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Fixed point theorems for (α, ψ)-Meir-Keeler-Khan mappings

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Abstract

In this paper, we establish fixed point theorems for a (α, ψ)-Meir-Keeler-Khan self mappings. The main result of our work is an extension of the theorem of Khan [M. S. Khan, Rend. Inst. Math. Univ. Trieste. Vol VIII, Fase., 10 (1976), 1–4]. We also give some consequences.

Keywords: Complete metric space, (c)-comparison function, fixed point, (α, ψ)-Meir-Keeler-Khan mapping, α-admissible mapping.
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1. Introduction and Preliminaries

The Banach fixed point theorem [3] (also known a contraction mapping principle) is an important tool in nonlinear analysis. It guarantees the existence and uniqueness of fixed points of self mappings on complete metric spaces and provides a constructive method to find fixed points. Many extensions of this principle have been done up to now, for a good read on this subject, we can quote, for example [4, 5, 7, 10, 11, 12, 14, 15, 16, 17] and the references therein.
In the sequel, $\mathbb{N}$ denotes the set of positive integers. Let Ω be the family of nondecreasing functions $\psi : [0, \infty[\longrightarrow [0, \infty[$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t > 0$, where ψ^n is the nth iterate of ψ.

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Remark 1.1. Every function $\psi \in \Omega$ is called a (c)-comparison function. It is easy to prove that if ψ is a (c)-comparison function, then $\psi(t) < t$ for any $t > 0$ and $\psi(0) = 0$.

Recently, Samet *et. al.*, [17] introduced the following concept.

Definition 1.2. Let (X, d) be a metric space, $f : X \longrightarrow X$ be a given mapping and $\alpha : X \times X \longrightarrow [0, \infty[$ be a function. We say that f is α-admissible if for all $x, y \in X$, we have

$$\alpha(x, y) \geq 1 \Longrightarrow \alpha(f(x), f(y)) \geq 1. \tag{1.1}$$

For some examples concerning the class of α-admissible mappings and other information on the subject, one can see [1, 2, 10, 17].

In (1976), M. S. Khan [8] proved the following fixed point theorem.

Theorem 1.3. *Let (X, d) be a complete metric space and let $f : X \longrightarrow X$ be a mapping satisfying the following condition:*

$$d(f(x), f(y)) \leq \mu \frac{d(x, f(x))d(x, f(y)) + d(y, f(y))d(y, f(x))}{d(x, f(y)) + d(y, f(x))}, \qquad \mu \in]0, 1[\tag{1.2}$$

Then f has a unique fixed point $u \in X$. Moreover, for all $x_0 \in X$, the sequence $\{f^n(x_0)\}$ converges to u.

Remark 1.4. It was shown by B. Fisher [6] that Theorem 1.3 is incorrect. In fact, it needed some extra conditions, that is,

$$d(x, f(y)) + d(y, f(x)) = 0 \text{ implies that } d(f(x), f(y)) = 0.$$

Thus, the correct version of Theorem 1.3 can be stated as follows.

Theorem 1.5. *Let (X, d) be a complete metric space and let $f : X \longrightarrow X$ be a mapping satisfying the following condition:*

$$d(f(x), f(y)) < \mu \frac{d(x, f(x))d(x, f(y)) + d(y, f(y))d(y, f(x))}{d(x, f(y)) + d(y, f(x))}, \quad \mu \in]0, 1[$$

$$\text{if} \quad d(x, f(y)) + d(y, f(x)) \neq 0 \tag{1.3}$$

$$\text{and}$$
$$d(f(x), f(y)) = 0 \text{ if } d(x, f(y)) + d(y, f(x)) = 0.$$

Then f has a unique fixed point $u \in X$. Moreover, for all $x_0 \in X$, the sequence $\{f^n(x_0)\}$ converges to u.

In his paper, B. Fisher [6] also presented examples which show the insufficiency of Khan's theorem. One of his examples is given below.

Example 1.6. Let $X = \{0, 1\}$. Define the metric d on X as

$$d(x, y) = |x - y|,$$

and the map f as

$$f(0) = 1, \quad f(1) = 0.$$

Then f satisfies the condition (1.2) with $\mu = \frac{1}{2}$ whenever

$$d(x, f(y)) + d(y, f(x)) \neq 0,$$

but f has no fixed point in X.

Some variations of Theorem 1.5 and its extensions are established by several authors (see [6, 9, 13, 15]).

In this paper, we derive new fixed points theorems of Meir-Keeler-Khan mappings that generalize Theorem 1.5 of B. Fisher. Our main results are given in Section 2. In Section 3, following the ideas of [5, 12, 16], the main results are applied to contractions of integral type.

2. Main Results

In this section, introducing the class of (α, ψ)-Meir-Keeler-Khan mappings, we study the existence of fixed point for a class of mappings via α-admissible mappings. Hereafter, all mappings $f : X \longrightarrow X$ which will be considered in the sequel of this paper satisfy

$$\forall x, y \in X, x \neq y \Longrightarrow d(x, f(y)) + d(y, f(x)) \neq 0.$$

Definition 2.1. Let (X, d) be a complete metric space and $f : X \longrightarrow X$. The mapping f is called (α, ψ)-Meir-Keeler-Khan mapping if there exist two functions $\psi \in \Omega$ and $\alpha : X \times X \longrightarrow [0, \infty[$ satisfying the following condition:
For each $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that

$$\begin{aligned} \epsilon \leq \psi\left(\frac{d(x, f(x))d(x, f(y)) + d(y, f(y))d(y, f(x))}{d(x, f(y)) + d(y, f(x))}\right) < \epsilon + \delta(\epsilon) \\ \Longrightarrow \alpha(x, y)d(f(x), f(y)) < \epsilon. \end{aligned} \tag{2.1}$$

Remark 2.2. It is easy to see that if $f : X \longrightarrow X$ is an (α, ψ)-Meir-Keeler-Khan mapping, then

$$\alpha(x, y)d(f(x), f(y)) \leq \psi\left(\frac{d(x, f(x))d(x, f(y)) + d(y, f(y))d(y, f(x))}{d(x, f(y)) + d(y, f(x))}\right), \tag{2.2}$$

for all $x, y \in X$.

Our first result is an existence theorem for fixed points of (α, ψ)-Meir-Keeler-Khan mappings.

Theorem 2.3. *Let (X, d) be a complete metric space and let $f : X \longrightarrow X$ be an (α, ψ)-Meir-Keeler-Khan mapping. Assume that*

(i) f is α-admissible;

(ii) There exists $x_0 \in X$ such that $\alpha(x_0, f(x_0)) \geq 1$;

(iii) f is continuous.

Then f has a fixed point in X, that is, there exists $u \in X$ such that $f(u) = u$.

Proof. Following (ii), there exists $x_0 \in X$ such that $\alpha(x_0, f(x_0)) \geq 1$. We define the sequence $\{x_k\}$ in X by $x_{k+1} = f(x_k)$ for all $k \geq 0$. If $x_{k_0} = x_{k_0+1}$ for some k_0, then x_{k_0} is a fixed point of f and the proof is done. We assume that $x_k \neq x_{k+1}$ for all $k \in \mathbb{N}$. The fact that f is α-admissible implies that

$$\alpha(x_0, x_1) = \alpha(x_0, f(x_0)) \geq 1 \Longrightarrow \alpha(f(x_0), f(x_1)) = \alpha(x_1, x_2) \geq 1.$$

By induction, we deduce that

$$\alpha(x_k, x_{k+1}) \geq 1 \text{ for all } k = 0, 1, ... \tag{2.3}$$

By (2.2) and (2.3) it follows that $\forall k \in \mathbb{N}$, we have

$$\begin{aligned} d(x_k, x_{k+1}) &= d(f(x_{k-1}), f(x_k)) \\ &\leq \alpha(x_{k-1}, x_k)d(f(x_{k-1}), f(x_k)) \\ &\leq \psi\left(\frac{d(x_{k-1}, x_k)d(x_{k-1}, x_{k+1}) + d(x_k, x_{k+1})d(x_k, x_k)}{d(x_{k-1}, x_{k+1}) + d(x_k, x_k)}\right) \\ &\leq \psi(d(x_{k-1}, x_k)), \end{aligned}$$

for all $k \in \mathbb{N}$. Inductively, we obtain

$$d(x_k, x_{k+1}) \leq \psi^k(d(x_0, x_1)).$$

Now, we prove that $\{x_k\}$ is a Cauchy sequence. Regarding the properties of the function ψ, for any $\epsilon > 0$ there exists $n(\epsilon) \in \mathbb{N}$ such that

$$\sum_{k \geq n(\epsilon)} \psi^k(d(x_0, x_1)) < \epsilon.$$

Let $n, m \in \mathbb{N}$ with $n > m > n(\epsilon)$. Applying the triangle inequality repeatedly, we get

$$\begin{aligned} d(x_m, x_n) &\leq \sum_{k=m}^{n-1} d(x_k, x_{k+1}) \leq \sum_{k=m}^{n-1} \psi^k(d(x_0, x_1)) \\ &\leq \sum_{k \geq n(\epsilon)} \psi^k(d(x_0, x_1)) < \epsilon. \end{aligned}$$

Hence, we deduce that $\{x_k\}$ is a Cauchy sequence in the complete metric space (X, d). Thus, there exists $u \in X$ such that $\lim_{k \longrightarrow \infty} x_k = u$. Since f is continuous,

$$u = \lim_{k \longrightarrow \infty} x_{k+1} = \lim_{k \longrightarrow \infty} f(x_k) = f(\lim_{k \longrightarrow \infty} x_k) = f(u),$$

which shows that $u \in X$ is a fixed point of f and completes the proof. □

In the next theorem, we establish a fixed point result without any continuity assumption on the mapping f.

Theorem 2.4. *Let (X, d) be a complete metric space and let $f : X \longrightarrow X$ be an (α, ψ)-Meir-Keeler-Khan mapping. Assume that*

(i) f is α-admissible;

(ii) There exists $x_0 \in X$ such that $\alpha(x_0, f(x_0)) \geq 1$;

(iii) if $\{x_k\}$ is a sequence in X such that $\alpha(x_k, x_{k+1}) \geq 1$ for all $k \in \mathbb{N}$ and $x_k \longrightarrow x \in X$ as $k \longrightarrow \infty$ then $\alpha(x_k, x) \geq 1$ for all $k \in \mathbb{N}$.

Then there exists $u \in X$ such that $f(u) = u$.

Proof. Following the proof of Theorem 2.3, we obtain the sequence $\{x_k\}$ in X defined by $x_{k+1} = f(x_k)$ for all $k \geq 0$ which converges to some $u \in X$. Now, using (2.3) together with condition (iii), we have $\alpha(x_k, u) \geq 1$ for all $k \in \mathbb{N}$. Next, assume that $d(u, f(u)) \neq 0$. Applying Remark 2.2, for each $k \in \mathbb{N}$, we have

$$\begin{aligned} d(u, f(u)) &\leq d(f(x_k), u) + d(f(x_k), f(u)) \\ &\leq d(x_{k+1}, u) + \alpha(x_k, u) d(f(x_k), f(u)) \\ &\leq d(x_{k+1}, u) + \psi\left(\frac{d(x_k, f(x_k)) d(x_k, f(u)) + d(u, f(u)) d(u, f(x_k))}{d(x_k, f(u)) + d(u, f(x_k))}\right). \end{aligned}$$

As $\psi(t) < t$, we obtain

$$d(u, f(u)) < d(x_{k+1}, u) + \frac{d(x_k, f(x_k)) d(x_k, f(u)) + d(u, f(u)) d(u, f(x_k))}{d(x_k, f(u)) + d(u, f(x_k))}.$$

Letting $k \longrightarrow \infty$ in the above inequality, we end up with

$$d(u, f(u)) \leq 0,$$

which obviously implies $d(u, f(u)) = 0$. Therefore $u \in X$ is a fixed point of f and the proof is done. □

Uniqueness of α-admissible mappings usually requires some extra conditions on the mapping itself or on the space on which the mapping is defined. One of these conditions can be defined as follows:

(U1) For all fixed points x and y of the mapping f, we have $\alpha(x, y) \geq 1$.

Alternatively, instead of the above condition, the following one can be used.

(U2) For all fixed points x and y of the mapping f there exists $z \in X$ such that $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$.

Theorem 2.5. *Adding the condition (U1) to the statement of Theorem 2.3 or Theorem 2.4, we obtain the uniqueness of the fixed point.*

Proof. The existence of a fixed point is obvious from the proof of Theorem 2.3(respectively Theorem2.4). Assume that the mapping f has more than one fixed points and let u and v be any two of them such that $u \neq v$. Then the condition (U1) implies $\alpha(u, v) \geq 1$. By the Remark 2.2 we have

$$\begin{aligned} d(u,v) &\leq \alpha(u,v)d(u,v) = \alpha(u,v)d(f(u),f(v)) \\ &\leq \psi\left(\frac{d(u,f(u))d(u,f(v)) + d(v,f(v))d(v,f(u))}{d(u,f(v)) + d(v,f(u))}\right) = \psi(0) = 0, \end{aligned} \tag{2.4}$$

due to the fact that $u = f(u)$ and $v = f(v)$ and by the definition of the function ψ. Therefore, $d(u, v) = 0$, which completes the uniqueness proof. □

Theorem 2.6. *Adding the condition (U2) to the statement of Theorem 2.3 or Theorem 2.4, we obtain the uniqueness of the fixed point.*

Proof. The existence of a fixed point is proved in Theorem 2.3(respectively Theorem2.4). To prove the uniqueness, let u and v be any two fixed points of f with $u \neq v$. By the condition (U2) there exists $z \in X$ such that

$$\alpha(u, z) \geq 1 \quad \text{and} \quad \alpha(v, z) \geq 1.$$

Define the sequence $\{z_n\}$ in X by $z_0 = z$, $z_{n+1} = f(z_n)$ for all $n \geq 0$. Since f is $\alpha-$admissible, and $u = f(u)$ and $v = f(v)$, we obtain

$$\alpha(u, z_n) \geq 1 \text{ and } \alpha(v, z_n) \geq 1, \text{ for all } n. \tag{2.5}$$

By the Remark 2.2, we have

$$\begin{aligned} d(u,z_{n+1}) &= d(Tu,Tz_n) \leq \alpha(u,z_n)d(Tu,Tz_n) \\ &\leq \psi\left(\frac{d(u,f(u))d(u,f(z_n)) + d(z_n,f(z_n))d(z_n,f(u))}{d(u,f(z_n)) + d(z_n,f(u))}\right) \\ &= \psi\left(\frac{d(z_n,z_{n+1})d(z_n,u)}{d(u,z_{n+1}) + d(z_n,u)}\right). \end{aligned} \tag{2.6}$$

The triangle inequality gives,

$$d(z_n, z_{n+1}) \leq d(u, z_{n+1}) + d(z_n, u),$$

and hence,

$$\frac{d(z_n,z_{n+1})d(z_n,u)}{d(u,z_{n+1}) + d(z_n,u)} \leq d(z_n,u).$$

Since ψ is nondecreasing we deduce

$$d(u,z_{n+1}) \leq \psi\left(\frac{d(z_n,z_{n+1})d(z_n,u)}{d(u,z_{n+1}) + d(z_n,u)}\right) \leq \psi(d(z_n,u)).$$

Iteratively, this inequality implies

$$d(u, z_{n1}) \leq \psi^{n+1}\left(d(u, z_0)\right), \tag{2.7}$$

for all n. Letting $n \to \infty$ in (2.7), we obtain

$$\lim_{n\to\infty} d(z_n, u) = 0. \tag{2.8}$$

In a similar way, one can show that

$$\lim_{n\to\infty} d(z_n, v) = 0. \tag{2.9}$$

By the uniqueness of the limit, we get $u = v$ and this completes the proof. □

In Theorem 2.4, if we take $\psi(t) = \lambda t$ where $\lambda \in]0, 1[$, and $\alpha(x, y) = 1$ for all $x, y \in X$, we obtain the following corollary.

Corollary 2.7. *Let (X, d) be a complete metric space and let $f : X \longrightarrow X$ be a mapping satisfying the following hypothesis:*
For any $\epsilon > 0$, there exists $\delta'(\epsilon) > 0$ such that

$$\begin{gathered}\frac{1}{\lambda}\epsilon \leq \frac{d(x, f(x))d(x, f(y)) + d(y, f(y))d(y, f(x))}{d(x, f(y)) + d(y, f(x))} < \frac{1}{\lambda}\epsilon + \delta'(\epsilon) \\ \Longrightarrow d(f(x), f(y)) < \epsilon.\end{gathered} \tag{2.10}$$

Then f has a unique fixed point $u \in X$. Moreover, for all $x_0 \in X$, the sequence $\{f^n(x_0)\}$ converges to u.

Remark 2.8. Let $\mu \in]0, 1[$ and choose $\lambda_0 \in]0, 1[$ with $\lambda_0 > \mu$. Fix $\epsilon > 0$. If we take

$$\delta'(\epsilon) = \epsilon(\frac{1}{\mu} - \frac{1}{\lambda_0})$$

in Corollary 2.7 and assume that

$$\frac{1}{\lambda_0}\epsilon \leq \frac{d(x, f(x))d(x, f(y)) + d(y, f(y))d(y, f(x))}{d(x, f(y)) + d(y, f(x))} < \frac{1}{\lambda_0}\epsilon + \delta'(\epsilon),$$

then, from (1.3), it follows that

$$\begin{aligned} d(f(x), f(y)) &\leq \mu\frac{d(x, f(x))d(x, f(y)) + d(y, f(y))d(y, f(x))}{d(x, f(y)) + d(y, f(x))} \\ &< \mu(\frac{1}{\lambda_0}\epsilon + \delta'(\epsilon)) \\ &= \mu(\frac{1}{\lambda_0}\epsilon + \epsilon(\frac{1}{\mu} - \frac{1}{\lambda_0})) \\ &= \epsilon. \end{aligned}$$

Hence (2.10) is satisfied which makes Theorem 1.5 an immediate consequence of Corollary 2.7.

Now, we denote by Ξ the set of all mappings $h : [0, +\infty[\longrightarrow [0, +\infty[$ satisfying:

(i) h continuous and nondecreasing;

(ii) $h(0) = 0$ and $h(t) > 0$ for all $t > 0$.

The following Corollary is given in [12].

Corollary 2.9 ([12])**.** *Let (X, d) be a complete metric space and $f : X \longrightarrow X$ be a mapping. Assume that there exist $h \in \Xi$ and $c \in]0, 1[$ satisfying*

$$h(d(f(x), f(y))) \leq c\ h(d(x, y)). \tag{2.11}$$

Then f has a unique fixed point $u \in X$ and for each $x \in X$, $\lim\limits_{n\longrightarrow+\infty} f^n(x) = u$.

Remark 2.10. In the case where $h(x) = x$ for all $x \in [0, +\infty[$, we obtain the Banach contraction principle [3]. If $h(x) = \displaystyle\int_0^x \varphi(t)dt$ where $\varphi : [0, +\infty[\longrightarrow [0, +\infty[$ is a Lebesgue measurable mapping which is summable (i.e., with finite integral) on each compact subset of $[0, +\infty[$, and for each $\epsilon > 0$, $\displaystyle\int_0^\epsilon \varphi(t)dt > 0$, then we get Branciari's result [5].

Example 2.11. The following positive functions h defined on $[0, +\infty[$ are increasing continuous and satisfying $h(x) = 0$ if and only if $x = 0$.

1. $h(x) = x^n \quad (n \geq 1)$;
2. $h(x) = \ln(1 + x)$;
3. $h(x) = \ln(1 + x) - \dfrac{x}{x + 1}$;
4. $h(x) = e^x - 1$;
5. $h(x) = x^{\frac{1}{x}} = e^{\frac{\ln x}{x}}$ for $x > 0$ and $h(0) = 0$ defined on $[0, 1]$;
6. $h(x) = \nu([0, x])$ where ν is a positive Radon measure defined on Borel sets of $[0, +\infty[$ such that $\nu([0, \epsilon]) > 0$ for all $\epsilon > 0$.

Remark 2.12. We observe that the Banach contraction principle can be obtained if we take the Borel measure defined on the σ-algebra of Borel sets of $[0, +\infty[$ in the item (6) in the above example, while the case of Branciari's result can be established by taking the Radon measure given by the integral of positive measurable function.

Remark 2.13. It is easy to see that every contraction satisfies (2.11) with $h(x) = x$, but the converse is, in general, false. Indeed, let

$$X = \{\frac{1}{n}\}_{n\geq 1} \bigcup \{0\},$$

be equipped with the usual metric $d(x, y) = |x - y|$ on $\mathbb{R}$ and $f : X \longrightarrow X$ be defined by

$$f(x) = \begin{cases} \dfrac{1}{n+1} & \text{if } x = \dfrac{1}{n}, n \in \mathbb{N}^*; \\ 0 & \text{if } x = 0. \end{cases}$$

A simple calculation proves that f satisfies (2.11) by taking $c = \dfrac{1}{2}$ and h is the function in the item (5) in Example 2.11, but unfortunately f is not a (strict) contraction since

$$\sup_{\{x,y\in X/x\neq y\}} \frac{d(f(x), f(y))}{d(x, y)} = 1,$$

(for more details, see [5]).

3. Consequences

In this section, following the idea of B. Samet [16], we will show that Corollary 2.7 together with Remark 2.8 allows us to obtain an integral version of Fisher's result.
We start by the following theorem.

Theorem 3.1. *Let (X, d) be a complete metric space, let f be a mapping from X into itself and let $\lambda \in]0, 1[$. Assume that there exists a function ρ from $[0, +\infty[$ into itself satisfying the following conditions:*

(i) $\rho(0) = 0$ and $\rho(t) > 0$ for every $t > 0$;

(ii) ρ is nondecreasing and right continuous;

(iii) For every $\epsilon > 0$, there exists $\delta'(\epsilon) > 0$ such that

$$\frac{1}{\lambda}\epsilon \leq \rho\left(\frac{d(x,f(x))d(x,f(y)) + d(y,f(y))d(y,f(x))}{d(x,f(y)) + d(y,f(x))}\right) < \frac{1}{\lambda}\epsilon + \delta'(\epsilon)$$

$$\Longrightarrow \rho\left(\frac{1}{\lambda}d(f(x),f(y))\right) < \frac{1}{\lambda}\epsilon,$$

for all $x, y \in X$.

Then (2.10) is satisfied.

Proof. Fix $\epsilon > 0$. Since $\rho\left(\frac{1}{\lambda}\epsilon\right) > 0$, by (iii), there exists $\theta > 0$ such that

$$\begin{aligned}\rho\left(\frac{1}{\lambda}\epsilon\right) \leq \rho\left(\frac{d(x,f(x))d(x,f(y)) + d(y,f(y))d(y,f(x))}{d(x,f(y)) + d(y,f(x))}\right) < \rho\left(\frac{1}{\lambda}\epsilon\right) + \theta \\ \Longrightarrow \rho\left(\frac{1}{\lambda}d(f(x),f(y))\right) < \rho\left(\frac{1}{\lambda}\epsilon\right).\end{aligned} \tag{3.1}$$

From the right continuity of ρ, there exists $\delta' > 0$ such that

$$\rho\left(\frac{1}{\lambda}\epsilon + \delta'\right) < \rho\left(\frac{1}{\lambda}\epsilon\right) + \theta.$$

Fix $x, y \in X$ such that

$$\frac{1}{\lambda}\epsilon \leq \frac{d(x,f(x))d(x,f(y)) + d(y,f(y))d(y,f(x))}{d(x,f(y)) + d(y,f(x))} < \frac{1}{\lambda}\epsilon + \delta'.$$

Since ρ is nondecreasing, we get

$$\begin{aligned}\rho\left(\frac{1}{\lambda}\epsilon\right) &\leq \rho\left(\frac{d(x,f(x))d(x,f(y)) + d(y,f(y))d(y,f(x))}{d(x,f(y)) + d(y,f(x))}\right) \\ &< \rho\left(\frac{1}{\lambda}\epsilon + \delta'\right) < \rho\left(\frac{1}{\lambda}\epsilon\right) + \theta.\end{aligned}$$

Then, by (3.1), we have

$$\rho\left(\frac{1}{\lambda}d(f(x),f(y))\right) < \rho\left(\frac{1}{\lambda}\epsilon\right),$$

which implies that $d(f(x),f(y)) < \epsilon$. Then (2.10) is satisfied which completes the proof. □

Corollary 3.2. *Let (X,d) be a complete metric space and let f be a mapping from X into itself. Let $h \in \Xi$ be such that, for each $\epsilon > 0$, there exists $\delta'(\epsilon)$ with*

$$\frac{1}{\lambda}\epsilon \leq h\left(\frac{d(x,f(x))d(x,f(y)) + d(y,f(y))d(y,f(x))}{d(x,f(y)) + d(y,f(x))}\right) < \frac{1}{\lambda}\epsilon + \delta'(\epsilon)$$

$$\Longrightarrow h\left(\frac{1}{\lambda}d(f(x),f(y))\right) < \frac{1}{\lambda}\epsilon.$$

Then (2.10) is satisfied.

Proof. The proof follows immediately from Theorem 3.1, since every continuous function $h : [0,+\infty[\longrightarrow [0,+\infty[$ is right continuous. □

As a consequence of this corollary, we can state the following result.

Corollary 3.3. *Let (X, d) be a complete metric space and let f be a mapping from X into itself. Let φ be a locally integrable function from $[0, +\infty[$ into itself such that $\int_0^t \varphi(s)ds > 0$ for all $t > 0$. Assume that for each $\epsilon > 0$ there exists $\delta'(\epsilon)$ such that*

$$\begin{aligned}\frac{1}{\lambda}\epsilon \leq \int_0^{\frac{d(x,f(x))d(x,f(y))+d(y,f(y))d(y,f(x))}{d(x,f(y))+d(y,f(x))}} \varphi(t)dt < \frac{1}{\lambda}\epsilon + \delta'(\epsilon) \\ \Longrightarrow \int_0^{\frac{1}{\lambda}d(f(x),f(y))} \varphi(t)dt < \frac{1}{\lambda}\epsilon.\end{aligned} \tag{3.2}$$

Then (2.10) *is satisfied.*

Now, we are able to obtain an integral version of Khan's result.

Corollary 3.4. *Let (X, d) be a complete metric space and let f be a mapping from X into itself. Let φ be a locally integrable function from $[0, +\infty[$ into itself such that $\int_0^t \varphi(s)ds > 0$ for all $t > 0$ and let $\lambda \in]0, 1[$. Assume that f satisfies the following condition.*
For all $x, y \in X$,

$$\int_0^{\frac{1}{\lambda}d(f(x),f(y))} \varphi(t)dt \leq \mu' \int_0^{\frac{d(x,f(x))d(x,f(y))+d(y,f(y))d(y,f(x))}{d(x,f(y))+d(y,f(x))}} \varphi(t)dt \tag{3.3}$$

where $\mu' \in]0, 1[$. Then f has a unique fixed point $u \in X$. Moreover, for any $x \in X$, the sequence $\{f^n(x)\}$ converges to u.

Proof. Let $\epsilon > 0$. It is easy to observe that (3.2) is satisfied for $\delta'(\epsilon) = \frac{\epsilon}{\lambda}(\frac{1}{\mu'} - 1)$. Then (2.10) is satisfied, which proves the existence and uniqueness of a fixed point. □

Remark 3.5. Note that Theorem 1.5 can be obtained from Corollary 3.4 by taking $\varphi \equiv 1$ and $\mu' = \frac{\mu}{\lambda}$ where $\lambda \in]0, 1[$ and $\lambda > \mu$.

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