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Fredholm, Semi-Fredholm Perturbations, and Essential Spectra

Khalid Latrach and Abdelkader Dehici

Département de Mathématiques, Université de Corse, 20250 Corte, France
E-mail: latrach@lotus.univ-corse.fr

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This paper is devoted to the investigation of the stability of the various essential spectra of closed densely defined linear operators under perturbations belonging to any two-sided ideal of the algebra of bounded linear operators contained in the set of Fredholm perturbations. Our results generalize many known ones in the literature and, in particular, extend and unify those obtained by K. Latrach (1999, *J. Math. Anal. Appl.* **233,** 607–623) and K. Latrach and A. Jeribi (1998, *J. Math. Anal. Appl.* **225,** 461–485). They are used to describe the essential spectra of one-dimensional transport equations with anisotropic scattering and abstract boundary conditions.

1. INTRODUCTION

Let X be a complex Banach space. By an operator A on X we mean a linear operator with domain $D(A) \subseteq X$ and range $R(A) \subseteq X$. We denote by $\mathscr{C}(X)$ (resp. $\mathscr{L}(X)$) the set of all closed, densely defined (resp. bounded) linear operators on X. The subset of all compact (resp. weakly compact) operators of $\mathscr{L}(X)$ is designated by $\mathscr{K}(X)$ (resp. $\mathscr{W}(X)$). For $A \in \mathscr{C}(X)$, we let $\sigma(A)$, $\rho(A)$, and $N(A)$ denote the spectrum, the resolvent set, and the null space of A, respectively. The nullity, $\alpha(A)$, of A is defined as the dimension $N(A)$ and the deficiency, $\beta(A)$, of A is defined as the codimension of $R(A)$ in X. The set of upper semi-Fredholm operators is defined by

$$\Phi_+(X) = \{A \in \mathscr{C}(X) : \alpha(A) < \infty \text{ and } R(A) \text{ is closed in } X\},$$

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and the set of lower semi-Fredholm operators is defined by

$$\Phi_-(X) = \{A \in \mathscr{C}(X) : \beta(A) < \infty \text{ (and } R(A) \text{ is closed in } X)\}.$$

Operators in $\Phi_\pm(X) := \Phi_+(X) \cup \Phi_-(X)$ are called semi-Fredholm operators on X while $\Phi(X) = \Phi_+(X) \cap \Phi_-(X)$ denotes the set of Fredholm operators on X. If $A \in \Phi(X)$, the number $i(A) = \alpha(A) - \beta(A)$ is called the index of A. A complex number λ is in Φ_{+A}, Φ_{-A}, $\Phi_{\pm A}$, or Φ_A if $\lambda - A$ is in $\Phi_+(X)$, $\Phi_-(X)$, $\Phi_\pm(X)$, or $\Phi(X)$, respectively.

For a self-adjoint operator in a Hilbert space, there seems to be only one reasonable way of defining the essential spectrum: the set of all points of the spectrum that are not isolated eigenvalues of finite algebraic multiplicity (see, for example, [14, 27, 28]). If X is a Banach space and $A \in \mathscr{C}(X)$, various notions of essential spectrum appear in the applications of spectral theory (see, for instance, [10, 12, 15, 18, 31, 37]). Most are enlargement of the continuous spectrum. They may be ordered as

$$\sigma_{e3}(A) := \sigma_{e1}(A) \cap \sigma_{e2}(A) \subseteq \sigma_{e4}(A) \subseteq \sigma_{e5}(A) \subseteq \sigma_{e6}(A),$$

where $\sigma_{ei}(A) = \mathbb{C} \setminus \rho_i(A)$ with $\rho_1(A) := \Phi_{+A}$, $\rho_2(A) := \Phi_{-A}$, $\rho_3(A) := \Phi_{\pm A}$, $\rho_4(A) := \Phi_A$, $\rho_5(A) := \{\lambda \in \rho_4(A),\ i(\lambda - A) = 0\}$, and $\rho_6(A)$ denotes the set of those $\lambda \in \rho_5(A)$ such that scalars near λ are in $\rho(A)$.

The subsets $\sigma_{e1}(\cdot)$ and $\sigma_{e2}(\cdot)$ are the Gustafson and Weidmann essential spectra [12]. $\sigma_{e3}(\cdot)$ is the Kato essential spectrum [18]. $\sigma_{e4}(\cdot)$ is the Wolf essential spectrum [12, 37]. $\sigma_{e5}(\cdot)$ is the Schechter essential spectrum [12, 31] and $\sigma_{e6}(\cdot)$ denotes the Browder essential spectrum [12, 15, 25]. Note that all these sets are closed and if X is a Hilbert space and A is self-adjoint on X, then all these sets coincide.

One of the central questions in the study of essential spectra of closed densely defined linear operators on Banach spaces consists in showing when different notions of essential spectrum coincide. Among the works in this direction we can quote, for example, [12, 14, 15, 19, 20, 22, 27, 31, 37] (see also the references therein). Recently, in [22] (see also [20]), motivated by the description of essential spectra of transport operators, the behavior of essential spectra of operators in $\mathscr{C}(X)$ under additive perturbations was discussed on L_p spaces. The analysis uses the concept of strictly singular operators which possess some nice properties on these spaces (cf. [24, 34]). In [19] this analysis was extended to operators on Banach spaces which possess the Dunford–Pettis property. This was accomplished by means of weakly compact perturbations. It is shown that if X has the Dunford–Pettis property, then the set of weakly compact operators behaves like that of strictly singular ones on L_p spaces. In particular, weakly compact operators are power compact and form a closed two-sided ideal, $\mathscr{W}(X)$, of $\mathscr{L}(X)$. Combining these facts with Proposition 4 in [26] (it asserts that if X

has the Dunford–Pettis property, then $\mathscr{W}(X) \subseteq \mathscr{S}(X) \cap C\mathscr{S}(X)$ where $\mathscr{S}(X)$ (resp. $C\mathscr{S}(X)$) denotes the ideal of strictly singular (resp. strictly cosingular) operators on X (cf. Section 2)) we show that perturbations by operators belonging to $\mathscr{W}(X)$ leave invariant the sets $\Phi_+(X)$, $\Phi_-(X)$, $\Phi_\pm(X)$, and $\Phi(X)$ which is a basic step in our approach.

The purpose of this work is to pursue the analysis started in [19, 22] and to extend it to general Banach spaces. More precisely, let $A \in \mathscr{C}(X)$ and let $\mathscr{I}(X)$ be an arbitrary two-sided ideal of $\mathscr{L}(X)$. If $\mathscr{I}(X) \subseteq \mathscr{F}(X)$, where $\mathscr{F}(X)$ denotes the set of Fredholm perturbations, then $\sigma_{ei}(A) = \sigma_{ei}(A + J)$ for all $J \in \mathscr{I}(X)$ and $i = 4, 5$ and if $\rho_{e5}(A)$ is connected and neither $\rho(A)$ nor $\rho(A + J)$ is empty then $\sigma_{e6}(A) = \sigma_{e6}(A + J)$. Moreover, if $\mathscr{I}(X)$ satisfies some additional (reasonable) conditions we get $\sigma_{ei}(A) = \sigma_{ei}(A + J)$ for $J \in \mathscr{I}(X)$ and $i = 1, 2, 3$ (see Theorem 3.1). In general, in applications (see, for example, [14, 21, 22, 27, 37]), these results are not applicable directly, so practical criterions which guarantee the invariance of the various essential spectra for perturbed linear operators are provided (Theorems 3.2 and 3.3). Also, we point out that the definition of the Schechter essential spectrum may be expressed in terms of operators belonging to any subset $\mathscr{I}(X)$ of $\mathscr{L}(X)$ provided that $\mathscr{K}(X) \subseteq \mathscr{I}(X) \subseteq \mathscr{F}(X)$. A spectral mapping theorem for $\sigma_{e5}(\cdot)$ is also derived. Our results extend and improve many known ones in the literature. In particular, those obtained in [19, 21, 22] are now special cases of those obtained here.

In the last section we will apply the results obtained in Section 3 to describe the essential spectra of the following integro-differential operator

$$\begin{aligned} A_H\psi(x, \xi) &= -\xi\frac{\partial\psi}{\partial x} - \sigma(\xi)\psi(x, \xi) + \int_{-1}^{1} \kappa(x, \xi, \xi')\psi(x, \xi')\, d\xi' \\ &= T_H\psi + K\psi \end{aligned}$$

$x \in [-a, a]$ for a parameter $0 < a < \infty$ and $\xi \in [-1, 1]$. It describes the transport of particles (neutron, photons, molecules of gas, etc.) in a slab with thickness $2a$. The function $\psi(x, \xi)$ represents the number density of gas particles having the position x and the direction cosine of propagation ξ. (The variable ξ may be thought of as the cosine of the angle between the velocity of particles and the x-direction.) The functions $\sigma(\cdot)$ and $\kappa(\cdot, \cdot, \cdot)$ are nonnegative measurable functions called, respectively, the collision frequency and the scattering kernel. The boundary conditions are modeled by

$$\psi_{|D^-} = H\psi_{|D^+},$$

where D^- (resp. D^+) is the incoming (resp. outgoing) part of the phase space boundary, $\psi_{|D^-}$ (resp. $\psi_{|D^+}$) is the restriction of ψ to D^- (resp.

D^+), and H is a bounded linear operator from a suitable function space on D^- to a similar one on D^+. There is a wealth of literature treating the transport equation with different boundary conditions (see, e.g., [3, 11, 16, 21, 22] and the references therein). The known boundary conditions (vacuum boundary conditions, specular reflections, periodic, diffuse reflections, generalized and mixed type boundary conditions) are specific examples of our general framework. Our analysis is based essentially on Theorem 3.2 and the knowledge of the essential spectra of T_0 where T_0 (i.e., $H = 0$) denotes the streaming operator with vacuum boundary conditions. We give sizable classes of boundary and collision operators for which the essential spectra of the operators T_0 and A_H coincide. Our results extend those obtained in [21, 22] to non-homogeneous regular collision operators.

2. PRELIMINARIES

DEFINITION 2.1. Let X be a Banach space. An operator $S \in \mathscr{L}(X)$ is called strictly singular if the restriction of S to any infinite-dimensional subspace of X is not an homeomorphism. Let $\mathscr{S}(X)$ denote the set of strictly singular operators on X.

For a detailed study of the properties of strictly singular operators we refer to [9, 17]. Note that $\mathscr{S}(X)$ is a closed two-sided ideal of $\mathscr{L}(X)$. In general, strictly singular operators are not compact and the strict singularity is not preserved under conjugation (see [9, 36]).

Let X be a Banach space. If N is a closed subspace of X, we denote by π_N the quotient map $X \to X/N$. The codimension of N, codim(N), is defined to be the dimension of the vector space X/N.

DEFINITION 2.2. Let X be a Banach space and $S \in \mathscr{L}(X)$. S is said to be strictly cosingular if there exists no closed subspace N of X with $\operatorname{codim}(N) = \infty$ such that $\pi_N S : X \to X/N$ is surjective. Let $C\mathscr{S}(X)$ denote the set of strictly cosingular operators on X.

This class of operators was introduced by Pelczynski [26]. It forms a closed two-sided ideal of $\mathscr{L}(X)$ (cf. [33]).

DEFINITION 2.3. Let X be a Banach space and $F \in \mathscr{L}(X)$. F is called a Fredholm perturbation if $U + F \in \Phi(X)$ whenever $U \in \Phi(X)$. F is called a upper (resp. lower) Fredholm perturbation if $F + U \in \Phi_+(X)$ (resp. $\Phi_-(X)$) whenever $U \in \Phi_+(X)$ (resp. $\Phi_-(X)$).

The sets of Fredholm, upper semi-Fredholm, and lower semi-Fredholm perturbations are denoted by $\mathscr{F}(X)$, $\mathscr{F}_+(X)$, and $\mathscr{F}_-(X)$, respectively.

Remark 2.1. Let $\Phi^b(X)$, $\Phi^b_+(X)$, and $\Phi^b_-(X)$ denote the sets $\Phi(X) \cap \mathscr{L}(X)$, $\Phi_+(X) \cap \mathscr{L}(X)$, and $\Phi_-(X) \cap \mathscr{L}(X)$, respectively. If in Definition 2.3 we replace $\Phi(X)$, $\Phi_+(X)$, and $\Phi_-(X)$ by $\Phi^b(X)$, $\Phi^b_+(X)$, and $\Phi^b_-(X)$ we obtain the sets $\mathscr{F}^b(X)$, $\mathscr{F}^b_+(X)$, and $\mathscr{F}^b_-(X)$. These classes of operators were introduced and investigated in [8]. In particular, it is shown that $\mathscr{F}^b_-(X)$ is closed, and $\mathscr{F}^b_+(X)$ and $\mathscr{F}^b(X)$ are closed two-sided ideals of $\mathscr{L}(X)$. In general, we have

$$\mathscr{K}(X) \subseteq \mathscr{S}(X) \subseteq \mathscr{F}^b_+(X) \subseteq \mathscr{F}^b(X),$$
$$\mathscr{K}(X) \subseteq C\mathscr{S}(X) \subseteq \mathscr{F}^b_-(X) \subseteq \mathscr{F}^b(X).$$

The containment $\mathscr{S}(X) \subseteq \mathscr{F}^b_+(X)$ is due to Kato [17] while the inclusion $C\mathscr{S}(X) \subseteq \mathscr{F}^b_-(X)$ was proved by Vladimirskii [33].

An operator $R \in \mathscr{L}(X)$ is said to be a Riesz operator if $\Phi_R = \mathbb{C} \setminus \{0\}$. For further information on the family of Riesz operators, $\mathscr{R}(X)$, we refer to [2, 15] and the references therein. We recall that Riesz operators satisfy the Riesz–Schauder theory of compact operators and $\mathscr{R}(X)$ is not an ideal of $\mathscr{L}(X)$ [2]. In [29], it is proved that $\mathscr{F}^b(X)$ is the largest ideal of $\mathscr{L}(X)$ contained in $\mathscr{R}(X)$. Hence, the inclusions above imply that $\mathscr{K}(X)$, $\mathscr{S}(X)$, $C\mathscr{S}(X)$, $\mathscr{F}^b_+(X)$, and $\mathscr{F}^b_-(X)$ are contained in $\mathscr{R}(X)$.

Let A be a closed linear operator on a Banach space X. For $x \in \mathscr{D}(A)$ (the domain of A),the graph norm of x is defined by

$$\|x\|_A = \|x\| + \|Ax\|.$$

It follows from the closedness of A that $\mathscr{D}(A)$ endowed with the norm $\|\cdot\|_A$ is a Banach space. In this new space, denoted by X_A, the operator A satisfies $\|Ax\| \le \|x\|_A$ and, consequently, is a bounded operator (acting from X_A into X).

Let J be a linear operator on X. If $\mathscr{D}(A) \subseteq \mathscr{D}(J)$, then J will be called A-defined. If J is an A-defined operator, we will denote by $\hat{J}$ the restriction of J to $\mathscr{D}(A)$. Moreover, if $\hat{J} \in \mathscr{L}(X_A, X)$, we say that J is A-bounded. One checks easily that if J is closed (or closable) (cf. [18, Remark 1.5, p. 191]), then J is A-bounded.

Remark 2.2. Let X be a Banach space and let $A \in \mathscr{C}(X)$. As mentioned above, $\mathscr{D}(A)$ provided with the graph norm is a Banach space denoted by X_A. Let J be an arbitrary A-bounded operator. Hence we can regard A and J as operators from X_A into X. They will be denoted by $\hat{A}$ and $\hat{J}$, respectively. These belong to $\mathscr{L}(X_A, X)$. Furthermore, we have the

obvious relations

$$\begin{cases} \alpha(\hat{A}) = \alpha(A),\ \beta(\hat{A}) = \beta(A), & R(\hat{A}) = R(A), \\ \alpha(\hat{A} + \hat{J}) = \alpha(A + J), & \\ \beta(\hat{A} + \hat{J}) = \beta(A + J) \quad \text{and} & R(\hat{A} + \hat{J}) = R(A + J). \end{cases} \tag{2.1}$$

We will continue this section by giving some lemmas which describe some properties of the sets $\mathscr{F}_+(X)$, $\mathscr{F}_-(X)$, and $\mathscr{F}(X)$ we will need in the sequel.

LEMMA 2.1. *Let $A \in \mathscr{C}(X)$. Then the following statements hold.*

(i) $A \in \Phi_+(X)$ *if and only if* $\alpha(A - K) < \infty$ *for all* $K \in \mathscr{K}(X)$.

(ii) $A \in \Phi_-(X)$ *if and only if* $\beta(A - K) < \infty$ *for all* $K \in \mathscr{K}(X)$.

This lemma is known for bounded semi-Fredholm operators (see [23, 29]). The proof of the statement (i) (resp. (ii)) is a straightforward adaptation of the proof of Theorem 23 in [29] (resp. Theorem 5.4 in [23]). So they are omitted.

LEMMA 2.2. *Let $F \in \mathscr{L}(X)$. Then the following statements hold.*

(i) $F \in \mathscr{F}_+(X)$ *if and only if* $\alpha(A - F) < \infty$ *for each* $A \in \Phi_+(X)$.

(ii) $F \in \mathscr{F}_-(X)$ *if and only if* $\beta(A - F) < \infty$ *for each* $A \in \Phi_-(X)$.

(iii) $F \in \mathscr{F}(X)$ *if and only if either* $\alpha(A - F) < \infty$ *or* $\beta(A - F) < \infty$ *for each* $A \in \Phi(X)$.

Proof. (i) Let $F \in \mathscr{F}_+(X)$ and let $A \in \Phi_+(X)$. Then $A - F \in \Phi_+(X)$ and consequently $\alpha(A - F) < \infty$. Conversely, assume that $F \notin \mathscr{F}_+(X)$. Then there exists $A \in \Phi_+(X)$ such that $A - F \notin \Phi_+(X)$. Therefore, by Remark 2.3, $\hat{A} - \hat{F} \notin \Phi_+^b(X_A, X)$. Next, applying [23, Lemma 4.3] we infer that there exists an operator K such that $\hat{K} \in \mathscr{K}(X_A, X)$ (i.e., K is A-compact) and $\alpha(\hat{A} - \hat{F} - \hat{K}) = \infty$, i.e., $\alpha(A - F - K) = \infty$ (use Eq. (2.1)). On the other hand, the use of [30, Theorem 6.2, p. 183] shows that $A + K \in \Phi_+(X)$, and therefore $\alpha(A - F - K) < \infty$. This contradicts the fact that $\alpha(A - F - K) = \infty$ and ends the proof of (i).

(ii) This may be checked in the same way as above; it suffices to replace [23, Lemma 4.3] by [23, Lemma 5.1].

(iii) Let $F \in \mathscr{F}(X)$. Hence, for each $A \in \Phi(X)$, $\alpha(A - F) < \infty$ and $\beta(A - F) < \infty$. Conversely, suppose that $\alpha(A - F) < \infty$ for each $A \in \Phi(X)$. Let μ be an arbitrary nonzero complex number and let A be an element of $\Phi(X)$. Then, by [30, Theorem 2.1, p. 167], $\frac{(A - K)}{\mu} \in \Phi(X)$

for each $K \in \mathscr{K}(X)$. Hence $\alpha(A - \mu F - K)$ is finite for all scalar μ. Thus, by Lemma 3.1(i), we see that $A - \mu F \in \Phi_+(X)$. Now arguing as in the proof of [9, Theorem 2.1, p. 117] (or [18, Theorem 5.22, p. 236]) and using the compactness of the interval [0, 1] we obtain $\beta(A - F) \leq \beta(A)$. Since $\beta(A) < \infty$, we get $\beta(A - F) < \infty$. Consequently, $A - F \in \Phi(X)$. This shows that $F \in \mathscr{F}(X)$.

If $\beta(A - F) < \infty$ for all $A \in \Phi(X)$, a similar proof as above using [9, Theorem 2.1, p. 117] (or [18, Theorem 5.22, p. 236]) shows that $\alpha(A - F) < \infty$ for all $A \in \Phi(X)$ which implies that $F \in \mathscr{F}(X)$. Q.E.D.

LEMMA 2.3. *Let X be a Banach space. Then*

(i) $\mathscr{F}_+(X)$ *and* $\mathscr{F}_-(X)$ *are closed in* $\mathscr{L}(X)$,

(ii) $\mathscr{F}(X) = \mathscr{F}^b(X)$.

Remark 2.3. In contrast to the result of the second item, whether or not $\mathscr{F}_+(X)$ (resp. $\mathscr{F}_-(X)$) is equal to $\mathscr{F}_+^b(X)$ (resp. $\mathscr{F}_-^b(X)$) seems to be unknown.

Remark 2.4. As a consequence of Lemma 2.3(ii), $\mathscr{F}(X)$ is a closed two-sided ideal of $\mathscr{L}(X)$. In fact, by [29], it is the largest closed two-sided ideal contained in $\mathscr{R}(X)$. On the other hand, Lemma 2.2 and [9, Theorem 2.1, p. 117] imply that $\mathscr{K}(X) \subseteq \mathscr{S}(X) \subseteq \mathscr{F}_+(X) \subseteq \mathscr{F}(X)$ and $\mathscr{K}(X) \subseteq \mathscr{F}_-(X) \subseteq \mathscr{F}(X)$.

As a consequence of Lemma 2.3(ii) is that $\mathscr{F}(X)$ is a closed two-sided ideal of $\mathscr{L}(X)$. On the other hand, Lemma 2.2 and [9, Theorem 2.1, p. 117] imply that $\mathscr{K}(X) \subseteq \mathscr{S}(X) \subseteq \mathscr{F}_+(X) \subseteq \mathscr{F}(X)$ and $\mathscr{K}(X) \subseteq \mathscr{F}_-(X) \subseteq \mathscr{F}(X)$.

Proof of Lemma 2.3. (i) Let (F_n) be a sequence of operators of $\mathscr{F}_+(X)$ (resp. $\mathscr{F}_-(X)$) such that (F_n) converges to F in $\mathscr{L}(X)$. If $A \in \Phi_+(X)$ (resp. $\Phi_-(X)$), then for n large enough, applying [9, Theorem 1.6, p. 112] we get $A - (F_n - F) \in \Phi_+(X)$ (resp. $\Phi_-(X)$). Next, using the fact that $F_n \in \mathscr{F}_+(X)$ (resp. $\mathscr{F}_-(X)$), together with the relation $A + F = A - (F_n + F) - F_n$ we conclude that $F \in \mathscr{F}_+(X)$ (resp. $\mathscr{F}_-(X)$).

(ii) Clearly, $\mathscr{F}(X) \subseteq \mathscr{F}^b(X)$ (because $\Phi^b(X) \subseteq \Phi(X)$). To prove the opposite inclusion, let $F \in \mathscr{F}^b(X)$. If $A \in \Phi(X)$, then by [30, Theorem 1.1, p. 162] there exist $A_0 \in \mathscr{L}(X)$ and $K \in \mathscr{F}_0(X)$ such that

$$AA_0 = I + K \quad \text{on } X, \tag{2.2}$$

where $\mathscr{F}_0(X)$ stands for the ideal of finite rank operators. Thus

$$(A + F)A_0 = I + K + FA_0 = I + \Xi. \tag{2.3}$$

Clearly, the fact that $A \in \Phi(X)$ implies that $\hat{A} \in \Phi^b(X_A, X)$ (use Remark 2.3). Also, (2.2) implies that AA_0 is a Fredholm operator. Next,

applying [30, Theorem 2.7, p. 171] we obtain that $A_0 \in \Phi^b(X, X_A)$. Similarly, since $\mathscr{F}^b(X)$ is a closed two-sided ideal of $\mathscr{L}(X)$ containing $\mathscr{F}_0(X)$, we conclude that $\Xi \in \mathscr{F}^b(X)$. This together with (2.3) implies that $(A + F)A_0 \in \Phi^b(X)$. Since $A_0 \in \Phi^b(X, X_A)$, it follows from [30, Theorem 2.5, p. 169] that $\hat{A} + \hat{F} \in \Phi^b(X_A, X)$. Now by [30, Lemma 1.7, p. 166] (or (2.1)) we see that $A + F \in \Phi(X)$. This shows that $F \in \mathscr{F}(X)$ which ends the proof. Q.E.D.

We now recall another definition of the Schechter essential spectrum which may be also encountered in the literature (see, for example, [30]). Indeed, let $A \in \mathscr{C}(X)$,

$$\sigma_{e5}(A) = \bigcap_{K \in \mathscr{K}(X)} \sigma(A + K). \tag{2.4}$$

The next proposition, owing to Schechter, shows the equivalence between (2.4) and the definition of $\sigma_{e5}(\cdot)$ given above.

PROPOSITION 2.1 [30, Theorem 5.4, p. 180]. *Let X be a Banach space and let $A \in \mathscr{C}(X)$. Then*

$$\lambda \notin \sigma_{e5}(A) \quad \textit{if and only if } \lambda \in \Phi_A^0,$$

where $\Phi_A^0 := \{\lambda \in \Phi(A);\ i(\lambda - A) = 0\}$.

We close this section by recalling the following definitions required below.

Let X be a Banach space. We say that X possesses the Dunford–Pettis property (for short, property D P) if for each Banach space Y every weakly compact operator $T : X \to Y$ takes weakly compact sets in X into norm compact sets of Y. It is well known that any L_1-space has the property D P. Also, if Ω is a compact Hausdorff space then $C(\Omega)$ has the property D P. For further examples we refer to [5, 6, pp. 494, 497, 508, and 511].

We say that X is weakly compactly generating (w.c.g) if the linear span of some weakly compact subset is dense in X. For the properties of w.c.g spaces we refer to [4]. In particular, all separable and all reflexive Banach spaces are w.c.g as well as $L_1(\Omega, d\mu)$ if (Ω, μ) is σ-finite.

We say that X is subprojective, if given any closed infinite dimensional subspace M of X, there exists a closed infinite dimensional subspace N contained in M and a continuous projection from X onto N. Clearly any Hilbert space is subprojective. The spaces c_0, l_p $(1 \le p < \infty)$, and L_p $(2 \le p < \infty)$ are also subprojective (cf. [36]).

We say that X is superprojective if every subspace V having infinite codimension in X is contained in a closed subspace W having infinite codimension in X and such that there is a bounded projection from X to

W. The spaces l_p $(1 < p < \infty)$ and L_p $(1 < p \leq 2)$ are superprojective (cf. [36]).

3. MAIN RESULTS

Let X be a fixed Banach space. Unless otherwise stated in all that follows $\mathscr{I}(X)$ will denote an arbitrary nonzero two-sided ideal of $\mathscr{L}(X)$ satisfying the condition

$$(H) \qquad \mathscr{I}(X) \subseteq \mathscr{F}(X).$$

Remark 3.1. It should be observed that if $\mathscr{I}(X)$ is a nonzero two-sided ideal satisfying (H), then

$$\mathscr{F}_0(X) \subseteq \mathscr{I}(X) \subseteq \mathscr{F}(X). \tag{3.1}$$

This follows from Lemma 2.3(ii) and [8, Proposition 4, p. 70].

We begin with the following proposition which is fundamental for our purpose. It generalizes many known perturbation results in the literature.

PROPOSITION 3.1. *Let $A \in \mathscr{C}(X)$ and let $\mathscr{I}(X)$ be any nonzero ideal of $\mathscr{L}(X)$ satisfying (H). If $J \in \mathscr{I}(X)$, then*

(i) *if $A \in \Phi(X)$, then $A + J \in \Phi(X)$ and $i(A + J) = i(A)$.*

Moreover,

(ii) *if $A \in \Phi_-(X)$ and $\mathscr{I}(X) \subseteq \mathscr{F}_+(X)$, then $A + J \in \Phi_+(X)$;*

(iii) *if $\mathscr{I}(X) \subseteq \mathscr{F}_-(X)$ or $[\mathscr{I}(X)]^* \subseteq \mathscr{F}_+(X^*)$, then $A + J \in \Phi_-(X)$ for all $A \in \Phi_-(X)$;*

(iv) *if $A \in \Phi_\pm(X)$ and $\mathscr{I}(X) \subseteq \mathscr{F}_+(X) \cap \mathscr{F}_-(X)$, then $A + J \in \Phi_\pm(X)$.*

Proof. Note that the statement (ii), the first part of (iii) and (iv) are trivial. The second part of (iii) may be checked as follows. Let $J \in \Phi_-(X)$. Clearly, $A^* \in \Phi_+(X^*)$. Moreover, the inclusion $[\mathscr{I}(X)]^* \subseteq \mathscr{F}_+(X^*)$ shows that $A^* + J^* \in \Phi_+(X^*)$. Next, this together with the fact that $\alpha(A^* + J^*) = \beta(A + J)$ (cf. [9, 18]), implies that $A + J \in \Phi_-(X)$.

(i) Let $A \in \Phi(X)$ and $J \in \mathscr{I}(X)$. The fact that $A + J \in \Phi(X)$ follows from Definition 2.3. Next, proceeding as in the proof of Lemma 2.3(ii) we see that there exist $A_0 \in \mathscr{L}(X)$ and $K \in \mathscr{F}_0(X)$ such that (2.3) holds. This leads to

$$(A + J)A_0 = I + K + JA_0 = I + \Theta. \tag{3.2}$$

Since $\mathscr{I}(X)$ is a closed two-sided ideal containing $\mathscr{F}_0(X)$, we have $\Theta \in \mathscr{I}(X) \subseteq \mathscr{F}(X)$. Then (2.4) and (3.2) imply that AA_0 and $(A+J)A_0$ are in $\Phi^b(X)$ and

$$i(AA_0) = i((A+R)A_0) = 0. \tag{3.3}$$

On the other hand, arguing as in Lemma 2.3(ii) we see that $A_0 \in \Phi^b(X, X_A)$ and $\hat{A} + \hat{J} \in \Phi^b(X_A, X)$. Next, applying the Atkinson theorem [30, Theorem 1.3, p. 163] to both AA_0 and $(A+J)A_0$ and using (3.3) we get

$$i(\hat{A}) = -i(A_0) \qquad \text{and} \qquad i(\hat{A} + \hat{J}) = -i(A_0)$$

which implies that $i(\hat{A}) = i(\hat{A} + \hat{J})$. Now, by (2.1), we have $A + J$ in $\Phi(X)$ and $i(A) = i(A+J)$. This concludes the proof. Q.E.D.

Our first result is the following theorem.

THEOREM 3.1. *Let $A \in \mathscr{C}(X)$ and let $\mathscr{I}(X)$ be any nonzero two-sided ideal of $\mathscr{L}(X)$. Assume that the condition (H) is satisfied, then*

(i) *if $J \in \mathscr{I}(X)$, then*

$$\sigma_{ei}(A) = \sigma_{ei}(A+J), \qquad i = 4, 5.$$

Moreover, if $C\sigma_{e5}(A)$ (the complement of $\sigma_{e5}(A)$) is connected and neither $\rho(A)$ nor $\rho(A+J)$ is empty, then

$$\sigma_{e6}(A) = \sigma_{e6}(A+J).$$

Further,

(ii) *if $\mathscr{I}(X) \subseteq \mathscr{F}_+(X)$, then*

$$\sigma_{e1}(A) = \sigma_{e1}(A+J) \qquad \forall J \in \mathscr{I}(X);$$

(iii) *if $\mathscr{I}(X) \subseteq \mathscr{F}_-(X)$ or $[\mathscr{I}(X)]^* \subset \mathscr{F}_+(X^*)$, then*

$$\sigma_{e2}(A) = \sigma_{e2}(A+J) \qquad \forall J \in \mathscr{I}(X);$$

(iv) *if $\mathscr{I}(X) \subseteq \mathscr{F}_+(X) \cap \mathscr{F}_-(X)$, then*

$$\sigma_{e3}(A) = \sigma_{e3}(A+J) \qquad \forall J \in \mathscr{I}(X).$$

Remark 3.2. (a) The ideal of finite rank operators $\mathscr{F}_0(X)$ is the minimal subset of $\mathscr{L}(X)$ for which the assertions (i), (ii), (iii), and (iv) are valid regardless of the Banach spaces.

(b) Theorem 3.1 may be regarded as an extension of [19, Theorem 3.1; 22, Theorem 3.1] to general Banach spaces. In fact, let (Ω, Σ, μ) be a

measure space; if $X = L_p(\mu)$, $1 \le p \le \infty$, and $A \in \mathscr{C}(X)$, then $\sigma_{ei}(A) = \sigma_{ei}(A + S)$ for all $S \in \mathscr{S}(X)$ and $i = 1, 4, 5$, and 6 [22]. The same holds true for $X = l_p$, $1 \le p < \infty$ or $C(\Xi)$, provided that Ξ is a compact Hausdorff space. Also, using weakly compact perturbations, similar results were obtained in [19] for $\sigma_{ei}(\cdot)$ with $i = 2, 4, 5$, and 6 on Banach spaces which possess the Dunford–Pettis property.

Proof of Theorem 3.1. The proofs of the statements (ii), (iii), (iv), and the first part of (i) for $i = 4$ use Proposition 3.1 and are immediate. So, they are omitted.

Next, we prove (i) for $i = 5$. If $\lambda \notin \sigma_{e5}(A)$, then by Proposition 2.1, $\lambda \in \Phi_A^0$. Hence, using Proposition 3.1(i) one has $\lambda \in \Phi_{A+J}^0$. Applying again Proposition 2.1 we see that $\lambda \notin \sigma_{e5}(A + J)$, i.e., $\sigma_{e5}(A + J) \subseteq \sigma_{e5}(A)$. Analogously, using Propositions 2.1 and 3.1(i) and arguing as above we derive easily the opposite inclusion $\sigma_{e5}(A) \subseteq \sigma_{e5}(A + J)$.

The proof of the statement (i) for $i = 6$ is essentially the same as that of the last assertion of Theorem 3.1 in [19]. Q.E.D.

Note that if A and B are bounded self-adjoint operators in a Hilbert space, the classical theorem of Weyl (see [14, 27, 28]) states that if $A - B$ is compact then $\sigma_{ess}(A) = \sigma_{ess}(B)$. (Here, as mentioned in the Introduction, all essential spectra coincide with the set of limit points of the spectrum denoted $\sigma_{ess}(\cdot)$.) Known generalizations of this result (see, for example, [27]) replace the compactness requirement of $A - B$ by the condition that $(\lambda - A)^{-1} - (\lambda - B)^{-1}$ is compact for $\lambda \in \rho(A) \cap \rho(B)$ and relax to various degrees the self-adjointness restriction on A and B. A generalization to closed densely defined linear operators was given in [31] for $\sigma_{ei}(\cdot)$ with $i = 4, 5$. For $\sigma_{e3}(\cdot)$ we refer to [18, Problem 5.38, p. 244]. The result below goes beyond these. We prove that the compactness of the operator $(\lambda - A)^{-1} - (\lambda - B)^{-1}$ is not necessary and it suffices that it belongs to any ideal of $\mathscr{L}(X)$ satisfying the condition (H).

THEOREM 3.2. *Let $A, B \in \mathscr{C}(X)$ and let $\mathscr{I}(X)$ be a nonzero two-sided ideal satisfying (H). If, for some $\lambda \in \rho(A) \cap \rho(B)$, $(\lambda - A)^{-1} - (\lambda - B)^{-1} \in \mathscr{I}(X)$, then*

(i) $\sigma_{ei}(A) = \sigma_{ei}(B)$, $i = 4, 5$.

Moreover,

(ii) *if $\mathscr{I}(X) \subseteq \mathscr{F}_+(X)$, then*

$$\sigma_{e1}(A) = \sigma_{e1}(B);$$

(iii) *if $\mathscr{I}(X) \subseteq \mathscr{F}_-(X)$ or $[\mathscr{I}(X)]^* \subset \mathscr{F}_+(X^*)$, then*

$$\sigma_{e2}(A) = \sigma_{e2}(B);$$

(iv) *if* $\mathscr{I}(X) \subseteq \mathscr{F}_+(X) \cap \mathscr{F}_-(X)$, *then*

$$\sigma_{e3}(A) = \sigma_{e3}(B).$$

Remark 3.3. (a) Earlier versions of Theorem 3.2 in the special cases where X has the Dunford–Pettis property or X is an L_p space $(1 \le p \le \infty)$ were established in [19, 22]. For self-adjoint operators on Hilbert spaces Theorem 3.2 is due to Wolf [37].

(b) Note that in applications (transport operators, Schrödinger operators, operators arising in dynamic populations, etc. (see [3, 11, 27])), the operator B is, in general, a bounded perturbation of A where $A \in \mathscr{C}(X)$ is the infinitesimal generator of a strongly continuous semigroup. Putting $J := B - A \in \mathscr{L}(X)$ and taking into account that $\lim_{\operatorname{Re}\lambda \to \infty} \|(\lambda - A)^{-1}\| \to 0$ (because A generates a C^0-semigroup), one sees that there exists $\lambda \in \rho(A)$ such that $r_\sigma((\lambda - A)^{-1}J) < 1$. Consequently, $\lambda \in \rho(B)$ and

$$(\lambda - B)^{-1} - (\lambda - A)^{-1} = \sum_{n \ge 1} \left[(\lambda - A)^{-1} J\right]^n (\lambda - A)^{-1}.$$

Obviously, if $(\lambda - A)^{-1}J \in \mathscr{I}(X)$ and the ideal $\mathscr{I}(X)$ is closed, then $(\lambda - B)^{-1} - (\lambda - A)^{-1} \in \mathscr{I}(X)$. Therefore, Theorem 3.2 applies to the operators A and B.

Proof of Theorem 3.2. Without loss of generality, we may suppose that $\lambda = 0$. Hence $0 \in \rho(A)$ and therefore

$$\mu - A = -\mu(\mu^{-1} - A^{-1})A, \qquad \mu \neq 0.$$

Since A is one to one and onto, then $\alpha(\mu - A) = \alpha(\mu^{-1} - A^{-1})$ and $R(\mu - A) = R(\mu^{-1} - A^{-1})$. This shows that $\mu \in \Phi_{+A}$ (resp. Φ_{-A}) if and only if $\mu^{-1} \in \Phi_{+A^{-1}}$ (resp. $\Phi_{-A^{-1}}$). Similarly, we have $\mu \in \Phi_A$ if and only if $\mu^{-1} \in \Phi_{A^{-1}}$.

Assume that $A^{-1} - B^{-1} \in \mathscr{I}(X)$. Hence using Proposition 3.1(i) we conclude that

$$\Phi_A = \Phi_B \qquad \text{and} \qquad i(\eta - A) = i(\eta - B) \text{ for each } \eta \in \Phi_A.$$

This proves (i).

If further $\mathscr{I}(X) \subseteq \mathscr{F}_+(X)$ (resp. $\mathscr{I}(X) \subseteq \mathscr{F}_-(X)$ or $[\mathscr{I}(X)]^* \subseteq \mathscr{F}_+(X^*)$) then Proposition 3.1(ii) (resp. Proposition 3.1(iii)) implies that $\Phi_{+A} = \Phi_{+B}$, (resp. $\Phi_{-A} = \Phi_{-B}$). This concludes the proof of (iii) (resp. (iv)).

To establish (iv), it suffices to observe that if $\mathscr{I}(X) \subseteq \mathscr{F}_+(X) \cap \mathscr{F}_-(X)$, then by Proposition 3.1(iv) we have

$$\Phi_{-A} \cup \Phi_{+A} = \Phi_{-B} \cup \Phi_{+B}.$$

This ends the proof. Q.E.D.

We have also the following useful stability result for the Wolf and the Schechter essential spectra.

THEOREM 3.3. *Let A, $B \in \mathscr{C}(X)$ and let $\mathscr{I}(X)$ be a nonzero two-sided ideal satisfying (H). Assume that there are A_0, B_0 in $\mathscr{L}(X)$ and J_1, J_2 in $\mathscr{I}(X)$ such that*

$$AA_0 = I - J_1, \tag{3.4}$$

$$BB_0 = I - J_2. \tag{3.5}$$

If $0 \in \Phi_A \cap \Phi_B$ and $A_0 - B_0$ is in $\mathscr{I}(X)$, then

$$\sigma_{e4}(A) = \sigma_{e4}(B). \tag{3.6}$$

If, further, $i(A) = i(B) = 0$, then

$$\sigma_{e5}(A) = \sigma_{e5}(B). \tag{3.7}$$

Remark 3.4. Theorem 3.3 generalizes [19, Theorem 3.4; 22, Theorem 3.4] to the general Banach spaces context.

Proof of Theorem 3.4. By (3.4) and (3.5), for any scalar λ, we have

$$(\lambda - A)A_0 - (\lambda - B)B_0 = J_1 - J_2 + \lambda(A_0 - B_0). \tag{3.8}$$

If $\lambda \notin \sigma_{e4}(B)$, then $\lambda \in \Phi_B$. Since B is closed, $\mathscr{D}(B)$ endowed with the graph norm is a Banach space (denoted by X_B, see Section 2). Using [30, Corollary 1.6, p. 166] we obtain $\lambda - \hat{B} \in \Phi^b(X_B, X)$. Moreover, since $J_2 \in \mathscr{I}(X)$, Eq. (3.5), Proposition 3.1(i), and [30, Theorem 2.7, p. 171] imply that $B_0 \in \Phi^b(X, X_B)$. Thus, $(\lambda - \hat{B})B_0 \in \Phi^b(X)$. If $A_0 - B_0 \in \mathscr{I}(X)$, then Eq. (3.8) together with Proposition 3.1(i) gives $(\lambda - \hat{A})A_0 \in \Phi^b(X)$ and

$$i\left[(\lambda - \hat{A})A_0\right] = i\left[(\lambda - \hat{B})B_0\right]. \tag{3.9}$$

On the other hand, since $A \in \mathscr{C}(X)$, using (3.4) and arguing as above we conclude that $A_0 \in \Phi(X, X_A)$. Thus, since $(\lambda - A)A_0 \in \Phi(X)$, the use of [30, Theorem 2.5, p. 169] shows that $\lambda - \hat{A} \in \Phi^b(X_A, X)$ and consequently $\lambda - A \in \Phi(X)$ (use [30, Corollary 1.6, p. 166]). Hence $\lambda \notin \sigma_{e4}(A)$, i.e., $\sigma_{e4}(A) \subseteq \sigma_{e4}(B)$. The opposite inclusion follows by symmetry and the proof of (3.6) is complete.

We now prove (3.7). If $\lambda \notin \sigma_{e5}(A)$, then Proposition 2.1 implies that $\lambda \in \Phi_A^0$. Next, since J_1 and J_2 belong to $\mathscr{I}(X)$ and $i(A) = i(B) = 0$, applying Proposition 3.1(i) to (3.4) and (3.5) and using the Atkinson theorem we get $i(A_0) = i(B_0) = 0$. This together with (3.9), the Atkinson theorem, and (2.1) gives

$$i(\lambda - A) = i(\lambda - B).$$

Since $\lambda \in \Phi_A^0$, we get $\lambda \in \Phi_B^0$. This proves $\sigma_{e5}(B) \subseteq \sigma_{e5}(A)$. The opposite inclusion follows by symmetry. Q.E.D.

In the following theorem we prove a sharper form of (2.4). To do so, we will assume that $\mathscr{I}(X)$ satisfies

$$\mathscr{K}(X) \subseteq \mathscr{I}(X) \subseteq \mathscr{F}(X). \tag{3.10}$$

THEOREM 3.4. *Let $A \in \mathscr{C}(X)$ and let $\mathscr{I}(X)$ be any nonzero two-sided ideal of $\mathscr{L}(X)$. If* (3.10) *is fulfilled, then*

$$\sigma_{e5}(A) = \bigcap_{J \in \mathscr{I}(X)} \sigma(A + J).$$

Remark 3.5. (a) Note that any subset $\mathscr{I}(X)$ of $\mathscr{L}(X)$ (not necessarily an ideal) satisfying (3.10) may characterize the Schechter essential spectrum. $\mathscr{K}(X)$ is then the minimal subset of $\mathscr{L}(X)$ (in the sense of the inclusion) for which Theorem 3.4 holds true. Hence Theorem 3.4 provides an improvement of the definition of $\sigma_{e5}(\cdot)$ valid for a somewhat large variety of subsets of $\mathscr{L}(X)$. Also, it may be viewed as an extension of [19, Theorem 3.2; 22, Theorem 3.2] to general Banach spaces.

(b) A result in the spirit of Theorem 3.4 for the Browder essential spectrum was established in [15]. It is proved that $\sigma_6(A)$ is invariant under perturbations of A by Riesz operators which commute with A.

Proof of Theorem 3.4. Set $\mathscr{O} := \bigcap_{J \in \mathscr{I}(X)} \sigma(A + J)$. We already know from (2.4) and (3.10) that $\mathscr{O} \subseteq \sigma_{e5}(A)$. It remains to show that $\sigma_{e5}(A) \subseteq \mathscr{O}$. If $\lambda \notin \mathscr{O}$, then there exists $J \in \mathscr{I}(X)$ such that $\lambda \in \rho(A + J)$. Hence, $\lambda \in \Phi^0_{(A+J)}$. On the other hand, since $(\lambda - A - J)^{-1} \in \mathscr{L}(X)$, then $(\lambda - A - J)^{-1}J \in \mathscr{I}(X)$. Therefore, by Proposition 3.1(i), one has $I + (\lambda - A - J)^{-1}J \in \Phi(X)$ and $i(I + (\lambda - A - J)^{-1}J) = 0$. Next, using the relation $\lambda - A = (\lambda - A - J)(I + (\lambda - A - J)^{-1}J)$ together with Atkinson's theorem [30, Theorem 1.3, p. 163] we get $\lambda \in \Phi_A^0$. Now, applying Proposition 2.1 we see that $\lambda \notin \sigma_{e5}(A)$, i.e., $\sigma_{e5}(A) \subseteq \mathscr{O}$. This completes the proof. Q.E.D.

PROPOSITION 3.2. *Let $A \in \mathscr{C}(X)$ and let $\mathscr{I}(X)$ be a nonzero two-sided ideal satisfying (H). If $\sigma_{e6}(A) = \sigma_{e5}(A)$, then, for each $J \in \mathscr{I}(X)$, there is at most a countable set $\mathscr{U}$ of complex numbers such that*

$$\sigma_{e6}(A + \zeta J) = \sigma_{e6}(A)$$

for $\zeta \notin \mathscr{U}$. If $C\sigma_{e6}(A)$ consists of a finite number of components, then $\mathscr{U}$ is discrete.

Proof. Let ζ be a complex number. Since $\zeta J \in \mathscr{I}(X)$, applying Theorem 3.1 we get $\sigma_{e5}(A + \zeta J) = \sigma_{e5}(A) = \sigma_{e6}(A)$. Let Σ be an arbitrary

component of $C\sigma_{e6}(A) = \rho_6(A)$ and let λ_0 be any point of Σ. By definition of $\rho_6(\cdot)$ (see Section 2), there is a neighborhood of λ_0, $\mathscr{V}_{\lambda_0}$, such that $\mathscr{V}_{\lambda_0} \setminus \{\lambda_0\} \subseteq \rho(A)$. Let $\lambda_1 \in \mathscr{V}_{\lambda_0} \setminus \{\lambda_0\} \subseteq \rho(A)$. Then, by Proposition 3.1(i), for all ζ the operator $\lambda_1 - A - \zeta J$ is a Fredholm operator with index equal to zero. Now, applying Proposition 3.1(i) we conclude that, for ζ not in a discrete set $\mathscr{U}$,

$$\alpha(\lambda_1 - A - \zeta J) = \beta(\lambda_1 - A - \zeta J) = 0,$$

i.e., $\lambda_1 \in \rho(A + \zeta J)$. Since $\Sigma \subseteq \Phi_{A+\zeta J}$, it cannot contain any point of the set $\sigma_{e6}(A + \zeta J)$. Since $C\sigma_{e6}(A)$ consists of at most a countable number of components, the proof is complete. Q.E.D.

The next result provides a spectral mapping theorem for the Schechter essential spectrum in a special case which occurs in applications (cf. [22]). Let us recall that the spectral mapping theorem holds true for $\sigma_{e1}(\cdot)$, $\sigma_{e2}(\cdot)$, $\sigma_{e4}(\cdot)$, and $\sigma_{e6}(\cdot)$ (cf. [10, 25]). However, a counter-example given in [10, p. 23] shows that, in general, it is false for $\sigma_{e3}(\cdot)$ and $\sigma_{e5}(\cdot)$.

PROPOSITION 3.3. *Let $\mathscr{I}(X)$ be a nonzero two-sided ideal satisfying (H) and let A_1 and A_2 be two elements of $\mathscr{C}(X)$ such that $A_1 - A_2 \in \mathscr{I}(X)$. If $\sigma_{e4}(A_1) = \sigma_{e5}(A_1)$ and f is a complex-valued function locally holomorphic on $\sigma(A_1) \cup \sigma(A_2) \cup \{\infty\}$ then*

$$f(\sigma_{e5}(A_k)) = \sigma_{e5}(f(A_k)), \qquad k = 1, 2.$$

In [22, Theorem 3.5] this proposition was proved for the case $X = L_p(d\mu)$ and $\mathscr{I}(X) = \mathscr{S}(L_p(d\mu))$.

Proof of Proposition 3.3. For $k = 1$ the result follows from the hypothesis $\sigma_{e5}(A_1) = \sigma_{e4}(A_1)$ and [10, Theorem 7(a)]. Consider now the case $k = 2$. The inclusion $\sigma_{e5}(f(A_2)) \subseteq f(\sigma_{e5}(A_2))$ follows from [10, Theorem 7(b)]. It remains to show $f(\sigma_{e5}(A_2)) \subseteq \sigma_{e5}(f(A_2))$. Let $\lambda \in f(\sigma_{e5}(A_2))$. Then there exists $\mu \in \sigma_{e5}(A_2)$ such that $\lambda = f(\mu)$. Hence using the hypothesis $\sigma_{e4}(A_1) = \sigma_{e5}(A_1)$ and Theorem 3.1, for $i = 4$, one sees that $\mu \in \sigma_{e4}(A_2)$. Next, applying the spectral mapping theorem for the Wolf essential spectrum [10, Theorem 7(a)] we obtain $f(\mu) \in \sigma_{e4}(f(A_2))$. Since $\sigma_{e4}(f(A_2)) \subseteq \sigma_{e5}(f(A_2))$, we infer that $f(\mu) \in \sigma_{e5}(f(A_2))$ which completes the proof. Q.E.D.

Remark 3.6. If instead of assuming that $A_1 - A_2 \in \mathscr{I}(X)$ we suppose that A_1 and A_2 satisfy the hypotheses of Theorem 3.2 or 3.3, the result of Proposition 3.3 remains valid.

Concluding Remarks. (1) As observed in Remark 3.1, if $\mathscr{I}(X)$ satisfies the hypothesis (H), then $\mathscr{F}_0(X) \subseteq \mathscr{I}(X)$. Hence the ideal of finite rank

operators is the minimal subset of $\mathscr{L}(X)$ (in the sense of the inclusion) for which the results of this paper (except Theorem 3.2) are valid.

(2) Let $A \in \mathscr{C}(X)$ and assume that X has the property D P. It is proved in [19, Sect. 3] that the ideal of weakly compact operators, $\mathscr{W}(X)$, leaves invariant the sets $\Phi_+(X)$, $\Phi_-(X)$, $\Phi_\pm(X)$, and $\Phi(X)$ under additive perturbations, i.e., $\mathscr{W}(X) \subseteq \mathscr{F}_+(X) \cap \mathscr{F}_-(X)$. Hence, for $\mathscr{I}(X) = \mathscr{W}(X)$, the assertions of Proposition 3.1 are valid and, consequently, the results obtained above hold true. Note that the tools used to prove these results depend essentially on the structure and the properties of the spaces considered and are somewhat different from those used in this paper.

(3) In [35] Weis proved that if X is a w.c.g Banach space, then

$$\mathscr{F}_+(X) = \mathscr{S}(X) \qquad \text{and} \qquad \mathscr{F}_-(X) = C\mathscr{S}(X). \tag{3.11}$$

Clearly if $\mathscr{I}(X) = \mathscr{S}(X)$ (resp. $\mathscr{I}(X) = C\mathscr{S}(X)$), then only the assertions (i) and (iv) (resp. (ii) and (iv)) of Proposition 3.1 are valid. Accordingly, solely partial results of those obtained in Section 3 hold true.

PROPOSITION 3.4. *If X is a w.c.g Banach space, then*

(i) *if X is superprojective, then $\mathscr{S}(X) \subseteq \mathscr{F}_+(X) \cap \mathscr{F}_-(X)$;*

(ii) *if X is subprojective, then $C\mathscr{S}(X) \subseteq \mathscr{F}_+(X) \cap \mathscr{F}_-(X)$.*

Proof. If X is superprojective, then, by [32, Theorem 6(i)], one sees that $\mathscr{S}(X) \subseteq C\mathscr{S}(X)$. Now the first result follows from Eq. (3.11). If X is subprojective, then the use of the second assertion of Theorem 6 in [32] gives $C\mathscr{S}(X) \subseteq \mathscr{S}(X)$. This together with Eq. (3.11) completes the proof. Q.E.D.

Proposition 3.4 shows that if X is w.c.g and superprojective (resp. subprojective) then, for $\mathscr{I}(X) = \mathscr{S}(X)$ (resp. $\mathscr{I}(X) = C\mathscr{S}(X)$), the statements of Proposition 3.1 hold true and therefore all results of this section are valid.

(4) Let (Ω, Σ, μ) be a positive measure space and let X_p denote the spaces $L_p(\Omega, d\mu)$ with $1 \le p < \infty$. Since $p \in [1, \infty)$, the spaces X_p are w.c.g and consequently we have $\mathscr{F}_+(X_p) = \mathscr{S}(X_p)$ and $\mathscr{F}_-(X_p) = C\mathscr{S}(X_p)$. Actually, for these spaces, we have a stronger result, namely that $\mathscr{S}(X_p)$ and $C\mathscr{S}(X_p)$ are both equal to $\mathscr{F}(X_p)$ (see [24; 34; 35, p. 430]). A detailed treatment of essential spectra of closed densely defined operators on L_p-spaces using the ideal of strictly singular operators and related results may be found in [22] (see also [20] for unbounded perturbations).

Note that the identity $\mathscr{S}(X_p) = C\mathscr{S}(X_p) = \mathscr{F}(X_p)$ is not specific to L_p-spaces. In fact, it is also fulfilled for $C(\Xi)$ (the Banach space of continuous scalar-valued functions on Ξ with the supremum norm) provided that Ξ is a compact Hausdorff space (see [24; 34; 35, p. 430]).

Essential spectra of closed densely defined operators on $C(\Xi)$ (where Ξ denotes a compact Hausdorff space) under bounded as well as unbounded additive perturbations were discussed in [20].

(5) Even though the description of the ideal structure of $\mathscr{L}(X)$ is a complex task, there exist some Banach spaces X for which $\mathscr{L}(X)$ has only one proper nonzero closed two-sided ideal. The first result in this direction was established by Calkin [1]. He proved that if X is a separable Hilbert space then $\mathscr{K}(X)$ is the unique proper nonzero closed two-sided ideal of $\mathscr{L}(X)$. An extension of this result was obtained by Gohberg *et al.* [8]. They proved the same result for $X = l_p$, $1 \leq p < \infty$, and $X = c_0$. In [13] Herman established the same result for a large class of Banach spaces, namely Banach spaces which have perfectly homogeneous block bases and satisfy $(+)$ (for the definition and more information about these spaces we refer to [13]). (Evidently the spaces l_p, $1 \leq p < \infty$, and c_0 belong to this class.) Obviously, if X has perfectly homogeneous block bases which satisfy $(+)$, then

$$\mathscr{K}(X) = \mathscr{F}_+(X) = \mathscr{F}_-(X) = \mathscr{F}(X).$$

Consequently, for this class of spaces the only subset of $\mathscr{L}(X)$, which permits us to obtain the results described in this section, is the ideal of compact operators.

(6) A Banach space X is said to be an h-space if each closed infinite dimensional subspace of X contains a complemented subspace isomorphic to X. Any Banach space isomorphic to an h-space is an h-space; c, c_0, and l_p $(1 \leq p < \infty)$ are h-spaces. In [36, Theorem 6.2], Whitley proved that if X is an h-space, then $\mathscr{S}(X)$ is the greatest proper ideal of $\mathscr{L}(X)$. This together with Remark 2.5 implies that

$$\mathscr{K}(X) \subseteq \mathscr{F}_+(X) = \mathscr{F}(X) = \mathscr{S}(X) \qquad \text{and}$$
$$\mathscr{K}(X) \subseteq \mathscr{F}_-(X) \subseteq \mathscr{F}(X) = \mathscr{S}(X).$$

This shows that the results of this section hold true for strictly singular perturbations on h-spaces.

4. APPLICATION TO TRANSPORT EQUATION

The purpose of this section is to apply Theorem 3.2 to study the essential spectra of the following one-speed neutron transport operator

with general boundary conditions in slab geometry (cf. [3, 11, 16])

$$A_H\psi(x,\xi) = -\xi\frac{\partial\psi}{\partial x}(x,\xi) - \sigma(\xi)\psi(x,\xi) + \int_{-1}^{1}\kappa(x,\xi,\xi')\psi(x,\xi')\,d\xi',$$

where $x \in [-a, a]$ for a parameter $0 < a < \infty$ and $\xi \in [-1, 1]$. Let us first make precise the functional setting of the problem. Let

$$X_p = L_p(D; dx\,d\xi),$$

where $D = [-a, a] \times [-1, 1]$ $(a > 0)$, and $p \in [1, \infty)$. Define the following sets representing the *incoming* and the *outgoing* boundary of the phase space D:

$$D^- = D_1^- \cup D_2^- = \{-a\} \times [0,1] \cup \{a\} \times [-1,0],$$
$$D^+ = D_1^+ \cup D_2^+ = \{-a\} \times [-1,0] \cup \{a\} \times [0,1].$$

Moreover, we introduce the following boundary spaces

$$L_p^- := L_p(D^-, |\xi|\,d\xi) \sim L_p(D_1^-, |\xi|\,d\xi) \oplus L_p(D_2^-, |\xi|\,d\xi) := L_{1,p}^- \oplus L_{2,p}^-,$$

endowed with the norm

$$\|\psi^-, L_p^-\| = \left(\|\psi_1^-, L_{1,p}^-\|^p + \|\psi_2^i, X_{2,p}^-\|^p\right)^{1/p} = \left[\int_0^1 |\psi(-a,\xi)|^p |\xi|\,d\xi + \int_{-1}^0 |\psi(a,\xi)|^p |\xi|\,d\xi\right]^{1/p}.$$

$$L_p^+ := L_p(D^+, |\xi|\,d\xi) \sim L_p(D_1^+, |\xi|\,d\xi) \oplus L_p(D_2^+, |\xi|\,d\xi) := L_{1,p}^+ \oplus L_{2,p}^+,$$

endowed with the norm

$$\|\psi^+, L_p^+\| = \left(\|\psi_1^0, L_{1,p}^+\|^p + \|\psi_2^+, L_{2,p}^+\|^p\right)^{1/p} = \left[\int_{-1}^0 |\psi(-a,\xi)|^p |\xi|\,d\xi + \int_0^1 |\psi(a,\xi)|^p |\xi|\,d\xi\right]^{1/p},$$

where $\sim$ means the natural identification of these spaces.

The boundary conditions may be written abstractly as an operator H relating the incoming and the outgoing fluxes, namely

$$\begin{cases} H : L^+_{1,p} \oplus L^+_{2,p} \to L^-_{1,p} \oplus L^-_{2,p} \\ H\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} := \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \end{cases}$$

with $H_{j,k} : L^+_{k,p} \to L^-_{j,p}$, $H_{j,k} \in \mathscr{L}(L^+_{k,p}; L^-_{j,p})$, $j, k = 1, 2$, defined such that, on natural identification, the boundary conditions can be written as $\psi^- = H(\psi^+)$.

We define now the streaming operator T_H with domain including the boundary conditions,

$$\begin{cases} T_H : D(T_H) \subseteq X_p \to X_p \\ \psi \to T_H\psi(x, \xi) = -\xi\dfrac{\partial\psi}{\partial x}(x, \xi) - \sigma(\xi)\psi(x, \xi) \\ D(T_H) = \Big\{\psi \in X_p, \xi\dfrac{\partial\psi}{\partial x} \in X_p, \psi_{|D^-} = \psi^- \in L^-_p, \\ \qquad\qquad \psi_{|D^+} = \psi^+ \in L^+_p; \text{ and } \psi^- = H(\psi^+)\Big\}, \end{cases}$$

where $\sigma(\cdot)$ is a nonnegative, measurable, and almost everywhere finite function on $(-1, 1)$, $\psi^+ = (\psi^+_1, \psi^+_2)^\mathsf{T}$, and $\psi^- = (\psi^-_1, \psi^-_2)^\mathsf{T}$ with ψ^+_1, ψ^+_2, ψ^-_1, and ψ^-_2 given by

$$\begin{cases} \psi^-_1(\xi) = \psi(-a, \xi), & \xi \in (0, 1); \\ \psi^-_2(\xi) = \psi(a, \xi), & \xi \in (-1, 0); \\ \psi^+_1(\xi) = \psi(-a, \xi), & \xi \in (-1, 0); \\ \psi^+_2(\xi) = \psi(a, \xi), & \xi \in (0, 1). \end{cases}$$

Remark 4.1. The derivative of ψ in the definition of T_H is meant in a distributional sense. Note that if $\psi \in D(T_H)$, then it is absolutely continuous with respect to x. Hence the restrictions of ψ to D^- and D^+ are meaningful. Note also that $D(T_H)$ is dense in X_p because it contains $C^\infty_0[(-a, a) \times (1, 1)]$.

Let $\varphi \in X_p$ and consider the resolvent equation for T_H

$$(\lambda - T_H)\psi = \varphi, \tag{4.1}$$

where λ is a complex number and the unknown ψ must be sought in $D(T_H)$. Let λ^* denote the real defined by

$$\lambda^* := \liminf_{|\xi| \to 0} \sigma(\xi).$$

Thus, for Re $\lambda > -\lambda^*$, the solution of (4.1) is formally given by

$$\psi(x, \xi) = \begin{cases} \psi(-a, \xi) e^{-\frac{(\lambda + \sigma(\xi))|a+x|}{|\xi|}} + \dfrac{1}{|\xi|} \displaystyle\int_{-a}^{x} e^{-\frac{(\lambda + \sigma(\xi))|x-x'|}{|\xi|}} \varphi(x', \xi)\, dx', \\ \qquad 0 < \xi < 1; \\ \psi(a, \xi) e^{-\frac{(\lambda + \sigma(\xi))|a-x|}{|\xi|}} + \dfrac{1}{|\xi|} \displaystyle\int_{x}^{a} e^{-\frac{(\lambda + \sigma(\xi))|x-x'|}{|\xi|}} \varphi(x', \xi)\, dx', \\ \qquad -1 < \xi < 0; \end{cases} \tag{4.2}$$

whereas $\psi(a, \xi)$ and $\psi(-a, \xi)$ are given by

$$\psi(a, \xi) = \psi(-a, \xi) e^{-2a\frac{(\lambda + \sigma(\xi))}{|\xi|}} + \frac{1}{|\xi|} \int_{-a}^{a} e^{-\frac{(\lambda + \sigma(\xi))|a-x|}{|\xi|}} \varphi(x, \xi)\, dx, \qquad 0 < \xi < 1; \tag{4.3}$$

$$\psi(-a, \xi) = \psi(a, \xi) e^{-2a\frac{(\lambda + \sigma(\xi))}{|\xi|}} + \frac{1}{|\xi|} \int_{-a}^{a} e^{-\frac{(\lambda + \sigma(\xi))\,|a+x|)}{|\xi|}} \varphi(x, \xi)\, dx, \qquad -1 < \xi < 0. \tag{4.4}$$

To allow the abstract formulation of (4.2), (4.3), and (4.4), let us define the following operators depending on the parameter λ,

$$\begin{cases} M_\lambda : L_p^- \to L_p^+, M_\lambda u := (M_\lambda^+ u, M_\lambda^- u) \text{ with} \\ (M_\lambda^+ u)(-a, \xi) := u(-a, \xi) \exp\left(-2a \dfrac{(\lambda + \sigma(\xi))}{|\xi|}\right), \qquad 0 < \xi < 1; \\ (M_\lambda^- u)(a, \xi) := u(a, \xi) \exp\left(-2a \dfrac{(\lambda + \sigma(\xi))}{|\xi|}\right), \qquad -1 < \xi < 0; \end{cases}$$

$$\begin{cases} B_\lambda : L_p^- \to X_p,\ B_\lambda u := \chi_{(-1,0)}(\xi) B_\lambda^- u + \chi_{(0,1)}(\xi) B_\lambda^+ u \text{ with} \\ (B_\lambda^+ u)(-a,\xi) := u(-a,\xi)\exp\left(-\dfrac{(\lambda+\sigma(\xi))|a+x|}{|\xi|}\right), \quad 0<\xi<1; \\ (B_\lambda^- u)(a,\xi) := u(a,\xi)\exp\left(-\dfrac{(\lambda+\sigma(\xi))|a-x|}{|\xi|}\right), \quad -1<\xi<0; \end{cases}$$

$$\begin{cases} G_\lambda : X_p \to L_p^+,\ G_\lambda \varphi := (G_\lambda^+ \varphi, G_\lambda^- \varphi) \text{ with} \\ G_\lambda^+ \varphi := \dfrac{1}{|\xi|}\displaystyle\int_{-a}^{1} \exp\left(-\dfrac{(\lambda+\sigma(\xi))|a-x|}{|\xi|}\right)\varphi(x,\xi)\,dx, \quad 0<\xi<1; \\ G_\lambda^- \varphi := \dfrac{1}{|\xi|}\displaystyle\int_{-a}^{a} \exp\left(-\dfrac{(\lambda+\sigma(\xi))|a+x|}{|\xi|}\right)\varphi(x,\xi)\,dx, \quad -1<\xi<0; \end{cases}$$

and

$$\begin{cases} C_\lambda : X_p \to X_p,\ C_\lambda \varphi := \chi_{(-1,0)}(\xi) C_\lambda^- \varphi + \chi_{(0,1)}(\xi) C_\lambda^+ \varphi \text{ with} \\ C_\lambda^+ \varphi := \dfrac{1}{|\xi|}\displaystyle\int_{-a}^{x} \exp\left(-\dfrac{(\lambda+\sigma(\xi))|x-x'|}{|\xi|}\right)\varphi(x',\xi)\,dx', \\ \qquad 0<\xi<1; \\ C_\lambda^- \varphi := \dfrac{1}{|\xi|}\displaystyle\int_{x}^{a} \exp\left(-\dfrac{(\lambda+\sigma(\xi))|x-x'|}{|\xi|}\right)\varphi(x',\xi)\,dx', \\ \qquad -1<\xi<0, \end{cases}$$

where $\chi_{(-1,0)}(\cdot)$ and $\chi_{(0,1)}(\cdot)$ denote, respectively, the characteristic functions of the intervals $(-1,0)$ and $(0,1)$. Let λ_0 denote the real defined by

$$\lambda_0 := \begin{cases} -\lambda^* & \text{if } \|H\| \le 1; \\ -\lambda^* + \dfrac{1}{2a}\log(\|H\|) & \text{if } \|H\| > 1. \end{cases}$$

Simple calculations using Hölder's inequality show that these operators are bounded on their respective spaces. In fact, for $\operatorname{Re}\lambda > -\lambda^*$, the norms of the operators M_λ, B_λ, G_λ, and C_λ are bounded above, respectively, by $e^{-2a(\operatorname{Re}\lambda+\lambda^*)}, [p(\operatorname{Re}\lambda+\lambda^*)]^{-1/p}, (\operatorname{Re}\lambda+\lambda^*)^{-1}$ where q denotes the conjugate of p.

Using these operators and the fact that ψ must satisfy the boundary conditions, Eqs. (4.3) and (4.4) can be written in the space L_p^+ in the operator form

$$\psi^+ = M_\lambda H \psi^+ + G_\lambda \varphi.$$

The solution of this equation reduces to the invertibility of the operator $\mathscr{U}(\lambda) := I - M_\lambda H$ (which is the case if $\operatorname{Re}\lambda > \lambda_0$). This gives

$$\psi^+ = \{\mathscr{U}(\lambda)\}^{-1} G_\lambda \varphi. \tag{4.5}$$

On the other hand, (4.2) can be rewritten as $\psi = B_\lambda H \psi^+ + C_\lambda \varphi$. Substituting (4.5) into the last equation we get $\psi = B_\lambda H \{\mathscr{U}(\lambda)\}^{-1} G_\lambda \varphi + C_\lambda \varphi$. Thus

$$(\lambda - T_H)^{-1} = B_\lambda H \{\mathscr{U}(\lambda)\}^{-1} G_\lambda + C_\lambda.$$

On the other hand, observe that the operator C_λ is nothing else but $(\lambda - T_0)^{-1}$ where T_0 designates the streaming operator with vacuum boundary conditions, i.e., $H = 0$. Obviously, if the operator $\{\mathscr{U}(\lambda)\}$ is boundedly invertible (for example, if $\operatorname{Re}\lambda > \lambda_0$), then $\lambda \in \rho(T_H) \cap \rho(T_0)$ and

$$(\lambda - T_H)^{-1} - (\lambda - T_0)^{-1} = \mathscr{R}_\lambda, \tag{4.6}$$

where $\mathscr{R}_\lambda := B_\lambda H \{\mathscr{U}(\lambda)\}^{-1} G_\lambda$.

The essential spectra of T_0 were analyzed in detail in [22, Remark 4.1]. In particular, we have

$$\sigma_{ei}(T_0) = \{\lambda \in \mathbb{C} \text{ such that } \operatorname{Re}\lambda \le -\lambda^*\} \quad \text{for } i = 1, \ldots, 6. \tag{4.7}$$

Remark 4.2. Note that if the boundary operator H is strictly singular, then $\mathscr{R}_\lambda$ is strictly singular too (use [17, Lemma 461]). Therefore, Theorem 3.2 and Eqs. (4.6), (4.7) imply that (compare to [22, Theorem 4.1])

$$\sigma_{ei}(T_H) = \{\lambda \in \mathbb{C} \text{ such that } \operatorname{Re}\lambda \le -\lambda^*\}, \quad i = 1, 2, 3, 4, \text{ and } 5.$$

Next we consider the transport operator $A_H = T_H + K$ where K is a bounded operator given by

$$\begin{cases} K : X_p \to X_p \\ \psi \to \displaystyle\int_{-1}^{1} \kappa(x, \xi, \xi') \psi(x, \xi')\, d\xi' \end{cases} \tag{4.8}$$

with $\kappa(\cdot, \cdot, \cdot)$ a measurable function from $[-a, a] \times [-1, 1] \times [-1, 1]$ into $\mathbb{R}^+$.

Observe that K acts only on the velocity variable ξ', so x may be viewed merely as a parameter in $[-a, a]$. Hence, we may consider K as a function $K(\cdot) : x \in [-a, a] \to K(x) \in \mathscr{L}(L_p([-1, 1], d\xi))$.

In the following we will make the assumptions

$$\begin{cases} \text{the function } K(\cdot) \text{ is strongly measurable,} & (4.9) \\ \text{there exists a compact subset } \mathscr{C} \subseteq \mathscr{L}\big(L_p([-1,1], d\xi)\big) \\ \qquad \text{such that } K(x) \in C \text{ a.e. on } [-a, a], & (4.10) \\ \text{and } K(x) \in \mathscr{K}\big(L_p([-1,1], d\xi)\big) \text{ a.e. on } [-a, a], & (4.11) \end{cases}$$

where $\mathscr{K}(L_p([-1,1], d\xi))$ denotes the set of compact operators on $L_p([-1,1], d\xi)$.

We now introduce the class $\mathscr{G}(X_p)$ of collision operators defined by

$$\mathscr{G}(X_p) = \Big\{K \in \mathscr{L}(X_p) \text{ such that } (\lambda - T_H)^{-1} K \in \mathscr{S}(X_p) \\ \text{for some } \lambda \in \rho(T_H)\Big\}.$$

DEFINITION 4.1. A collision operator K, in the form (4.8), is said to be regular if it satisfies the assumptions (4.9), (4.10), and (4.11) above. We denote by $\mathscr{R}(X_p)$ the set of all regular collision operators on X_p.

Note that if K is regular on X_p then, by [20, Proposition 3.2], $(\lambda - T_H)^{-1}K$ is compact on X_p for $1 < p < \infty$ (resp. weakly compact on X_1). Therefore, using the inclusion $\mathscr{K}(X_p) \subseteq \mathscr{S}(X_p)$ (resp. the fact that the set of weakly compact operators on X_1 coincide with $\mathscr{S}(X_1)$, see [26]), we obtain the inclusion $\mathscr{R}(X_p) \subset \mathscr{G}(X_p)$. On the other hand, it is easy to see that the set of collision operators whose kernels of the form $\kappa(\xi, \xi') = f(\xi)g(\xi')$ where $f \in L_p([-1,1], d\xi)$ and $g \in L_q([-1,1], d\xi)$, $\frac{1}{p} + \frac{1}{q} = 1$, is contained in $\mathscr{R}(X_p)$. This shows that $\mathscr{G}(X_p) \neq \varnothing$.

We are now ready to prove:

THEOREM 4.1. *Let $p \in [1, \infty)$ and assume that the collision operator $K \in \mathscr{G}(X_p)$. Then*

$$\sigma_{ei}(A_H) = \sigma_{ei}(T_H), \qquad \textit{for } i = 1, \ldots, 5.$$

Moreover, if H is a strictly singular boundary operator, then

$$\sigma_{ei}(A_H) = \{\lambda \in \mathbb{C} \textit{ such that } \operatorname{Re} \lambda \leq -\lambda^*\}, \qquad \textit{for } i = 1, \ldots, 5.$$

Proof. Let $\lambda \in \rho(T_H)$ be such that $r_\sigma[(\lambda - T_H)^{-1}K] < 1$. Then $\lambda \in \rho(T_H + K)$ and

$$(\lambda - A_H)^{-1} - (\lambda - T_H)^{-1} = \sum_{n \geq 1} \left[(\lambda - T_H)^{-1}K\right]^n (\lambda - T_H)^{-1}. \tag{4.12}$$

Since $K \in \mathscr{G}(X_p)$, Eq. (4.12) shows that $(\lambda - A_H)^{-1} - (\lambda - T_H)^{-1} \in \mathscr{S}(X_p)$. Hence Theorem 3.2 gives the first assertion. On the other hand, the use of Eq. (4.5) allows us to write (4.12) in the form

$$(\lambda - A_H)^{-1} - (\lambda - T_0)^{-1} = \mathscr{R}_\lambda + \sum_{n \geq 1} \left[(\lambda - T_H)^{-1}K\right]^n (\lambda - T_H)^{-1}.$$

Therefore the strict singularity of H implies that of $(\lambda - A_H)^{-1} - (\lambda - T_0)^{-1}$. Now the second item follows from (4.7) and Theorem 3.2. Q.E.D.

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