakly compact on* $X_1$.

*Remark 4.1* Let $\lambda$ be such that $Re\lambda > \lambda_K$. We know from Eq. (2.6) that

$$(\lambda - S_K)^{-1}B = \sum_{n\ge 0} Q_\lambda K (P_\lambda K)^n \Pi_\lambda B \; + \; \Xi_\lambda B.$$

To prove the compactness (resp. the weak compactness) of $(\lambda - S_K)^{-1}B$ on $X_p$ (resp. $X_1$), it suffices to show that the operators $\Pi_\lambda B$ and $\Xi_\lambda B$ are compact (resp. weakly compact) on $X_p$ (resp. $X_1$). □

**Lemma 4.2** Assume that $(\mathcal{A}3)$ holds true. Then the operators $\Pi_\lambda B$ and $\Xi_\lambda B$ are compact on $X_p$ and weakly compact on $X_1$.

*Proof* Since $(\mathcal{A}3)$ is satisfied, then it follows from Lemma 4.1 that $B$ can be approximated, in the uniform topology by a sequence $B_n$ of finite rank operators on $L_p([0, c], dv)$ which converges, in the operator norm, to $B$. Then it suffices to establish the result for a finite rank operator, that is $\kappa_n(\mu, v, v') = \sum_{j=1}^{n} \eta_j(\mu)\theta_j(v)\beta_j(v')$ where $\eta_j(.) \in L^\infty([0, a], d\mu)$, $\theta_j(.) \in L_p([0, c], dv)$ and $\beta_j(.) \in L_q([0, c], dv)$ ($q$ denotes the conjugate of $p$). So, we infer from the linearity and the stability of the compactness by summation that it suffices to prove the result for an operator $B$ whose kernel is in the form $\kappa(\mu, v, v') = \eta(\mu)\,\theta(v)\beta(v')$ where $\eta(.) \in L^\infty([0,a], d\mu)$, $\theta(.) \in L_p([0,c], dv)$ and $\beta(.) \in L_q([0,c], dv)$.

Consider $g \in X_p$,

$$\begin{cases} (\Pi_\lambda Bg)(v) = \displaystyle\int_0^c \int_0^a \frac{1}{v}\eta(\mu)\theta(v)\, e^{-\frac{1}{v}\int_\mu^1 (\lambda+\sigma(\tau,v))d\tau} \beta(v')g(\mu, v')\, d\mu dv' \\ \qquad\qquad\quad = J_\lambda\, Ug \end{cases}$$

where $U$ and $J_\lambda$ denote the following bounded operators

$$\begin{cases} U : X_p & \longrightarrow & L_p([0,a], d\mu) \\ \qquad \varphi & \longrightarrow & (U\varphi)(\mu) = \displaystyle\int_0^c \beta(v)\, \varphi(\mu, v)\, dv \end{cases}$$

$$\begin{cases} J_\lambda : L_p([0,a], d\mu) & \longrightarrow & X_p^1 \\ \qquad \psi & \longrightarrow & \displaystyle\int_0^a \frac{\eta(\mu)\,\theta(v)}{v}\, e^{-\frac{1}{v}\int_\mu^a (\lambda+\sigma(\tau,v))d\tau} \psi(\mu)\, d\mu. \end{cases}$$

We first consider the case $p \in (1, \infty)$. It is then sufficient to check that $J_\lambda$ is compact. This will follow from Theorem 11.6 in [10, p. 275] if we show

$$\int_0^c \left[\int_0^a \left|\frac{1}{v}\,\eta(\mu)\,\theta(v)\, e^{-\frac{1}{v}\int_\mu^a (\lambda+\sigma(\tau,v))d\tau}\right|^q d\mu\right]^{\frac{p}{q}} v dv < +\infty$$

($J_\lambda$ is then a Hille-Tamarkin operator). To do so, let us first observe that we have

$$\begin{aligned} \int_0^a \left|\frac{1}{v}\,\eta(\mu)\,\theta(v)\, e^{-\frac{1}{v}\int_\mu^a (\lambda+\sigma(\tau,v))d\tau}\right|^q d\mu &\le \|\eta\|_\infty^q \frac{|\theta(v)|^q}{v^q} \int_0^a e^{-q\frac{(Re\lambda+\underline{\sigma})}{v}(a-\mu)} d\mu \\ &\le \|\eta\|_\infty^q \frac{|\theta(v)|^q}{q(Re\lambda+\underline{\sigma})v^{(q-1)}} \end{aligned}$$

which leads to

$$\left[\int_0^a \left|\frac{1}{v}\,\eta(\mu)\,\theta(v)\, e^{-\frac{1}{v}\int_\mu^a (\lambda+\sigma(\tau,v))d\tau}\right|^q d\mu\right]^{\frac{p}{q}} \le \|\eta\|_\infty^p \frac{|\theta(v)|^p}{(q(Re\lambda+\underline{\sigma}))^{\frac{p}{q}}}\, v^{(\frac{p}{q}-p)}.$$

Integrating in $v$ from 0 to $c$ we obtain

$$\begin{aligned} &\int_0^c \left[\int_0^a \left|\frac{1}{v}\,\eta(\mu)\,\theta(v)\ \ e^{-\frac{1}{v}\int_\mu^a (\lambda+\sigma(\tau,v))\mathrm{d}\tau}\right|^q \mathrm{d}\mu\right]^{\frac{p}{q}} v\mathrm{d}v \\ &\le \int_0^c \|\eta\|_\infty^p \frac{|\theta(v)|^p}{(q(Re\lambda+\underline{\sigma}))^{\frac{p}{q}}}\, v^{(\frac{p}{q}-p)} v\mathrm{d}v. \\ &\le \|\eta\|_\infty^p \frac{\|\theta\|^p}{(q(Re\lambda+\underline{\sigma}))^{\frac{p}{q}}}. \end{aligned}$$

Now we consider the case $p = 1$. Let $\lambda$ be such that $Re\lambda > -\underline{\sigma} + \frac{c}{a}\ln(\mathcal{U})$. As above, according to Lemma 3.1 it suffices to establish the result for an operator $B$

with kernel of the form $\kappa(\mu, v, v') = \eta(\mu)\theta(v)\beta(v')$, where $\eta \in L^\infty([0, a], d\mu)$, $\theta \in L_1([0, c], dv)$ and $\beta \in L^\infty([0, c], dv)$. The operator $\Pi_\lambda B$ writes in the form $\Pi_\lambda B = \Gamma_\lambda R_\beta$ where $R_\beta$ and $\Gamma_\lambda$ are the two bounded operators given by

$$R_\beta \,:\, X_1 \longrightarrow L_1([0, a], d\mu),\, u \longrightarrow (R_\beta\varphi)(\mu) := \textstyle\int_0^c \beta(v)\varphi(\mu, v)dv$$

and

$$\begin{cases} \Lambda_\lambda : L_1([0, a], d\mu) \longrightarrow X_1^1, \\ \qquad\qquad \varphi \longrightarrow \dfrac{1}{v}\displaystyle\int_0^a \eta(\mu')\theta(v)e^{-\frac{1}{v}\int_{\mu'}^a(\lambda+\sigma(\tau,v))d\tau}\,\varphi(\mu')\,d\mu'. \end{cases}$$

Thus it suffices to prove that $\Lambda_\lambda$ is weakly compact. To this end, let $\mathcal{O}$ be a bounded subset of $L_1([0, a], d\mu)$ and let $\varphi \in \mathcal{O}$. We have

$$\int_E |(\Lambda_\lambda\varphi)(v)|v dv \le \|\eta\|_\infty\|\varphi\| \int_E |\theta(v)|dv,$$

for all measurable subsets of $[0, c]$. Next, applying Corollary 11 in [4, p. 294] we infer that the set $\Lambda_\lambda(\mathcal{O})$ is weakly compact, since $\lim_{|E|\to 0}\int_E |\theta(v)|dv = 0$, where $|E|$ is the measure of $E$.

A similar reasoning allows us to reach the same results for the operator $\Xi_\lambda B$. This completes the proof. □

*Proof of Theorem 4.1* This follows from Lemma 4.2 and Remark 4.1. □

## 5 Existence of the Leading Eigenvalues of $A_K$

Denote by $L_p(dv)$ the space of functions $L_p[(0, c); dv]$. Notice that $L_p(dv)$ is a subspace of $X_p^0$ and the imbedding $L_p(dv) \hookrightarrow X_p^0$ is continuous. By $\overline{B}$ we mean the integral operator on $X_p$ whose kernel is given by $\overline{r}(\mu, v, v') = \dfrac{r(\mu, v, v')}{v}$.

**Theorem 5.1** *Suppose that the operator $\overline{B}$ is bounded on $X_p$ and $K$ is bounded from $X_p^0$ into $L_p(dv)$ with $\|K\| < 1$. Then $\sigma(A_K) \cap \{\lambda \in \mathbb{C} : Re\lambda > -\underline{\sigma}\} = \emptyset$ for a small enough.*

*Proof* Let $\psi \in X_p$ and put $\varphi = B\psi$. Then we have

$$|\Xi_\lambda\varphi(\mu, v)|^p \le a^{\frac{p}{q}} \int_0^a \frac{|\varphi(\mu, v)|^p}{v^p}\, d\mu$$

and so,

$$\begin{aligned} \int_0^a\int_0^c |\Xi_\lambda\varphi(\mu, v)|^p dv d\mu &\le a^{(\frac{p}{q}+1)} \int_0^a\int_0^c \frac{|\varphi(\mu, v)|^p}{v^p}\, dv d\mu \\ &= a^p \int_0^a\int_0^c |\overline{B}\psi(\mu, v)|^p dv d\mu \end{aligned}$$

where $q$ is the conjugate of $p$. Thus, we can write

$$\left[\int_0^a\int_0^c |\Xi_\lambda\varphi(\mu, v)|^p dv d\mu\right]^{\frac{1}{p}} \le a\,\|\overline{B}\|\,\|\psi\|$$

which gives the estimate

$$\|\Xi_\lambda B\| \leq a\|\overline{B}\|. \tag{5.1}$$

On the other hand, the operator $\Pi_\lambda$ satisfies the following inequality

$$|\Pi_\lambda\varphi(\mu, v)| \leq \frac{1}{v}\int_0^a e^{-\frac{1}{v}(Re\lambda+\underline{\sigma})(a-\mu)}\, |\varphi(\mu, v)|d\mu \leq \frac{1}{v}\int_0^a |\varphi(\mu, v)|d\mu.$$

Using Hölder's inequality we obtain

$$|\Pi_\lambda\varphi(\mu, v)| \leq \frac{a^{1/q}}{v}\Big[\int_0^a |\varphi(\mu, v)|^p d\mu\Big]^{1/p} \leq a^{1/q}\Big[\int_0^a \frac{|\varphi(\mu, v)|^p}{v^p} d\mu\Big]^{1/p}.$$

Finally, we have the estimate

$$\|\Pi_\lambda B\| \leq a^{1/q}\|\overline{B}\|. \tag{5.2}$$

Next, the hypothesis on $K$ together with the estimate $\|P_\lambda\| \leq e^{-\frac{a}{c}(Re\lambda+\underline{\sigma})}$ gives

$$\|P_\lambda K\| < 1 \text{ uniformly on } \{\lambda \in \mathbb{C} : \ \mathrm{Re}\lambda \geq -\underline{\sigma}\}$$

which implies

$$\|(I - P_\lambda K)^{-1}\| \leq \frac{1}{1 - \|K\|}, \quad \text{for } \mathrm{Re}\lambda \geq -\underline{\sigma}. \tag{5.3}$$

Moreover, a simple calculation leads to

$$\|Q_\lambda\|_{\mathcal{L}(L_p(dv), X_p)} \leq a^{1/p}. \tag{5.4}$$

Now combining (5.1), (5.2), (5.3), (5.4) together with the hypothesis on $K$ ($\|Ku\|_{L_p(dv)} \leq \rho\ \|u\|_{X_p^0},\ \rho > 0$), we may write

$$\begin{aligned}\|(\lambda - S_K)^{-1}B\| &\leq \frac{a^{1/p}\rho\ a^{1/q}\|\overline{B}\|}{1 - \|K\|} + a\ \|\overline{B}\| \\ &= \Big[\frac{\rho + 1 - \|K\|}{1 - \|K\|}\Big]\|\overline{B}\|\ a = f(a).\end{aligned}$$

Clearly, $f$ is a continuously increasing function on $[0, \infty[$ which satisfies $f(0) = 0$ and $\lim_{a\mapsto\infty} f(a) = +\infty$. Hence there exists $a_0 > 0$ such that $f(a_0) < 1$. This completes the proof. □

In what follows, we turn our attention to the bounded part of the transport operator $A_K$ which we denote by $\mathcal{N}$. We will discuss the relationship between the real eigenvalues of $A_K$ and those of $\mathcal{N}$. For the sake of simplicity we will deal here with the homogeneous case, i.e. $\sigma(\mu, v) = \sigma(v)$ and $r(\mu, v, v') = r(v, v')$. Hence the bounded part of $A_K$ is then defined by

$$\begin{cases} \mathcal{N} : L_p([0, c];\ dv) \longrightarrow L_p([0, c];\ dv) \\ \varphi \longrightarrow (\mathcal{N}\varphi)(v) = -\sigma(v)\,\varphi(v) + \displaystyle\int_0^c r(v, v')\varphi(v')dv' \end{cases}$$

In the following we denote by $P(A_K)$ (resp. $P(\mathcal{N})$) the set

$$\sigma(A_K) \cap \{\lambda \in \mathbb{C} :\ \mathrm{Re}\lambda > \lambda_K\}\ \text{(resp. } \sigma(\mathcal{N}) \cap \{\lambda \in \mathbb{C} :\ \mathrm{Re}\lambda > \lambda_K\}).$$

**Theorem 5.2** *Suppose that $B$ is a positive regular operator on $X_p$, and $K \leq Id$. Then, if $P(\mathcal{N}) = \emptyset$, then $P(A_K) = \emptyset\ \forall a > 0$ and the leading eigenvalue of $A_K$ is less than or equal to that of $\mathcal{N}$. Moreover, the latter is less than or equal to $-\underline{\sigma} + r_\sigma(B)$.*

*Proof* Since $B$ is regular, then according to Theorem 4.1, for all $\lambda$ such that $Re\lambda > -\underline{\sigma}$, $(\lambda - S_K)^{-1}B$ is power compact on $X_p$, $1 \leq p < +\infty$. Applying Theorem III in [30] we conclude that $A_K$ has a leading eigenvalue $\overline{\lambda}$ with a corresponding nonnegative eigenfunction $\overline{\psi}$, i.e. $A_K\overline{\psi} = \overline{\lambda}\,\overline{\psi}$. This equation may be written as

$$-v\frac{\partial\overline{\psi}}{\partial\mu}(\mu, v) - (\overline{\lambda} + \sigma(v))\overline{\psi}(\mu, v) + \int_0^c r(v, v')\overline{\psi}(\mu, v')dv' = 0. \tag{5.5}$$

Set

$$\overline{\varphi}(v) = \int_0^a \overline{\psi}(\mu, v)d\mu.$$

It is clear that $\overline{\varphi} \geq 0$ and $\overline{\varphi} \neq 0$. By integrating (5.5) with respect to $\mu$, we get

$$-v\ [\overline{\psi}(a, v) - \overline{\psi}(0, v)] - \sigma(v)\overline{\varphi}(v) + \int_0^c r(v, v')\overline{\varphi}(v')dv' = \overline{\lambda}\,\overline{\varphi}(v).$$

Taking into account of the hypotheses and the sign of $\overline{\psi}$ we obtain

$$-v\ [\overline{\psi}(a, v) - \overline{\psi}(0, v)] = -v[\overline{\psi}^1 - \overline{\psi}^0] = -v(I - K)\overline{\psi}^1 \leq 0\ \ \forall v \in [0, c]. \tag{5.6}$$

Now, Eqs. (5.5) and (5.6) lead to

$$-\sigma(v)\,\overline{\varphi}\ +\ B\,\overline{\varphi}\ \geq\ \overline{\lambda}\overline{\varphi}$$

and therefore

$$\int_0^c \frac{r(v, v')}{\overline{\lambda} + \sigma(v)}\,\overline{\varphi}\ \geq\ \overline{\varphi}. \tag{5.7}$$

Let $\lambda \in ]-\underline{\sigma}, +\infty[$ and define the operator $B_\lambda$ on $L_p([0, c];\ dv)$ by

$$\begin{cases} B_\lambda : L_p([0, c];\ dv) \longrightarrow L_p([0, c];\ dv) \\ \qquad\qquad \varphi \longrightarrow (B_\lambda\varphi)(v) = \displaystyle\int_0^c \frac{r(v, v')}{\lambda + \sigma(v)}\,\varphi(v')dv'. \end{cases}$$

Since $B$ is a positive regular operator on $X_p$, then $B_\lambda$ is positive and compact on $L_p([0,c];\ dv)$. It follows from Corollary 2.1 that $r_\sigma(B_\lambda)$ is an eigenvalue of $B_\lambda$ depending continuously on $\lambda$. On the other hand, using Eq. (5.7) and Theorem 2.2 we conclude that $r_\sigma(B_{\overline{\lambda}}) \geq 1$. Since $\lim_{\lambda \mapsto +\infty} r_\sigma(B_\lambda) = 0$, then there exists $\lambda_0 \geq \overline{\lambda}$ such that $r_\sigma(B_{\lambda_0}) = 1$. Consequently, there exists $\varphi_0 \neq 0$ and $\varphi_0 \geq 0$ in $L_p([0,c];\ dv)$ such that

$$B_{\lambda_0}\varphi_0 = \varphi_0. \tag{5.8}$$

This leads to $\mathcal{N}\varphi_0 = \lambda_0\varphi_0$ and proves the first part of the theorem.

On the other hand, (5.8) may be written in the form

$$\int_0^c r(v,v')\varphi_0(v')dv' = (\lambda_0 + \sigma(v))\varphi_0(v) \geq (\lambda_0 + \underline{\sigma})\varphi_0(v).$$

Since $\varphi_0 \neq 0$ and $\varphi_0 \geq 0$, applying Theorem 2.2 we conclude that $r_\sigma(B) \geq \underline{\sigma} + \lambda_0$ which ends the proof. □

**Corollary 5.1** Suppose that the hypotheses of Theorem 5.2 hold. If the operator $\mathcal{N}$ is subcritical (i.e. $P(\mathcal{N}) \subseteq \{\lambda \in \mathbb{R} : \lambda < 0\}$), then the transport operator $A_K$ is subcritical $\forall\ a > 0$.

*Remark 5.1* Let $\lambda$ be in $\rho(A_K) \cap \rho(A_0)$ such that $r_\sigma((\lambda - S_K)^{-1}B) < 1$. Then

$$(\lambda - S_K - B)^{-1} = \sum_{n \geq 0} [\ (\lambda - S_K)^{-1}B\ ]^n (\lambda - S_K)^{-1}.$$

The positivity of $B$ and the fact that $(\lambda - S_K)^{-1} \geq (\lambda - S_0)^{-1} \geq 0$ imply that

$$[\ (\lambda - S_K)^{-1}B\ ]^n(\lambda - S_B)^{-1} \geq [\ (\lambda - S_0)^{-1}S\ ]^n(\lambda - S_0)^{-1} \geq 0$$

and therefore,

$$R(\lambda, A_K) \geq R(\lambda, A_0) \geq 0. \tag{5.9}$$

Next, using (5.9) and Proposition 2.5 in [24, p. 67], it follows that if $P(A_0) \neq \emptyset$, then $P(A_K) \neq \emptyset$.

## 6 The Strict Monotonicity of the Leading Eigenvalue of $A_K$

The objective of this section is to study the strict growth properties of the leading eigenvalue with respect to the parameters of the equation. We start our study by discussing the incidence of the boundary operators on the monotony of the leading eigenvalue. To this end, we consider two positive boundary operators $K_1$ and $K_2$ satisfying $K_1 \leq K_2$ and $K_1 \neq K_2$. We denote by $\lambda(K)$ the leading eigenvalue of $A_K$ (when it exists).

**Theorem 6.1** *Suppose that the assumption* $(\mathcal{A}3)$ *is satisfied and* $\lambda(K_1)$ *exists, then* $\lambda(K_2)$ *exists and* $\lambda(K_1) \leq \lambda(K_2)$. *Further, if one of the following conditions* (i) *and* (ii) *is satisfied, then* $\lambda(K_1) < \lambda(K_2)$.

(i) *There exists an integer* $n \geq 1$ *such that* $(\Xi_{\lambda(K_1)})]^n$ *is strictly positive.*
(ii) *There exists an integer* $n \geq 1$ *such that* $\left(Q_{\lambda(K_1)} K_2 (I - P_{\lambda(K_1)} K_2)^{-1} \Pi_{\lambda(K_1)} B\right)^n$ *is strictly positive.*

*Remark 6.1* More practical criterions are given in Corollary 6.1.

*Proof of Theorem 6.1* Since $K_1 \leq K_2$, then $\lambda_{K_1} \leq \lambda_{K_2}$. The positivity of the operators $K_1$, $K_2$, $B$ and the fact that $K_1 \leq K_2$ imply that, for all $\lambda > \lambda_{K_2}$, $(\lambda - S_{K_1})^{-1} B \leq (\lambda - S_{K_2})^{-1} B$ and therefore $r_\sigma((\lambda - S_{K_1})^{-1} B) \leq r_\sigma((\lambda - S_{K_2})^{-1} B)$. On the other hand, by Theorem 4.1, $(\lambda - S_{K_1})^{-1} B$ is power compact on $X_p$, $1 \leq p < +\infty$. So, using Gohberg-Shmul'yan's theorem and arguing as in the proof of Theorem III in [30], we infer that $P(A_{K_1})$ consists of at most eigenvalues with finite algebraic multiplicity. On the other hand, it is clear that $\lambda \in P(A_{K_1})$ if and only if 1 is an eigenvalue of $(\lambda - S_{K_1})^{-1} B$. Accordingly, since $\lambda(K_1) \in P(A_{K_1})$, we have

$$r_\sigma[(\lambda(K_1) - S_{K_1})^{-1} B] \geq 1. \tag{6.1}$$

Set $\chi_1 = (\lambda(K_1) - S_{K_1})^{-1} B$ and $\chi_2 = (\lambda(K_1) - S_{K_2})^{-1} B$. By Theorem 4.1, $\chi_2$ is power compact on $X_p$. Moreover, if one of the conditions above is satisfied, then $\chi_2$ has a strictly positive power. Now, the fact that $\chi_1 \leq \chi_2$, (6.1) and Theorem 2.1 give

$$r_\sigma(\chi_2) = r_\sigma[(\lambda(K_1) - S_{K_2})^{-1} B] > 1$$

But the function $]s(S_{K_2}), +\infty[\ni \lambda \to r_\sigma[(\lambda - S_{K_2})^{-1} B]$ is strictly decreasing. Hence, there exists a unique $\lambda' > \lambda(K_1)$ such that $r_\sigma[(\lambda' - S_{K_2})^{-1} B] = 1$. This immediately implies that $\lambda' = \lambda(K_2)$ which completes the proof. □

We deduce the following corollary which provides a practical criteria of monotonicity of $\lambda(K)$.

**Corollary 6.1** Suppose that $B$ satisfies the hypothesis $(A3)$ and $\lambda(K_1)$ exists. Then $\lambda(K_2)$ exists and $\lambda(K_1) \leq \lambda(K_2)$. Further, if one of the following conditions is satisfied, then $\lambda(K_1) < \lambda(K_2)$.

(i) $K_2$ is strictly positive and $Ker(B) \cap \{\varphi \in X_p, \ \varphi \geq 0\} = \{0\}$.
(ii) There exists an integer $n \geq 1$ such that $(P_{\lambda(K_1)} K_2)^n$ is strictly positive and $Ker(B) \cap \{\varphi \in X_p, \ \varphi \geq 0\} = \{0\}$.

The proof of this corollary is similar to that of Theorem 6.1. It uses the fact that, for $\lambda > -\underline{\sigma}$, the operators $P_\lambda$ and $Q_\lambda$ are two multiplication operators by strictly positive functions.

In the following, we shall study the strict monotonicity of the leading eigenvalue of $A_K$ with respect to the collision operators. In fact, consider $B_1$ and $B_2$ two operators satisfying the hypothesis $(\mathcal{A}3)$, $B_1 \leq B_2$ and $B_1 \neq B_2$. We denote by $\lambda(B)$ the leading eigenvalue of $A_K = S_K + B$ (when it exists).

**Proposition 6.1** Assume that $B_1$ and $B_2$ satisfy $(\mathcal{A}3)$ and $\lambda(B_1)$ exists. Then $\lambda(K_2)$ exists and $\lambda(B_1) \leq \lambda(B_2)$. Further, if one of the following conditions is satisfied, then $\lambda(B_1) < \lambda(B_2)$.

(i) There exists an integer $n \geq 1$ such that $[\Xi_{\lambda(B_1)} B_2]^n$ is strictly positive.
(ii) There exists an integer $n \geq 1$ such that $[Q_{\lambda(B_1)} K(I - P_{\lambda(B_1)} K)^{-1} \Pi_{\lambda(B_1)} B_2]^n$ is strictly positive.

*Proof* Since $B_1$ is regular, as in the proof of Theorem 6.1, we have $P(S_K + B_1) \neq \emptyset$ and $\lambda(B_1) \in P(S_K + B_1)$. This implies that

$$r_\sigma[(\lambda(B_1) - S_K)^{-1} B_1] \geq 1. \tag{6.2}$$

Set $\chi_1 = (\lambda(B_1) - S_K)^{-1} B_1$ and $\chi_2 = (\lambda(B_1) - S_K)^{-1} B_2$. Clearly $\chi_1 \leq \chi_2$ and, by Theorem 4.1, $\chi_2$ is power compact on $X_p$. Moreover, if one of the conditions above is satisfied, then $\chi_2$ has a strictly positive power. Using (6.2) and applying Theorem 2.1 we conclude that

$$r_\sigma(\chi_2) = r_\sigma[(\lambda(B_1) - S_K)^{-1} B_2] > 1$$

Since the function $]\lambda_K, +\infty[ \ni \lambda \to r_\sigma[(\lambda - S_K)^{-1} B_2]$ is strictly decreasing, there exists a unique $\lambda' > \lambda(B_1)$ such that $r_\sigma[(\lambda' - S_K)^{-1} B_2] = 1$. This implies that $\lambda' = \lambda(B_2)$ which completes the proof. □

As an immediate consequence of Proposition 6.1, we have:

**Corollary 6.2** Assume that $\lambda(B_1)$ exists, then $\lambda(B_2)$ exists and $\lambda(B_1) \leq \lambda(B_2)$. Further, if one of the following conditions is satisfied, then $\lambda(B_1) < \lambda(B_2)$.

(i) $K$ is strictly positive and $Ker(B_2) \cap \{\varphi \in X_p, \ \ \varphi \geq 0\} = \{0\}$.
(ii) There exists an integer $n \geq 1$ such that $(P_{\lambda(B_1)} K)^n$ is strictly positive and $Ker(B_2) \cap \{\varphi \in X_p, \ \ \varphi \geq 0\} = \{0\}$.

## 7 Essential Spectra of $A_K$

The aim of this section is to describe in detail the various essential spectra of the operator $A_K$ for large classes of transition and collision operators. For the reader's convenience, we first recall some notations and definitions, referring for instance to [5, 6, 9, 13, 27].

Let $X$ be a Banach space. We denote by $\mathcal{C}(X)$ (resp. $\mathcal{L}(X)$) the set of all closed, densely defined (resp. bounded) linear operators on $X$. The subset of all compact operators of $\mathcal{L}(X)$ is designated by $\mathcal{K}(X)$. An operator $A \in \mathcal{C}(X)$ is said to be in $\Phi_+(X)$ if its range, $R(A)$, is closed in $X$ and the dimension $\alpha(A)$ of the null space of $A$, $N(A)$, is finite. It is said to be in $\Phi_-(X)$ if $R(A)$ is closed in $X$ and the codimension $\beta(A)$ of $R(A)$ is finite. Operators in $\Phi_\pm(X) := \Phi_+(X) \cup \Phi_-(X)$ are called semi-Fredholm operators. For such operators the index is defined as $i(A) = \alpha(A) - \beta(A)$. The set of Fredholm operators is defined by $\Phi(X) := \Phi_+(X) \cap \Phi_-(X)$. A complex number $\lambda$ is in $\Phi_{+A}$, $\Phi_{-A}$, $\Phi_{\pm A}$ or $\Phi_A$ if $\lambda - A$ belongs to $\Phi_+(X)$, $\Phi_-(X)$, $\Phi_\pm(X)$ or $\Phi(X)$, respectively.

**Definition 7.1** Let $X$ be a Banach space and let $F \in \mathcal{L}(X)$. $F$ is called a Fredholm perturbation if $U + F \in \Phi(X)$ whenever $U \in \Phi(X)$. Let $\mathcal{F}(X)$ denote the set of Fredholm perturbations on $X$.

Various notions of essential spectrum appear in the applications of spectral theory, most are enlargement of the continuous spectrum. For $A \in \mathcal{C}(X)$, let $\rho_1(A) := \Phi_{+A}$, $\rho_2(A) := \Phi_{-A}$, $\rho_3(A) := \Phi_{+A} \cup \Phi_{-A}$, $\rho_4(A) := \Phi_A$, $\rho_5(A)$ the set of those $\lambda \in \Phi_A$ such that $i(\lambda - A) = 0$ and $\rho_6(A)$ the set of those $\lambda \in \rho_5(A)$ such that all scalars near $\lambda$ are in $\rho(A)$. Following [9], we let $\sigma_{ei}(A) = \mathbb{C}\backslash\rho_i(A)$, $1 \leq i \leq 6$. These are called essential spectra of $A$. Note that, in general, we have

$$\sigma_{e3}(A) \subseteq \sigma_{e4}(A) \subseteq \sigma_{e5}(A) \subseteq \sigma_{e6}(A).$$

But if $X$ is a Hilbert space and $A$ is self-adjoint, then

$$\sigma_{e1}(A) = \sigma_{e2}(A) = \sigma_{e3}(A) = \sigma_{e4}(A) = \sigma_{e5}(A) = \sigma_{e6}(A).$$

A simple consequence of these definitions is that $\sigma_{ei}(.)$, $i = 1, ..., 6$, are closed subsets of the complex plane.

**Definition 7.2** Let $X$ be a Banach space and let $T \in \mathcal{L}(X)$. $T$ is said to be strictly singular, if for every infinite dimensional subspace $M$ of $X$, the restriction of $T$ to $M$ is not a homeomorphism. We denote by $\mathcal{S}(X)$ the set of all strictly singular operators on $X$.

For the properties of strictly singular operators we refer to [7, 12]. In general, strictly singular operators are not compact and $\mathcal{S}(X)$ is a closed two-sided ideal of $\mathcal{L}(X)$ containing $\mathcal{K}(X)$. If $X$ is a Hilbert space, then $\mathcal{S}(X) = \mathcal{K}(X)$.

A detailed analysis of essential spectra on general Banach spaces by means of the concept of Fredholm perturbations was done in [17]. On the other hand, when dealing with the spaces $L_p(d\mu) := L_p(\Omega, \Sigma, d\mu)$, where $(\Omega, \Sigma, \mu)$ denotes a positive measure space, we have

$$\mathcal{F}(L_p(d\mu)) = \mathcal{S}(L_p(d\mu)) \tag{7.1}$$

(cf. [17, p. 292]). Using (7.1), we can state the following result which is a special case of Theorem 3.3 in [17].

**Proposition 7.1** Let $A$ and $B$ be two elements of $\mathcal{C}(L_p(d\mu))$. If, for some $\lambda \in \rho(A) \cap \rho(B)$, $(\lambda - A)^{-1} - (\lambda - B)^{-1} \in \mathcal{S}(L_p(d\mu))$, then

$$\sigma_{ei}(A) = \sigma_{ei}(B), \quad i = 1, \ldots, 5.$$

Moreover, if $C\sigma_{e5}(A)$ [the complement of $\sigma_{e5}(A)$] is connected and neither $\rho(A)$ nor $\rho(B)$ is empty, then

$$\sigma_{e6}(A) = \sigma_{e6}(B).$$

After these preparations, we are now in a position to discuss the invariance properties of essential spectra of transport operators.

We know from Section 2 (Eq. (2.6)) that, if $Re\lambda > \lambda_K$, then $\lambda \in \rho(S_K)$ and $(\lambda - S_K)^{-1}$ is given by

$$(\lambda - S_K)^{-1} = \sum_{n\geq 0} Q_\lambda K(P_\lambda K)^n \Pi_\lambda + \Xi_\lambda.$$

On the other hand, the operator $\Xi_\lambda$ is nothing else but $(\lambda - S_0)^{-1}$, i.e. $K = 0$. So, if $Re\lambda > \lambda_K$, then $\lambda \in \rho(S_K) \cap \rho(S_0)$ and

$$(\lambda - S_K)^{-1} - (\lambda - S_0)^{-1} = \mathcal{V}_\lambda. \tag{7.2}$$

where $\mathcal{V}_\lambda := \sum_{n\geq 0} Q_\lambda K(P_\lambda K)^n \Pi_\lambda$.

Let $\lambda \in \mathbb{C}$ be such that $Re\lambda \leq -\lambda^*$. The solution of the eigenvalue problem $(\lambda - S_0)\psi = 0$ is formally given by

$$\psi(\mu, v) = k(v) e^{-\frac{1}{v}(\lambda+\sigma(v))\mu}. \tag{7.3}$$

Moreover, $\psi$ must satisfy the boundary conditions, i.e., $\psi^0 = 0$. So, we obtain $k(v) = 0$ and consequently, $\psi = 0$. This shows that the point spectrum of the operator $S_0$ is empty, i.e., $\sigma_p(S_0) = \emptyset$.

Let $S_0^*$ denotes the dual operator of $S_0$. It is given by

$$\begin{cases} S_0^* \,:\, D(S_0^*) \subset X_q \longrightarrow X_q \\ \qquad \psi \;\longrightarrow S_0^*\psi(\mu, v) = v\dfrac{\partial \psi}{\partial \mu}(\mu, v) - \sigma(\mu, v)\psi(\mu, v) \\ D(S_0^*) = \{\psi \in W_q \text{ such that } \psi^1 = 0\}, \end{cases}$$

where $q$ is the conjugate of $p$. Consider now the eigenvalue problem $(\lambda - S_0^*)\psi = 0$ with $Re\lambda \leq -\lambda^*$ (because $\sigma(S_0) = \sigma(S_0^*)$). In view of the boundary conditions, a straightforward computation shows that the problem above admits only the trivial solution, i.e. $\sigma_p(S_0^*) = \emptyset$. Now using the inclusion $\sigma_r(S_0) \subseteq \sigma_p(S_0^*)$ we conclude that $\sigma_r(S_0) = \emptyset$. This leads to the following lemma.

**Lemma 7.1** With the notations introduced above, we have

$$\sigma(S_0) = \sigma_c(S_0) = \{\lambda \in \mathbf{C} \,:\, Re\lambda \leq -\lambda^*\}.$$

As an immediate consequence of Lemma 7.1 and the fact that all essential spectra are enlargements of the continuous spectrum we have

$$\sigma_{ei}(S_0) = \{\lambda \in \mathbb{C} \,:\, Re\lambda \leq -\underline{\sigma}\} \text{ for } i = 1, ..., 6. \tag{7.4}$$

Note that the perturbation of the boundary conditions of the operator $S_0$ leads to the Eq. (7.2) above. So, if the transition operator $K$ is strictly singular (in applications, $K$ is compact or weakly compact), then by Lemma 461 in [12], $\mathcal{V}_\lambda$ is strictly singular too. So Lemma 7.1, Proposition 7.1 and (7.3) give

$$\sigma_{ei}(S_K) = \{\lambda \in \mathbb{C} \,:\, Re\lambda \leq -\underline{\sigma}\},\ i = 1, 2, 3, 4 \text{ and } 5. \tag{7.5}$$

Recall that the transport operator $A_K$ is defined as a bounded perturbation of $S_K$, i.e. $A_K = S_K + B$ where $B$ is the operator defined by (4.1). We now introduce the class $\mathcal{G}(X_p)$ of collision operators defined by

$$\mathcal{G}(X_p) = \Big\{ B \in \mathcal{L}(X_p) \ : \ (\lambda - S_K)^{-1} B \in \mathcal{S}(X_p) \text{ for some } \lambda \in \rho(S_K) \Big\}.$$

Clearly if $B$ is a collision operator on $X_p$ satisfying $(A3)$, then it follows from Theorem 4.1 that $(\lambda - S_K)^{-1}B$ is compact on $X_p$ for $1 < p < \infty$ (resp. weakly compact on $X_1$). Hence, using the inclusion $\mathcal{K}(X_p) \subseteq \mathcal{S}(X_p)$ (resp. the fact that the set of weakly compact operators on $X_1$ coincide with $\mathcal{S}(X_1)$ (cf. [25])), we infer that $B \in \mathcal{G}(X_p)$. In particular, the set of collision operators with kernels in the form $r(v, v') = f(v)\ g(v')$ with $f \in L_p([0, c],\ dv)$ and $g \in L_q([0, c],\ dv)$, $q = \frac{p}{p-1}$, is contained in $\mathcal{G}(X_p)$. This shows that $\mathcal{G}(X_p) \neq \emptyset$.

Let $\lambda \in \rho(S_K)$ be such that $r_\sigma((\lambda - S_K)^{-1}B) < 1$, then $\lambda \in \rho(S_K + B)$ and

$$(\lambda - A_K)^{-1} - (\lambda - S_K)^{-1} = \sum_{n\geq 1} [(\lambda - S_K)^{-1}B]^n (\lambda - S_K)^{-1}. \tag{7.6}$$

**Theorem 7.1** *Let $p \in [1, \infty)$. If the collision operator $B \in \mathcal{G}(X_p)$, then*

$$\sigma_{ei}(A_K) = \sigma_{ei}(S_K), \quad \textit{for} \ \ i = 1, ..., 5.$$

*Moreover, if $K$ is strictly singular, then*

$$\sigma_{ei}(A_K) = \{\lambda \in \mathbb{C} \ : \ Re\lambda \leq -\underline{\sigma}\}, \quad \textit{for} \ \ i = 1, ..., 5.$$

*Proof* Since $B \in \mathcal{G}(X_p)$, according to (7.4) and Theorem 4.1, $(\lambda - A_H)^{-1} - (\lambda - S_K)^{-1} \in \mathcal{G}(X_p)$. Then, the first claim follows from Proposition 7.1. To establish the second claim, observe that Eqs. (7.2) and (7.5) give

$$(\lambda - A_K)^{-1} - (\lambda - S_0)^{-1} = \mathcal{V}_\lambda + \sum_{n\geq 1} [(\lambda - S_K)^{-1}B]^n\, (\lambda - S_K)^{-1}.$$

Next, if $K$ is strictly singular, then $\mathcal{V}_\lambda$ is strictly singular too. This together with Theorem 4.1 leads to $(\lambda - A_K)^{-1} - (\lambda - S_0)^{-1} \in \mathcal{S}(X_p)$. Again the use of Lemma 7.1 and Proposition 7.1 gives the result. □

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## References

1. Anselone, P. M., Palmer, T. W.: Collectively compact sets of linear operators. Pacific J. Math., **25**, 417-422 (1968)
2. Dautray, R., Lions, J. L.: Analyse Mathématique et Calcul Numérique, Vol. 9. Masson, Paris, (1988)

3. P. Dodds, P., Fremlin, D. R.: Compact operators in Banach lattices. Israel J. Math., **34**, 287-320 (1979)
4. Dunford, N., Schwartz, J. T.: Linear Operators: Part I. Wiley-Interscience, New York, (1958)
5. Gohberg, I. C., Krein, G.: Fundamental theorems on deficiency numbers, root numbers and indices of linear operators. Amer. Math. Soc. Transl. Ser. 2, **13**, 185-264 (1960)
6. Gohberg, I. C., A. Markus, A., Feldman, I. A.: Normaly solvable operators and ideals associated with them. Amer. Math. Soc. Transl. Ser. 2, **61**, 63-84 (1967)
7. Goldberg, S.: Unbounded Linear Operators. McGraw-Hill, New-York, (1966)
8. Greenberg, W., Van der Mee, C., Protopopescu, V.: Boundary Value Problems in Abstract Kinetic Theory. Birkhäuser, Basel, (1987)
9. Gustafson, K., Weidmann, J.: On the essential spectrum. J. Math. Anal. Appl., **25**, 121-127 (1969)
10. Jörgens, K.: Linear Integral Operators. Pitman Advanced Publishing Program, (1982)
11. Kaper, H. G., Lekkerkerker, C. G., Hejtmanek, J.: Spectral Methods in Linear Transport Theory. Birkhäuser Basel,(1982)
12. Kato, T.: Perturbation theory for nullity, deficiency and other quantities of linear operators. J. Anal. Math., **6**, 261-322 (1958)
13. Kato, T.: Perturbation Theory for Linear Operators. Springer-Verlag, New York, (1966)
14. Krasnoselskii, M. A.: Positive Solutions of Operator Equations. Noordhoff, Groningen, (1964)
15. Latrach, K.: Compactness properties for linear transport operator with abstract boundary conditions in slab geometry. Transport Theory Statist. Phys., **22** 39-64 (1993)
16. Latrach, K.: Compactness results for transport equations and applications. Math. Models Methods Appl. Sci., **11** 181-1202 (2001)
17. Latrach, K., A. Dehici, A.: Fredholm, semi-Fredholm perturbations and essential spectra. J. Math. Anal. Appl., **259** 277-301 (2001)
18. Latrach, K., Jeribi, A.: A nonlinear boundary value problem arising in growing cell population. Nonlinear Anal. TMA, **36** 843-862 (1999)
19. Latrach, K., Jeribi, A.: Sur une équation de transport intervenant en dynamique des populations. C. R. Acad. Sci. Paris, Sér. I Math. **325** 1087-1090 (1997)
20. Lebowitz, J. L., Rubinow, S. I.: A theory for the age and generation time distribution of a microbial population. J. Math. Biol., **1** 17-36 (1974)
21. Marek, I.: Frobenius theory of positive operators : Comparison theorems and applications. SIAM. J. Appl. Math., **19** 607-628 (1970)
22. Meyer-Nieberg, P.: Banach Lattices. Springer-Verlag, New York, (1991)
23. Mokhtar-Kharroubi, M.: Propriétés spéctrales de l'opérateur de transport dans le cas anisotrope. Thèse de Doctorat de 3ème cycle, Université Paris **6**, (1983)
24. Nagel (ed.), R.: One-parameter Semigroups of Positive Operators. Lecture Notes in Math., 1184, Springer Verlag, (1986)
25. Pelczynski, A.: Strictly singular and strictly cosingular operators. Bull. Acad. Polon. Sci., **13** 31-41 (1965)
26. Rotenberg, M.: Transport theory for growing cell populations. J. Theor. Biol. **103** 181-199 (1983)
27. Schechter, M.: On the essential spectrum of an arbitrary Operator. J. Math. Anal. Appl., **13**, 205-215 (1966)
28. Van der Mee, C.: A transport equation modelling cell growth. In: P. Tautu (ed.), Stochastic Modelling in Biology. Relevant Mathematical Concepts and Recent Applications, Word Sientific, Singapore, pp. 381-398 (1990)
29. Van der Mee, C., Zweifel, P.: A Fokker-Plank equation for growing cell populations. J. Math. Biol. **25**, 61-72 (1987)
30. Vidav, I.: Existence and uniqueness of nonnegative eigenfunction of the Boltzmann operator. J. Math. Anal. Appl., **22** 144-155 (1968)
31. Zerner, M.: Quelques propriétés spectrales des opérateurs positifs. J. Funct. Anal., **72** 381-417 (1987)