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The Maximum Norm Analysis of a Nonmatching Grids Method for a Class of Parabolic Equation*

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ABSTRACT: Motivated by the idea which has been introduced by M. Haiour and S.Boulaaras (Proc. Indian Acad. Sci. (Math. Sci.) Vol. 121,No. 4, November 2011,pp.481–493), we provide a maximum norm analysis of Euler combined with finite element Schwarz alternating method for a class of parabolic equation on with nolinear source terms two overlapping subdomains with nonmatching grids. We consider a domain which is the union of two overlapping subdomains where each subdomain has its own independently generated grid. The two meshes being mutually independent on the overlap region, a triangle belonging to one triangulation does not necessarily belong to the other one. Under a stability analysis on Euler scheme which given by our work in (App. Math. Comp., 217, 6443–6450 (2011)), we establish, on each subdomain, an optimal asymptotic behavior between the discrete Schwarz sequence and the asymptotic solution of parabolic differential equations.

Key Words: Maximum norm analysis, Nonmatching grids method, Schwarz sequence, Parabolic differential equations.

Contents

* This presented work is in memory of the first author father (1910–1999) Mr. Mahmoud ben Mouha Boulaaras.

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1. Introduction

This paper deals with the error analysis in the maximum norm, in the context of the nonmatching grids method, of the following evolutionary equation: find $u \in L^2\left(0, T; H_0^1(\Omega)\right) \cap C^2\left(0, T, H^{-1}(\Omega)\right)$ solution of

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + \alpha u = f(u), \quad \text{in } \Sigma, \\ u = 0 \text{ in } \Gamma/\Gamma_0, \\ \dfrac{\partial u}{\partial \eta} = \varphi \text{ in } \Gamma_0,\ u(.,0) = u_0, \quad \text{in } \Omega, \end{cases} \tag{1.1}$$

where Σ is a set in $\mathbb{R}^2 \times \mathbb{R}$ defined as $\Sigma = \Omega \times [0, T]$ with $T < +\infty$, where Ω is a smooth bounded domain of $\mathbb{R}^2$ with boundary Γ.

The function $\alpha \in L^\infty(\Omega)$ is assumed to be non-negative satisfies

$$\alpha \leq \beta, \quad \beta > 0. \tag{1.2}$$

The function $f(\cdot)$ is a nonlinear and Lipschitz functions with Lipschitz constant c and satisfying the following condition

$$\begin{cases} f \in L^2\left(0, T, L^2(\Omega)\right) \cap C^1\left(0, T, H^{-1}(\Omega)\right) \\ c < \beta. \end{cases} . \tag{1.3}$$

Let $(.,.)_\Omega$ be the scalar product in $L^2(\Omega)$ and $(.,.)_{\Gamma_0}$ be the scalar product in $L^2(\Gamma_0)$, where Γ_0 is the part of the boundary defined in [25] as impulse control problem:

$$\Gamma_0 = \left\{x \in \partial\Omega = \Gamma \text{ such that } \forall \xi > 0,\ x + \xi \notin \bar{\Omega}\right\}.$$

Schwarz method has been invented by Herman Amandus Schwarz in 1890. This method has been used to solve the stationary or evolutionary boundary value problems on domains which consists of two or more overlapping sub-domains (see [1], [9], [24], [26]). We refer to ([1], [9]-[11]), and the references therein for the analysis of the Schwarz alternating method for elliptic obstacle problems and to the proceedings of the annual domain decomposition conference beginning with [17]. For results on maximum norm error analysis of overlapping nonmatching grids methods for elliptic problems we refer, for example, to [6].

In [9], we studied the overlapping domain decomposition method combined with a finite element approximation for elliptic equation related for Laplace operator Δ, where on uniform norm of an overlapping Schwarz method on nonmatching grids has been used, where we proved that the discretization on every subdomain converges on uniform norm norm. Furthermore, a result of asymptotic behavior in uniform norm has been given. In this paper, similar to that in [9], we extend the last work for evolutionary equation with mixed boundary conditions, where we provide a maximum norm analysis of a theta scheme combined with finite element

Schwarz alternating method for a linear parabolic equations on two overlapping subdomains with nonmatching grids. We consider a domain which is the union of two overlapping subdomains where each subdomain has its own independently generated grid. The two meshes being mutually independent on the overlap region, a triangle belonging to one triangulation does not necessarily belong to the other one. Under a stability analysis on the theta scheme which given by our work in [3], we establish, on each subdomain, an optimal asymptotic behavior between the discrete Schwarz sequence and the asymptotic solution of parabolic differential equations with respect the nonlinearity of the right hand side.

The outline of the paper is as follows: In section 2, we introduce some necessary notations, then we prove a full-discrete weak formulation of the presented problem using Euler time scheme combined with a finite element method. In section 3 we state a continuous alternating Schwarz sequences and define their respective finite element counterparts in the context of nonmatching overlapping grids. Section 4 is devoted to the asymptotic behavior of the method.

2. The discrete parabolic equation

The problem (1.1) can be reformulated into the following continuous parabolic variational equation: find $u \in L^2\left(0, T, H_0^1(\Omega)\right)$ solution of

$$\begin{cases} \left(\dfrac{\partial u}{\partial t}, v\right)_\Omega + a(u, v) = (f(u), v)_\Omega + (\varphi, v)_{\Gamma_0}, \\ u = 0 \text{ in } \Gamma/\Gamma_0, \\ \dfrac{\partial u}{\partial \eta} = \varphi \text{ in } \Gamma_0, \\ u(x, 0) = u_0 \text{ in } \Omega, \end{cases} \tag{2.1}$$

where $a(.,.)$ is the bilinear form defined as:

$$a(u, u) = (\nabla u, \nabla u)_\Omega - (\alpha u, u)_\Omega \tag{2.2}$$

2.1. The spatial discretization

We discretize the problem (2.1) with respect to time by using Euler scheme. Therefore, we search a sequence of elements $u^k \in H_0^1(\Omega)$ which approaches $u(t_k)$, $t_k = k\Delta t$, with initial data $u^0 = u_0$.

Thus, we have for $k = 1, ..., n$

$$\begin{cases} \left(\dfrac{u^k - u^{k-1}}{\Delta t}, v\right) + a\left(u^k, v\right) = \left(f\left(u^k\right), v\right) + (\varphi, v)_{\Gamma_0}, \\ u = 0 \text{ in } \Gamma/\Gamma_0, \\ \dfrac{\partial u}{\partial \eta} = \varphi \text{ in } \Gamma_0, \\ u(x, 0) = u_0 \text{ in } \Omega. \end{cases} \tag{2.3}$$

2.2. The spatial discretization

Let Ω be decomposed into triangles and τ_h denote the set of all those elements $h > 0$ is the mesh size. We assume that the family τ_h is regular and quasi-uniform. We consider the usual basis of affine functions φ_l, $l = \{1, ..., m(h)\}$ defined by $\varphi_l(M_s) = \delta_{ls}$ where M_s is a vertex of the considered triangulation. We introduce the following discrete spaces V^h of finite element

$$V^h = \left\{ \begin{array}{l} v \in L^2\left(0, T, H_0^1(\Omega)\right) \cap C\left(0, T, H_0^1\left(\bar{\Omega}\right)\right), \text{ such that} \\ \\ v\mid_K \in P_1,\ K \in \tau_h, \text{ and } u(.,0) = u_0 \text{ in } \Omega, \\ \\ u = 0 \text{ in } \Gamma/\Gamma_0,\ u(x,0) = u_0 \text{ in } \Omega. \end{array} \right\} \tag{2.4}$$

where r_h is the usual interpolation operator defined by

$$v \in L^2\left(0, T, H_0^1(\Omega)\right) \cap C\left(0, T, H_0^1\left(\bar{\Omega}\right)\right),\ r_h v = \sum_{i=1}^{m(h)} v(M_i)\,\varphi_i(x) \tag{2.5}$$

and P_1 denotes the space of polynomials with degree at most 1.

In the sequel of the paper, we shall make use of the discrete maximum principle assumption (dmp). In other words, we shall assume that the matrices $(A)_{ps} = a\left(\varphi_p, \varphi_s\right)$ is M-matrices (cf. [13]).

We discretize in space the problem (2.3), i.e. that we approach the space H_0^1 by a space discretization of finite dimensional $V_h \subset H_0^1$, we get the following discrete PQVIs.

$$\left\{ \begin{array}{l} \left(\dfrac{u_h^k - u_h^{k-1}}{\Delta t}, v_h\right) + a\left(u_h^k, v_h\right) \geq \left(f\left(u_h^k\right), v_h\right) + (\varphi, v)_{\Gamma_0}, \\ \\ u_h = 0 \text{ in } \Gamma/\Gamma_0, \\ \\ \dfrac{\partial u_h}{\partial \eta} = \varphi \text{ in } \Gamma_0, \\ \\ u_h^0(x) = u_{h0} \text{ in } \Omega, \end{array} \right. \tag{2.6}$$

which implies

$$\left\{ \begin{array}{l} \left(\dfrac{u_h^k}{\Delta t}, v_h\right) + a\left(u_h^k, v_h\right) \geq \left(f\left(u_h^k\right) + \dfrac{u_h^{k-1}}{\Delta t}, v_h\right) + (\varphi, v)_{\Gamma_0}, \\ \\ u_h = 0 \text{ in } \Gamma/\Gamma_0, \\ \\ \dfrac{\partial u_h}{\partial \eta} = \varphi \text{ in } \Gamma_0, \\ \\ u_h^0(x) = u_{h0} \text{ in } \Omega. \end{array} \right. \tag{2.7}$$

Then, the problem (2.7) can be reformulated into the following coercive discrete system of elliptic quasi-variational inequalities (EQVIs)

$$\begin{cases} b\left(u_h^k, v_h\right) = \left(f\left(u_h^k\right) + \lambda u_h^{k-1}, v_h\right) + (\varphi, v)_{\Gamma_0}, \ u_h^k \in V^h, \\ u_h = 0 \text{ in } \Gamma/\Gamma_0, \\ \dfrac{\partial u_h}{\partial \eta} = \varphi \text{ in } \Gamma_0, \\ u_h^0(x) = u_{h0} \text{ in } \Omega, \end{cases} \tag{2.8}$$

such that

$$\begin{cases} b\left(u_h^k, v_h\right) = \lambda\left(u_h^k, v_h\right) + a\left(u_h^k, v_h\right), \ u_h^k \in V^h, \\ \lambda = \dfrac{1}{\Delta t} = \dfrac{1}{k} = \dfrac{T}{n}, \ k = 1, ..., n. \end{cases} \tag{2.9}$$

2.3. An iterative discrete algorithm

As we have chosen before in the iterative semi-discrete algorithm $u_h^0 = u_{h0}$ the solution of the following full-discrete equation

$$b\left(u_h^0, v_h\right) = \left(g^0, v_h\right), \ v_h \in V^h, \tag{2.10}$$

where g^0 is a linear and a regular function.

Now we give the full following discrete algorithm

$$u_h^k = T_h u^{k-1}, k = 1, .., n, \tag{2.11}$$

where u_h^k is the solution of the problem (2.8).

Let $F^{k-1}(w) = f\left(u_h^k\right) + \lambda w$, $\tilde{F}^{k-1}(\tilde{w}) = f\left(\tilde{u}\ _h^k\right) + \lambda\tilde{w} \in L^\infty(\Omega)$ be the corresponding right-hand sides to the EQVIs.

Lemma 2.1. *[cf. 4,6] Under the previous assumption and the dmp we have, if*

$$F^{k-1}(w) \geqq F^{k-1}(\tilde{w}),$$

then

$$u_h^k = \partial\left(F^{k-1}(w)\right) \geqq \tilde{u}_h^k = \partial\left(F^{k-1}(\tilde{w})\right).$$

We shall first recall some results related to coercive quasi variational inequalities that are necessarily in proving some useful qualitative properties.

Proof. The proof of the Lemma is very similar to that in ([7] and [10]) for free boundary problem. □

Definition 2.2. *ζ_h^k is said to be a subsolution for the system of EQVIs (2.8) if*

$$\begin{cases} b\left(\zeta_h^k,\varphi_s\right) \leq \left(f\left(\zeta_h^k\right)+\lambda\zeta_h^{k-1},\varphi_s\right)+\left(\varphi,\varphi_s\right)_{\Gamma_0}, \ \forall\varphi_s, \quad s=1,...,m\left(h\right), \\ u_h = 0 \ in \ \Gamma/\Gamma_0, \\ \dfrac{\partial u_h}{\partial\eta} = \varphi \ in \ \Gamma_0, \\ u_h^0\left(x\right) = u_{h0} \ in \ \Omega, \end{cases}$$

Notation 1. *Let X_h be the set of discrete subsolutions. Then, we have the following theorem.*

Theorem 2.3. *Under the discrete maximum principle, the solution of the system of EQVI (2.8) is the maximum element of X_h.*

Proof. We denote by $\varphi^+ = \max(\varphi, 0)$, $\varphi^- = \max(-\varphi, 0)$.

Let $w_h \in V_h$ be a solution of the following of the full discrete system of parabolic quai variational inequalities using Euler time scheme combined with a finite element spatial approximation (cf. [3,4])

$$\begin{cases} b\left(w_h,\breve{v}_h\right) = \left(f\left(w_h\right)+\lambda w_h,\tilde{v}_h\right)+\left(\varphi,\tilde{v}_h\right)_{\Gamma_0}, \ \forall\tilde{v}_h \in V_h, \\ u_h = 0 \text{ in } \Gamma/\Gamma_0, \\ \dfrac{\partial u_h}{\partial\eta} = \varphi \text{ in } \Gamma_0, \\ u_h^0\left(x\right) = u_{h0} \text{ in } \Omega, \end{cases} \tag{2.12}$$

where $\breve{v}_h = \sum\limits_{s=1}^{m(h)} \tilde{v}_s\varphi_s$.

Since $\tilde{v}$ is a trial function, we choose $\tilde{v}_h = w_h - v_h$ and $v_h > 0$. Thus

$$b\left(w_h,\varphi_s\right) \leq \left(f\left(z_h\right)+\lambda w_h,\varphi_s\right), \tag{2.13}$$

that is to say $w_h \in X_h$.

On the other hand; let z_h be a subsolution, such that

$$w_h \leq z_h. \tag{2.14}$$

Then we have

$$b\left(z_h,\varphi_s\right) \leq \left(f\left(w_h\right)+\lambda w_h,\varphi_s\right).$$

Setting $v_h = \left(z_h - w_h\right)^+ \geq 0$ as a trial function. Yields

$$b\left(z_h,\left(z_h-w_h\right)^+\right) \leq \left(f\left(z_h\right)+\lambda w_h,\left(z_h-w_h\right)^+\right)$$

and since w_h is a subsolution too, we have

$$b\left(w_h, (z_h - w_h)^+\right) \leq \left(f\left(z_h\right) + \lambda w_h, (z_h - w_h)^+\right).$$

Thus, we deduce

$$-b\left((z_h - w_h)^+, (z_h - w_h)^+\right) \geq 0.$$

Under the coerciveness of the bilinear, we can get

$$(z_h - w_h)^+ = 0.$$

Therefore

$$z_h \leq w_h. \tag{2.15}$$

Thus, from (2.14) and (2.15) we obtain $z_h = w_h$. □

Theorem 2.4. *see [9] . Under suitable regularity of the solution of problem (1.1), there exists a constant C independent of h such that*

$$\|\zeta_h^\infty - \zeta\| \leq Ch^2 \left|\log h\right|. \tag{2.16}$$

Lemma 2.5. *(see [20]) Let $w \in H^1(\Omega) \cap C\left(\bar{\Omega}\right)$ satisfies $a(w, \phi) + \lambda(w, \phi) \geq 0$ or all nonnegative $\phi \in H^1(\Omega)$ and $w \geq 0$ on Γ, then $w \geq 0$ on $\bar{\Omega}$.*

Notation 2. *$(F^{k-1}, \varphi); (\widetilde{F}^{k-1}, \widetilde{\varphi})$ be a pair of data and $\zeta = \partial(F^{k-1}, \varphi); \widetilde{\zeta} = \partial(\widetilde{F}^{k-1}, \widetilde{\varphi})$ the corresponding solutions to (2.3) .*

Proposition 2.6. *Under the previous notation, we have*

$$\|\zeta_h - \zeta\|_{L\infty(\Omega)} \leq \max\{c\left\|u^k - \widetilde{u}^k\right\|_{L\infty(\Omega)} + \lambda\left\|u^{k-1} - \widetilde{u}^{k-1}\right\|_{L\infty(\Omega)}, \|\varphi - \widetilde{\varphi}\|_{L\infty(\Gamma)}\}. \tag{2.17}$$

Proof. First, putting

$$\mu^k = \max\{c\left\|u^k - \widetilde{u}^k\right\|_{L\infty(\Omega)} + \lambda\left\|u^{k-1} - \widetilde{u}^{k-1}\right\|_{L\infty(\Omega)}, \|\varphi - \widetilde{\varphi}\|_{L\infty(\Gamma)}\}, \tag{2.18}$$

then

$$\begin{aligned}
\widetilde{F}^k &\leq F^k + \left\|F^k - \widetilde{F}^k\right\|_{L\infty(\Omega)} \\
&\leq F^k + \max\{c\left\|u^k - \widetilde{u}^k\right\|_{L\infty(\Omega)} + \lambda\left\|u^{k-1} - \widetilde{u}^{k-1}\right\|_{L\infty(\Omega)}, \|\varphi - \widetilde{\varphi}\|_{L\infty(\Gamma)}\} \\
&\leq F^k + \lambda\mu^k.
\end{aligned}$$

So

$$b\left(\widetilde{\zeta}^k, \phi\right) \leq b\left(\zeta^k, \phi\right) + \lambda\left(\mu^k, \phi\right), \text{ for all } \phi \geq 0,\ \phi \in H_0^1(\Omega) \tag{2.19}$$

and thus
$$b\left(\widetilde{\zeta}^{k},\phi\right)\leq b\left(\zeta^{k}+\mu^{k},\phi\right)=\left(F^{k}+\lambda\mu^{k},\phi\right).$$

On the other hand, we have
$$\zeta^{k}+\phi-\widetilde{\zeta}^{k}\geq 0 \text{ on } \Gamma_{0}. \tag{2.20}$$
So
$$b(\zeta^{k}+\phi-\widetilde{\zeta}^{k}\geq 0. \tag{2.21}$$
By using the result of lemma 2.1, we get
$$\widetilde{\zeta}^{k}+\phi-\zeta^{k}\geq 0 \text{ on } \overline{\Omega} \tag{2.22}$$
Similarly, interchanging the roles of the couples (F^{k},φ) and $(\widetilde{F}^{k},\widetilde{\varphi}^{k})$, we get
$$\widetilde{\zeta}^{k}+\phi-\zeta^{k}\geq 0 \text{ on } \overline{\Omega}, \tag{2.23}$$
which completes the proof. □

Remark 2.7. *Proposition 2.6 stays true for the discrete case.*

Lemma 2.8. *([20]) Let $w\in V_{h}$ satisfy $b(w^{k},\phi_{s})>0$ for $s=1,2...,m(h)$and $w^{\theta,k}\geq 0$ on Γ_{0}.then $w^{\theta,k}\geq 0$ on $(\overline{\Omega})$.*

Notation 3. (F^{k},φ); $(\widetilde{F}^{k},\widetilde{\varphi}^{k})$ *be a pair of data and* $\zeta_{h}^{k}=\partial(F^{k},\varphi);\widetilde{\zeta}_{h}^{k}=\partial(\widetilde{F}^{k},\widetilde{\varphi})$ *the corresponding solutions to (2.3) .*

Proposition 2.9. *Let DMP hold, we have*

$$\left\|\zeta_{h}^{k}-\widetilde{\zeta}_{h}^{k}\right\|_{L\infty(\Omega)}\leq \max\{c\left\|u_{h}^{k}-\widetilde{u}_{h}^{k}\right\|_{L\infty(\Omega)}+\lambda\left\|u_{h}^{k-1}-\widetilde{u}_{h}^{k-1}\right\|_{L\infty(\Omega)},\left\|\varphi-\widetilde{\varphi}\right\|_{L\infty(\Gamma)}\}. \tag{2.24}$$

Proof. The proof is similar to that of the continuous case. □

3. Schwarz Alternating Methods for parabolic equation

We decompose (Ω) in two overlapping smooth subdomain Ω_{1} and Ω_{2} such that $\Omega=\Omega_{1}\cup\Omega_{2}$, we denote by $\partial\Omega_{i}$ the boundary of Ω_{i} and $\Gamma_{i}=\partial\Omega_{i}\cap\Omega_{j}$ and assume that the intersection of $\overline{\Gamma}_{i}$ and $\overline{\Gamma}_{j}$;$i\neq j$ is empty. Let
$$V_{i}^{(w_{j}^{k})}=\left\{\begin{array}{l} v\in\left(L^{2}\left(0,T,H_{0}^{1}\left(\Omega\right)\right)\cap C\left(0,T,H_{0}^{1}\left(\bar{\Omega}\right)\right)\right)\\ \\ \text{such that } v=w_{j} \text{ on } \Gamma_{i}.\end{array}\right.$$

We associate with problem (2.8) the following system: find $(u_1^k, u_2^k) \in V_1^k \times V_2^k$ solution to

$$\begin{cases} b_1(u_1^k, v) = (F^{\theta,k}, v)_{\Omega 1} + (\varphi^k, v)_{\Gamma_{01}}, \\ \\ b_2(u_2^k, v) = (F^{\theta,k}, v)_{\Omega 2} + (\varphi^k, v)_{\Gamma_{02}}, \end{cases} \tag{3.1}$$

where

$$b_i(u_i^k, v) = \int\limits_{\Omega_i} (\bigtriangledown u^k . \nabla v^k + \alpha u^k . v^k) dx \tag{3.2}$$

and

$$u_i^k = u^k / \Omega_i ; i = 1, 2$$

3.1. The Continuous Schwartz Sequences

Let u_0 be an initialization in $C_0\ \left(\overline{\Omega}\right)$,i.e., continuous functions vanishing on $\partial\Omega$ such that

$$b(u_0, v) = (F^k, v). \tag{3.3}$$

Starting from $u_0 = u_0/\Omega_2$, we respectively define the alternating Schwarz sequences$\left(u_1^{n+1}\right)$on Ω_1 such that

$u_1^{k,n+1} \in V_1^{\left(u_2^{k,n}\right)}$ solves of

$$b_1(u_1^{k,n+1}, v) = (F_1^k, v), \tag{3.4}$$

where

$$F_1^k = f^k \left(u_1^{k,n+1}\right) + \lambda u_1^{k-1,n+1}$$

and $(u_2^{k,n+1})$on Ω_2 such that $u_2^{k,n+1} \in V_2^{\left(k, u_1^{\theta,k,n+1}\right)}$ solves

$$b_2(u_2^{k,n+1}, v) = (F_1^k, v), \tag{3.5}$$

where

$$F_2^k = f^k \left(u_2^{k,n+1}\right) + \lambda u_2^{k-1,n+1}$$

Theorem 3.1. *[4] The sequences $(u_h^{n+1}); (u_h^{n+1})$, $n \geq 0$ produced by the Schwarz alternating method converge geometrically to a solution u of the elliptic obstacle problem. More precisely, there exist $k_1, k_2 \in (0, 1)$ which depend on (Ω_1, γ_2) and $(\Omega_2, \gamma 1)$ such that for all $n \geq 0$,*

$$\sup_{\overline{\Omega}_1} \left| u_h - u^{2n+1} \right| \leq \delta_1^n \delta_2^n \ \sup_{\gamma_1} \left| u_h - u_h^0 \right| \tag{3.6}$$

and

$$\sup_{\overline{\Omega}_2} \left| u_h - u^{2n} \right| \leq \delta_1^n \delta_2^{n-1} \sup_{\gamma_2} \left| u_h - u_h^0 \right| . \tag{3.7}$$

3.2. The discrete Schwartz sequences

As we have defined before, for $i=1,2$, let τ^{h_i} be a standard regular and quasi-uniform finite element triangulation in $\Omega_i; h_i$, being the mesh size. The two meshes being mutually independent $\Omega_1 \cap \Omega_2$, a triangle belonging to one triangulation does not necessarily belong to the other and for every $w \in C(\Omega_i)$, we set

$$V_{hi}^{(w_j^{\theta,k})} = \left\{ \begin{array}{l} v \in \left(L^2\left(0,T,H_0^1\left(\Omega\right)\right) \cap C\left(0,T,H_0^1\left(\bar{\Omega}\right)\right)\right) \\ \\ \text{such that } v=\phi \text{ on } \Gamma_{01}\cap\Gamma_{02};\ v=\pi_{h_i}\left(w\right) \text{ on } \Gamma_{0i}, \end{array} \right\}$$

where π_{h_i}denote an interpolation operator on Γ_{0i}.

Now, we define the discrete counterparts of the continuous Schwarz sequences defined in (3.4) and (3.5) .

Indeed, let u_{0h} be the discrete analog of u_0, defined in (3.3), we respectively, define by $u_{1h}^{k,n+1} \in V_{h1}^{(u_{2h}^{k,n})}$ such that

$$b_1(u_{1h}^{k,n+1},v) = (F_1^k,v), \forall v \in V_h^{(\varphi)};\ n \geq 0 \tag{3.8}$$

and $u_{2h}^{k,n+1} \in V_{h2}^{(u_{1h}^{k,n+1})}$ such that

$$b_2(u_{2h}^{k,n+1},v) = (F_2^k,v), \forall v \in V_h^{(\varphi)};\ n \geq 0. \tag{3.9}$$

4. Maximum norm analysis of asymptotic behavior

4.1. Error analysis for the stationary case

We begin by introducing two discrete auxiliary sequences and prove a fundamental lemma.

4.1.1. Two auxiliary Schwarz sequences. For $w_{2h}^0 = u_{2h}^0$, we define the sequences $w_{1h}^{\infty,n+1}$ and $w_{2h}^{\infty,n+1}$ such that $u_{1h}^{\infty,n+1} \in V_{h1}^{(u_2^{;\infty,n})}$ solves

$$b_1(w_{1h}^{\infty,n+1},v) = (F_1^k,v), \forall v \in V_{h1}^{(\varphi)}; n \geq 0, \tag{4.1}$$

and $w_{2h}^{\infty,n+1} \in V_{2h}^{(u_{1h}^{\infty,n+1})}$ solves

$$b_2(w_{2h}^{\infty,n+1},v) = (F_2^k,v), \forall v \in V_{h2}^{(\varphi)}; n \geq 0, \tag{4.2}$$

respectively. It is then clear that $w_{1h}^{\infty,n+1}$ and $w_{2h}^{\infty,n+1}$ are the finite element approximation of $u_1^{\infty,n+1}$ and $u_2^{\infty,n+1}$ defined in (4.1), (4.2), respectively. Then, as $F^k\left(.\right)$ is continuous,$\left\Vert F^k\left(u_i^{k,n+1}\right)\right\Vert_\infty \leq \lambda \left\Vert u_i^{k,n+1}\right\Vert_\infty$, (independent i of n). Therefore, making use of standard maximum norm estimates for linear parabolic problems, we have

$$\left\Vert u_i^{k,n} - u_{ih}^{k,n}\right\Vert_{L\infty(\Omega_i)} \leq Ch^2\left|\log h\right|, \tag{4.3}$$

where C is a constant independent of both h and n.

Notation 4. *From now on, we shall adopt the following notations:* $|.|_1 = |.|_{L\infty(\Gamma_1)}$, $|.|_2 = |.|_{L\infty(\Gamma_2)}$, $\|.\|_1 = \|.\|_{L\infty(\Gamma_1)}$, $\|.\|_2 = \|.\|_{L\infty(\Gamma_2)}$, *and we set* $\pi_{h_1} = \pi_{h_2} = \pi_h$.

4.2. Iterative discrete algorithm

We give our following discrete algorithm

$$u_{ih}^{k,n+1} = T_h u_{ih}^{k-1,n+1}, k = 1, ..., p, \ u_{ih}^{k,n+1} \in V_{hi}^{(u_2^{k,n})} \tag{4.4}$$

where u_h^k is the solution of the problem (2.8) and the first iteration u_h^0 is solution of (3.3).

Proposition 4.1. *[5]Under the previous hypotheses and notations, we have the following estimate of convergence*

$$\left\| u_h^{k,n+1} - u_h^{\infty} \right\|_{\infty} \leq \left(\frac{\lambda + c}{\beta + \lambda} \right)^k \| u_h^{\infty} - u_{h_0} \|_{\infty} . \tag{4.5}$$

Lemma 4.2. *Let* $\rho = \dfrac{\lambda + c}{\beta + \lambda}$. *Then, under assumption (1.2), there exists a constant* C *independent of both* h *and* n *such that*

$$\left\| u_i^{\infty,n+1} - u_{ih}^{\infty,n+1} \right\|_i \leq \frac{Ch^2 |\log h|}{1 - \rho}, \quad i = 1, 2. \tag{4.6}$$

Proof. We know from standard error estimate on uniform norm for linear problem [19] that there exists a constant C independent of h such that

$$\left\| u^0 - u_h^0 \right\|_{L=(\Omega)} \leq Ch^2 |\log h| . \tag{4.7}$$

Since $\dfrac{1}{2} < \rho < 1$, then $1 < \rho / (1 - \rho)$ and

$$\left\| u_2^0 - u_{2h}^0 \right\|_2 \leq Ch^2 |\log h| \leq \frac{\rho C h^2 |\log h|}{1 - \rho}. \tag{4.8}$$

Let us now prove (4.6) by induction. Indeed for $n = 1$, using the result of Proposition 1, we have in Ω_1

$$\begin{aligned}
\left\| u_1^{k,1} - u_{1h}^{k,1} \right\|_1 &\leq \left\| u_1^{k,1} - w_{1h}^{k,1} \right\|_1 + \left\| w_1^{k,1} - u_{1h}^{k,1} \right\|_1 \\
&\leq Ch^2 |\log h| + \left\| w_1^{k,1} - u_{1h}^{k,1} \right\|_1 \\
&\leq Ch^2 |\log h| + \max\{\rho \left\| u_1^{k,1} - u_{1h}^{k,1} \right\|_1, \left\| u_2^0 - u_{2h}^0 \right\|_2\}.
\end{aligned}$$

We then have to distinguish between two cases

$$\max\{\rho \left\| u_1^{k,1} - u_{1h}^{k,1} \right\|_1, \left\| u_2^0 - u_{2h}^0 \right\|_2\} = \rho \left\| u_1^{k,1} - u_{1h}^{k,1} \right\|_1 \tag{4.9}$$

or

$$\max\{\rho \left\| u_1^{k,1} - u_{1h}^{k,1} \right\|_1, \left\| u_2^0 - u_{2h}^0 \right\|_2\} = \left\| u_2^0 - u_{2h}^0 \right\|_2. \tag{4.10}$$

(4.9) implies

$$\begin{cases} \left\| u_1^{k,1} - u_{1h}^{\theta,k1} \right\|_1 \leq Ch^2 \left|\log h\right| + \rho \left\| u_1^{k,1} - u_{1h}^{k,1} \right\|_1, \\ \left\| u_2^0 - u_{2h}^0 \right\|_2 \leq \rho \left\| u_1^{k,1} - u_{1h}^{k,1} \right\|_1, \end{cases}$$

then

$$\begin{cases} \left\| u_1^{k,1} - u_{1h}^{k,1} \right\|_1 \leq \dfrac{Ch^2 \left|\log h\right|}{1-\rho}. \\ \left\| u_2^0 - u_{2h}^0 \right\|_2 \leq \rho \left\| u_1^{k,1} - u_{1h}^{k,1} \right\|_1 \leq \dfrac{\rho Ch^2 \left|\log h\right|}{1-\rho}. \end{cases}$$

(4.10) implies

$$\begin{cases} \left\| u_1^{k,1} - u_{1h}^{k,1} \right\|_1 \leq Ch^2 \left|\log h\right| + \left\| u_2^0 - u_{2h}^0 \right\|_2 \\ \qquad\qquad \leq \left\| u_2^0 - u_{2h}^0 \right\|_2, \end{cases}$$

so, by multiplying (4.10) by ρ we get

$$\rho \left\| u_1^{k,1} - u_{1h}^{k,1} \right\|_1 \leq \rho Ch^2 \left|\log h\right| + \rho \left\| u_2^0 - u_{2h}^0 \right\|_2. \tag{4.11}$$

So, $\rho \left\| u_1^{k,1} - u_{1h}^{k,1} \right\|_1$ is bounded by both

$$\rho Ch^2 |\log h| + \rho \left\| u_2^0 - u_{2h}^0 \right\|_2$$

and

$$\left\| u_2^0 - u_{2h}^0 \right\|_2,$$

this implies that

$$\rho \left\| u_2^0 - u_{2h}^0 \right\|_2 \leq \rho Ch^2 \left|\log h\right| + \rho \left\| u_2^0 - u_{2h}^0 \right\|_2, \tag{4.12}$$

or

$$\rho Ch^2 \left|\log h\right| + \rho \left\| u_2^0 - u_{2h}^0 \right\|_2 \leq \left\| u_2^0 - u_{2h}^0 \right\|_2, \tag{4.13}$$

that is (4.12) implies

$$\left\| u_2^0 - u_{2h}^0 \right\|_2 \leq \frac{\rho Ch^2 \left|\log h\right|}{1-\rho} \tag{4.14}$$

and (4.13) implies

$$\left\|u_2^0 - u_{2h}^0\right\|_2 \geq \frac{\rho C h^2 |\log h|}{1-\rho}. \tag{4.15}$$

It follows that only the case (4.12) is true, that is,

$$\left\|u_2^0 - u_{2h}^0\right\|_2 \leq \frac{\rho C h^2 |\log h|}{1-\rho}, \tag{4.16}$$

then

$$\begin{aligned}
\rho \left\|u_1^{k,1} - u_{1h}^{k,1}\right\|_1 &\leq Ch^2 |\log h| + \left\|u_2^0 - u_{2h}^0\right\|_2 \\
&\leq Ch^2 |\log h| + \frac{\rho C h^2 |\log h|}{1-\rho} \\
&\leq \frac{Ch^2 |\log h|}{1-\rho}.
\end{aligned}$$

So, in both cases (4.9) and (4.10), we have

$$\left\|u_1^{k,1} - u_{1h}^{k,1}\right\|_1 \leq \frac{Ch^2 |\log h|}{1-\rho}. \tag{4.17}$$

Similarly, we have in Ω_2

$$\begin{aligned}
\left\|u_2^{k,1} - u_{2h}^{k,1}\right\|_2 &\leq Ch^2 |\log h| + \left\|w_2^{k,1} - u_{2h}^{k,1}\right\|_2 \\
&\leq Ch^2 |\log h| + \max\{\rho \left\|u_2^{k,1} - u_{2h}^{k,1}\right\|_2, \left\|u_1^{k,1} - u_{1h}^{k,1}\right\|_1\}.
\end{aligned}$$

So

$$\max\{\rho \left\|u_2^{k,1} - u_{2h}^{k,1}\right\|_2, \left\|u_1^{k,1} - u_{1h}^{k,1}\right\|_1\} = \rho \left\|u_2^{\theta,k,1} - u_{2h}^{k,1}\right\|_2 \tag{4.18}$$

or

$$\max\{\rho \left\|u_2^{k,1} - u_{2h}^{k,1}\right\|_2, \left\|u_1^{k,1} - u_{1h}^{k,1}\right\|_1\} = \left\|u_1^{k,1} - u_{1h}^{k,1}\right\|_1. \tag{4.19}$$

cases (4.18) implies

$$\begin{aligned}
\left\|u_2^{k,1} - u_{2h}^{k,1}\right\|_2 &\leq Ch^2 |\log h| + \rho \left\|u_2^{k,1} - u_{2h}^{k,1}\right\|_2, \\
\left\|u_1^{k,1} - u_{1h}^{k,1}\right\|_1 &\leq \rho \left\|u_2^{k,1} - u_{2h}^{k,1}\right\|_2
\end{aligned}$$

so

$$\begin{cases} \left\|u_2^{k,1}-u_{2h}^{k,1}\right\|_2 \leq \dfrac{Ch^2\left|\log h\right|}{1-\rho}, \left\|u_1^{k,1}-u_{1h}^{k,1}\right\|_1 \\ \qquad \leq \rho\left\|u_2^{k,1}-u_{2h}^{k,1}\right\|_2 \\ \qquad \leq \dfrac{\rho Ch^2\left|\log h\right|}{1-\rho} \leq \dfrac{Ch^2\left|\log h\right|}{1-\rho}, \end{cases}$$

while case (4.19) implies

$$\begin{cases} \left\|u_2^{k,1}-u_{2h}^{k,1}\right\|_2 \leq Ch^2\left|\log h\right| + \left\|u_1^{k,1}-u_{1h}^{k,1}\right\|_1, \\ \rho\left\|u_2^{k,1}-u_{2h}^{k,1}\right\|_2 \leq \left\|u_1^{k,1}-u_{1h}^{k,1}\right\|_1. \end{cases} \tag{4.20}$$

So, by multiplying (4.20) by ρ we get

$$\rho\left\|u_2^{k,1}-u_{2h}^{k,1}\right\|_2 \leq \rho Ch^2\left|\log h\right| + \rho\left\|u_1^{k,1}-u_{1h}^{k,1}\right\|_1. \tag{4.21}$$

Hence $\rho\left\|u_2^{k,1}-u_{2h}^{k,1}\right\|_2$ is bounded by both

$$\rho Ch^2|logh| + \rho\left\|u_1^{k,1}-u_{1h}^{k,1}\right\|_1$$

and

$$\left\|u_1^{k,1}-u_{1h}^{k1}\right\|_1,$$

then

$$\left\|u_1^{k,1}-u_{1h}^{k,1}\right\|_1 \leq \rho Ch^2\left|\log h\right| + \rho\left\|u_1^{k,1}-u_{1h}^{k,1}\right\|_1 \tag{4.22}$$

or

$$Ch^2\left|\log h\right| + \rho\left\|u_1^{k,1}-u_{1h}^{k,1}\right\|_1 \leq \left\|u_1^{k,1}-u_{1h}^{k,1}\right\|_1, \tag{4.23}$$

which (4.22) implies

$$\left\|u_1^{k,1}-u_{1h}^{k,1}\right\|_1 \leq \frac{\rho Ch^2\left|\log h\right|}{1-\rho} < \frac{Ch^2\left|\log h\right|}{1-\rho} \tag{4.24}$$

or (4.23) implies

$$\frac{\rho Ch^2\left|\log h\right|}{1-\rho} \leq \left\|u_1^{k,1}-u_{1h}^{k,1}\right\|_1 < \frac{Ch^2\left|\log h\right|}{1-\rho}. \tag{4.25}$$

Hence, (4.22) and (4.23) are true because they both coincide with (4.17). So, there is either a contradiction and thus case (4.18) is impossible or case (4.19) is possible only if

$$\left\|u_1^{k,1}-u_{1h}^{k,1}\right\|_1 = \rho Ch^2\left|\log h\right| + \rho\left\|u_1^{k,1}-u_{1h}^{k,1}\right\|_1, \tag{4.26}$$

that is

$$\left\|u_1^{k,1}-u_{1h}^{k,1}\right\|_1=\frac{\rho Ch^2\left|\log h\right|}{1-\rho}, \tag{4.27}$$

thus

$$\begin{aligned}\left\|u_2^{k,1}-u_{2h}^{k,1}\right\|_2 &\leq Ch^2\left|\log h\right|+\left\|u_1^{k,1}-u_{1h}^{k,1}\right\|_1\\ &\leq Ch^2\left|\log h\right|+\frac{\rho Ch^2\left|\log h\right|}{1-\rho}\\ &\leq \frac{Ch^2\left|\log h\right|}{1-\rho},\end{aligned}$$

that is, both cases (4.18) and (4.19) imply

$$\left\|u_2^{k,1}-u_{2h}^{k,1}\right\|_2\leq\frac{Ch^2\left|\log h\right|}{1-\rho}. \tag{4.28}$$

Now, let us assume that

$$\left\|u_2^{k,n}-u_{2h}^{k,n}\right\|_2\leq\frac{Ch^2\left|\log h\right|}{1-\rho} \tag{4.29}$$

and prove that

$$\begin{cases}\left\|u_1^{k,n+1}-u_{1h}^{k,n+1}\right\|_1\leq\dfrac{Ch^2\left|\log h\right|}{1-\rho}\\ \left\|u_2^{k,n+1}-u_{2h}^{k,n+1}\right\|_2\leq\dfrac{Ch^2\left|\log h\right|}{1-\rho}\end{cases}$$

□

Theorem 4.3. *Let $h=max\,(h_1,h_2)$. Then, for n large enough, there exists a constant C independent of both h and n such that*

$$\left\|u_i^{k,n+1}-u_{ih}^{k,n+1}\right\|_1\leq\frac{ch^2\left|\log h\right|}{1-\rho},\quad\forall i=1,2. \tag{4.30}$$

Proof. Let us give the proof for $i=1$. The one for $i=2$ is similar and so will be omitted. Indeed, Let $\delta=\delta_1\delta_2$, then making use of Theorem 2 and Lemma 3, we get

$$\begin{aligned}\left\|u_1^{k}-u_{1h}^{k,n+1}\right\|_1 &\leq \left\|u_1^{k}-u_{1}^{k,n+1}\right\|_1+\left\|u_1^{k,n+1}-u_{1h}^{k,n+1}\right\|_1\\ &\leq \delta_1^n\delta_2^n\left|u^0-u\right|_1+\frac{ch^2\left|\log h\right|}{1-\rho}\\ &\leq \delta^{2n}\left|u^0-u\right|_1+\frac{ch^2\left|\log h\right|}{1-\rho}.\end{aligned}$$

So, for n large enough, we have

$$\delta^{2n} \leq h^2 \tag{4.31}$$

and thus

$$\begin{aligned}\left\| u_1^k - u_{1h}^{k,n+1} \right\|_1 &\leq ch^2 + ch^2 \left|\log h\right| \\ &\leq ch^2 \left|\log h\right|,\end{aligned}$$

which is the desired result. □

4.3. Asymptotic behavior

This section is devoted to the proof of main result of the present paper, where we prove the theorem of the asymptotic behavior in L^∞-norm for parabolic variational inequalities, where we evaluate the variation in L^∞ between $u_h\left(T\right)$, the discrete solution calculated at the moment $T = p\Delta t$ and u^∞, the asymptotic continuous solution of (2.1)

Theorem 4.4. *According to the results of the Proposition 3 and the Theorem 3, we have*

$$\left\| u_{1h}^{p,n+1} - u^\infty \right\|_\infty \leq C\left[h^2 \left|\log h\right| + \left(\frac{\lambda + c}{\beta + \lambda}\right)^p\right] \tag{4.32}$$

and

$$\left\| u_{2h}^{\theta,p,n+1} - u^\infty \right\|_\infty \leq C\left[h^2 \left|\log h\right| + \left(\frac{\lambda + c}{\beta + \lambda}\right)^p\right], \tag{4.33}$$

where C is a constant independent of h and k.

Proof. We have

$$\left\| u_h^{p,2n+1} - u^\infty \right\|_\infty \leq \left\| u_h^{p,2n+1} - u_h^\infty \right\|_\infty + \left\| u_h^\infty - u^\infty \right\|_\infty.$$

Using the Proposition 4.1 and the Theorem 4.3, we have for $\theta \geq \dfrac{1}{2}$

$$\left\| u_h^{p,2n+1} - u^\infty \right\|_\infty \leq C\left[h^2 \left|\log h\right| + \left(\frac{\lambda + c}{\beta + \lambda}\right)^p\right].$$

□

Remark 4.5. *It can be seen in the previous estimates (4.32) and (4.33), $\left(\dfrac{\lambda + c}{\beta + \lambda}\right)^p$ goes to 0 when p tend to infinity. Therefore, the estimation order for both the coercive and noncoercive problems is*

$$\left\| u^\infty - u_{1h}^{\infty,n+1} \right\|_{L^\infty\left(\bar{\Omega}_1\right)} \leq Ch^2 \left|\log h\right|$$

and

$$\left\| u^\infty - u_{2h}^{\infty,n+1} \right\|_{L^\infty\left(\bar{\Omega}_2\right)} \leq Ch^2 \left|\log h\right|.$$

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