SOME FIXED POINT RESULTS FOR NONEXPANSIVE MAPPINGS
IN BANACH SPACES

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Abstract. In this paper, we establish some fixed point results for nonexpansive mappings defined on a finite intersection of closed, convex and bounded subsets in Banach spaces. In particular, these results can be applied to obtain the same contribution for Suzuki mappings.

1. Introduction

In Banach spaces, the investigation of the existence of fixed point for nonexpansive mappings defined on weakly compact, convex subsets is a major field in functional analysis and it is well known since the contributions of W. Kirk, F.E. Browder and D. Göhde (1965-1966) that the role of the geometry of Banach spaces played in this direction. The notion of normal structure set on a Banach space was the first tool introduced to study the existence of fixed point results. Thus, the existence of such fixed points for nonexpansive mappings defined on compact and convex subsets which is an immediate consequence of Schauder theorem (since nonexpansive mappings are continuous) can be deduced easily from the fact that compact subsets have normal structures. Unfortunately, it’s not the case of weakly compact subsets in Banach spaces. The first example of a bounded subset which does not possess the normal structure was given by D. Alspach (1981) and constructed in the Banach space $L^1([0,1])$. This example is given by the set

$$C = \left\{ f \in L^1([0,1]), \int_0^1 f(t) dt = 1, 0 \leq f \leq 2 \right\},$$

the last author was able to construct a selfmapping nonexpansive $T$ on $C$ (more precisely, an isometry) without fixed points. It is easy to see that $C$ is bounded and convex and it’s weak compactness comes from the fact that the intervals in $L^1([0,1])$ have this property. D. Alspach gave not only an example of a weakly compact subset without normal structure, but he solved also an open problem for a long time.

Concerning the W. F. P. P (weak fixed point property). We say that a Banach space $X$ has the weak fixed point property if for every weakly compact, closed subset of it and for every nonexpansive selfmapping $T$ on $C$, $T$ has at least a fixed point in $C$. Alspach’s example enable us to assert that not all Banach spaces have W. F. P. P. The uniformly convex Banach spaces have W. F. P. P; indeed, it was proved that, bounded and convex subsets of such reflexive spaces have normal structures which

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imply as an immediate consequence that Hilbert spaces and $L^p([0,1]), 1 < p < \infty$ spaces have W. F. P. P. But the problem that if every reflexive Banach space has or not W. F. P. P (or just F. P. P concerning just closed, bounded and convex subsets) is always open. However, reflexive closed subspaces of $L^1([0,1])$ have W. F. P. P (see [10]), which enable us to ask if this is true for every reflexive lattice Banach spaces. If in a Banach space $X$, the weak compactness is equivalent to the compactness (such spaces are called having Schur property) then it is easy to assert that these spaces have W. F. P. P.

One of the important problems concerning W. F. P. P is related to Banach spaces having unconditional basis solved partially by P. K. Lin (1985) [19] who proved that every Banach space $X$ having an unconditional basis with unconditional basic constant $\lambda < \sqrt{33} - 3\over 2$, then $X$ has the property W. F. P. P. But for this fact, we have no an equivalence, indeed the James Banach $J$ (for which $dim(J^{**}J) = 1$) has not an unconditional basis (however, this space posses closed subspaces isomorphic to $l^2$) but it has W. F. P. P. Also, we don’t know the situation when the unconditional basic constant $\lambda > \sqrt{33} - 3\over 2$. One of surprising remark concerning all known Banach spaces and their relation with W. F. P. P is that or the space with all its closed subspaces have W. F. P. P or if has not W. F. P. P property then necessarily it has closed subspaces having this property. But, we don’t know if there exist a Banach space $X$ such that it and all is closed subspaces does not have W.F. P. P. This hereditarily of nonlinearity property can be compared to the problem of unconditional basic problem or the problem of indecomposability of Banach spaces and their closed subspaces solved by T. Gowers and B. Maury (1993) [20].

In our paper, inspired by some techniques given in P. K. Lin [19], we prove the existence of fixed points of nonexpansive mappings defined on a finite intersection of a closed, bounded and convex subsets of an arbitrary Banach spaces provided that the sum of their diameters is strictly less than their number or their maximum is strictly less than 1 with more assumptions together with the existence of finite number of bounded linear operators for which the norms and some associated norms are less than or equal 1 and satisfying that their null spaces have a non empty intersection with a retraction of correspond subset $(A_i)_{i=1}^{n+1}$ by the union of the others and we prove in particular that these results hold for the case of $C_\lambda$-mapping (or Suzuki mappings).

2. Preliminaries

**Definition 2.1.** Let $X$ be a Banach space and $C$ a nonempty subset of $X$ and let $T : C \rightarrow C$ be a selfmapping on $C$. We say that $T$ is a nonexpansive if for all $x, y \in C$, we have

$$\|Tx - Ty\| \leq \|x - y\|.$$  

**Remark 2.2.** It is easy to observe that every nonexpansive mapping $T$ is uniformly continuous, hence is continuous and the composition of any two nonexpansive mappings is also nonexpansive. Thus the family of nonexpansive selfmappings on a convex subset can be seen as a semigroup. This idea allow many authors to
study common fixed points for semitopological actions on convex subsets of Banach spaces extending well known results in this direction. For this subject, we quote for example the works of Professor A. M. Lau [11, 12, 13, 14, 15, 16, 17, 18].

**Definition 2.3.** Let $X$ be a Banach space and $C$ a closed, bounded and convex subset of $X$, we define the function $r_C : C \rightarrow \mathbb{R}$ by

$$r_C(x_1) = \sup \{\|x_1 - x_2\| : x_2 \in C\}$$

The continuity of the norm and the triangle inequality show that $r_C$ is a continuous convex function. Moreover, the radius $r(C)$ and the diameter of $C$ denoted by $diam(C)$ are defined by

$$r(C) = \inf \{r_C(x) : x \in C\} \text{ and } diam(C) = \sup \{r_C(x) : x \in C\}$$

**Lemma 2.4.** Let $C$ be a a closed, bounded and convex subset of a Banach space $X$ and $T$ a nonexpansive selfmapping on $C$.

1. Suppose that the closed convex hull $\text{co}(T(C)) = C$ and by $\beta > r(C)$. Then the set

$$\{x \in C, r_C(x) \leq \beta\}$$

is a nonempty, closed, convex and $T$-invariant subset of $C$ (in other words, $T(C) \subset C$).

2. Suppose that $C$ is minimal. Then $r_C$ is a constant function and $r(C) = diam(C)$. If $C$ is a bounded, closed and convex subset of Banach space $X$ and $T : C \rightarrow C$ is a nonexpansive selfmapping on $C$. It is easy to prove the existence of a sequence $(x_n)_n \subset C$ such that $\lim_{n \rightarrow +\infty} \|Tx_n - x_n\| = 0$. Indeed, it suffices to take $x_n$ the fixed point of the mapping $T_n : C \rightarrow C$ given by $T_n(x) = \frac{1}{n}z_0 + (1 - \frac{1}{n})T(x), n \geq 1, z_0 \in C$ which satisfies the Banach contraction principle (with a contractions constant equal to $1 - \frac{1}{n}$). Such sequence $(x_n)_n$ is said to be an approximate fixed point sequence for $T$.

**Remark 2.5.** As indicated above, if $T$ is a nonexpansive selfmapping on $C$, then $T_n$ satisfies the Banach contraction principle for all $n \geq 2$. This reasoning does not hold for other generalized nonexpansive mappings. Indeed, for example if $T$ is a Kannan selfmapping on $C$ defined by

$$\|Tx - Ty\| \leq \frac{1}{2} (\|x - Tx\| + \|y - Ty\|),$$

the mapping $T_n = \frac{1}{n}I + (1 - \frac{1}{n})T$ in this case does not necessarily satisfy Kannan’s contraction principle.

The following lemma d to L. A. Karlovitz [8]and K. Goebel play a crucial role to investigate the existence of fixed points for nonexpansive mappings.

**Remark 2.6.** Let $K$ be a minimal weakly compact convex subset for a nonexpansive selfmapping on $K$ and let $(x_n)_n \subset C$ be an approximate fixed point sequence. Then, $\forall x \in K$, we have
\[
\lim_{n \to +\infty} \|x_n - x\| = \text{diam}(K) \quad (\text{diameter of } K).
\]

The following corollary is an immediate consequence of Lemma 2.4.

**Corollary 2.7.** Let \( K \) be a minimal weakly compact convex subset for a nonexpansive selfmapping \( T \) on \( K \). Assume that \( 0 \in K \) and \( \text{diam}(K) = 1 \), then

\[
\forall \epsilon > 0, \exists \delta > 0 \text{ such that } \|y\| > 1 - \epsilon \text{ whenever } \|Ty - y\| < \delta.
\]

3. **Main Results**

Our first result in this direction is given by the following Theorem.

**Theorem 3.1.** Let \( X \) be a Banach space and let \( A_1, A_2, ..., A_{n+1} \) be a closed, bounded and convex subsets of \( X \) such that \( \bigcap_{i=1}^{n+1} A_i \neq \emptyset \) with \( \sum_{i=1}^{n+1} \text{diam}(A_i) < n \).

Assume that there exists \( j_0 \in \{1, 2, ..., n+1\} \) such that \( A_{j_0} \) is weakly compact and there exist bounded linear operators on \( X, S_1, S_2, ..., S_n \) satisfying that

1. \( \|S_i\| \leq 1 \forall i = 1, ..., n \), \( \|I - S_i\| \leq 1 \forall i = 1, ..., n \) and
2. \( \|nI - \sum_{i=1}^{n} S_i\| \leq 1 \).

Then, if each \( A_i, i = 1, ..., n+1 \) is invariant under a nonexpansive \( T \), then \( T \) has a fixed point in \( \bigcap_{i=1}^{n+1} A_i \).

**Proof.** Assume that for all \( i \in \{1, ..., n+1\} \), \( A_i \) is invariant under \( T \), in other words, \( T(A_i) \subset A_i \) for all \( i = 1, ..., n+1 \). Since each \( A_i \) is bounded, closed and convex, then \( \bigcap_{i=1}^{n+1} A_i \) is bounded, closed and convex subset of \( X \) and it is nonempty by hypothesis. Moreover, since \( A_{j_0} \) is weakly compact, then \( \bigcap_{i=1}^{n+1} A_i \) is weakly compact and invariant under \( T \), indeed, since

\[
T(\bigcap_{i=1}^{n+1} A_i) \subset T(A_i) \subset A_i, \forall i = 1, ..., n+1
\]

Hence

\[
T\left(\bigcap_{i=1}^{n+1} A_i\right) \subset \bigcap_{i=1}^{n+1} A_i.
\] (1)

Assume that \( T \) has no fixed points in \( \bigcap_{i=1}^{n+1} A_i \), then \( \bigcap_{i=1}^{n+1} A_i \) must contain an approximate fixed point \((x_n)_n\). By assumption 2, we get the existence of
\[ z_i \in \text{Ker}(S_i) \cap \left( A_i \setminus \bigcup_{k \neq i} A_k \right) \] $i = 1, \ldots, n$ \\

and \\

\[ z_{n+1} \in \text{ker} \left( nI - \sum_{i=1}^{n} S_i \right) \cap \left( A_{n+1} \setminus \bigcup_{i=1}^{n} A_i \right). \]

It follows that for all \( w_0 \in \bigcap_{i=1}^{n+1} A_i \), we have \\

\[
\|w_0\| = \frac{1}{n} \left[ \left\| \left( nI - \sum_{i=1}^{n} S_i \right) (w_0) + \sum_{i=1}^{n} S_i(w_0) \right\| \right] \\
\leq \frac{1}{n} \left[ \left\| \left( nI - \sum_{i=1}^{n} S_i \right) (w_0) \right\| + \sum_{i=1}^{n} \|S_i(w_0)\| \right] \\
\leq \frac{1}{n} \left[ \left\| \left( nI - \sum_{i=1}^{n} S_i \right) (w_0 - z_{n+1}) \right\| + \sum_{i=1}^{n} \|S_i(w_0 - z_i)\| \right] \\
\leq \frac{1}{n} \left[ \|nI - \sum_{i=1}^{n} S_i\| \|w_0 - z_{n+1}\| + \sum_{i=1}^{n} \|w_0 - z_i\| \right] \\
\leq \frac{1}{n} \left[ \text{diam}(A_{n+1}) + \sum_{i=1}^{n} \text{diam}(A_i) \right] \\
\leq \frac{1}{n} \sum_{i=1}^{n+1} \text{diam}(A_i) < 1.
\]

Our second result in this section is given by the following theorem

**Theorem 3.2.** Let \( X \) be a Banach space and let \( (A_i)_{i=1}^{n+1} \) be a closed, bounded and convex subset of \( X \) which are invariant under a nonexpansive mapping \( T \). Assume that \( \bigcap_{i=1}^{n+1} A_i \neq \emptyset \) and there exists \( k_0 \in \{1, 2, \ldots, n+1\} \) such that \( A_{k_0} \) is weakly compact. If there exist bounded linear operators \( (S_i)_{i=1}^{n+1} \) on \( X \) satisfying that \\

(1) \( \max(\|I - nS_i\|, i = 1, 2, \ldots, n) \leq \alpha_1 \) for \( \alpha_1 > 1 \) and \\

\[ \left\| I - n \sum_{i=1}^{n} S_i \right\| \leq \alpha_2 \ (\alpha_2 > 0). \]

(2) \( \text{Ker}(S_i - I) \cap \left( A_i \setminus \bigcup_{k \neq i} A_k \right) \neq \emptyset \ \forall i = 1, \ldots, n \) and \\

\[ \text{Ker} \left( I - n \sum_{i=1}^{n} S_i \right) \cap \left( A_i \setminus \bigcup_{k \neq i} A_k \right) \neq \emptyset. \]

If \\

\[ \max (\text{diam}(A_i))_{i=1}^{n+1} \in \left[ \frac{2}{\alpha_n + 1}, 1 \right], \]

then \\

\[ z_i \in \text{Ker}(S_i) \cap \left( A_i \setminus \bigcup_{k \neq i} A_k \right) \] $i = 1, \ldots, n$ \\

and \\

\[ z_{n+1} \in \text{ker} \left( nI - \sum_{i=1}^{n} S_i \right) \cap \left( A_{n+1} \setminus \bigcup_{i=1}^{n} A_i \right). \]
and
\[
\text{diam}(A_{n+1}) < \frac{n [2 - (\alpha_1 + 1)] \max (\text{diam}(A_i))}{\alpha_2 + 1}.
\]

Then \( T \) has a fixed point in \( \bigcap_{i=1}^{n+1} A_i \).

**Proof.** Assume that \( T \) has no fixed points, then \( \bigcap_{i=1}^{n+1} A_i \) has an invariant minimal weakly compact subset. Let \( w_0 \in \bigcap_{i=1}^{n+1} A_i \) and \( x_i \in A_i \) \( (i = 1, 2, \ldots, n+1) \). Without loss of generality, we can assume that \( \|w_0\| = 1 \). By Hahn-Banach theorem, let \( f_0 \in X^* \) such that \( f_0(w_0) = 1 = \|f_0\| \). Hence
\[
1 - f_0(x_i) = f_0(w_0 - x_i) \leq \|f_0\|\|w_0 - x_i\| \leq \text{diam}(A_i). \tag{2}
\]

So
\[
1 - \text{diam}(A_i) \leq f_0(x_i), \quad i = 1, 2, \ldots, n+1.
\]

Let
\[
\alpha_0 = f_0 \left[ \left( I - \sum_{i=1}^{n} S_i \right) (w_0) \right]
\]

Then
\[
1 - \alpha_0 = f_0(w_0) - f_0 \left[ \left( I - \sum_{i=1}^{n} S_i \right) (w_0) \right]
\]
\[
= f_0 \left( \sum_{i=1}^{n} S_i \right) (w_0)
\]
\[
= f_0(S_1(w_0)) + f_0(S_2(w_0)) + \ldots + f_0(S_n(w_0))
\]

Hence, there exists necessarily \( l_0 \in \{1, 2, \ldots, n\} \) such that \( f_0(S_{l_0}(w_0)) \leq \frac{1 - \alpha_0}{n} \).

Now, since \( \|I - nS_i\| \leq \alpha_1 \) and \( \|I - n \sum_{i=1}^{n} S_i\| \leq \alpha_2 \), we have for
\[
x \in \text{Ker} \left( I - n \sum_{i=1}^{n} S_i \right) \bigcap (A_{n+1} \setminus \bigcup_{i=1}^{n} A_i)
\]

the following inequality
\[
n(1 - \alpha_0) - \text{diam}(A_{n+1}) \leq n[f_0(S_1(w_0)) + f_0(S_2(w_0)) + \ldots + f_0(S_n(w_0))] - f_0(w_0 - x).
\]
The linearity of \( f_0 \) implies that
\[
n(1 - \alpha_0) - \text{diam}(A_{n+1}) \leq f_0[(nS_1 + ... + nS_n)(w_0)] - f_0(w_0 - x)
\]
\[
= f_0 \left( \left( n \sum_{i=1}^{n} S_i - I \right)(w_0 - x) \right)
\]
\[
\leq \|f_0\| \left\| I - n \sum_{i=1}^{n} S_i \right\| \|w_0 - x\|
\]
\[
\leq \alpha_2 \text{ diam}(A_{n+1})
\]
(3)

On the other hand, we have
\[
\alpha_0 + (1 - \text{diam}(A_{l_0})) = (1 - \text{diam}(A_{l_0})) + 1 - (1 - \alpha_0)
\]
Afterwords, for \( z \in \left( A_{l_0} \setminus \bigcup_{k \neq l_0} A_k \right) \cap \ker(S_{l_0} - I) \), we infer that
\[
\alpha_0 + (1 - \text{diam}(A_{l_0})) = 1 - \text{diam}(A_{l_0}) + 1 - (1 - \alpha_0)
\]
\[
\leq f_0(w_0) + f_0(z) - n f_0(S_{l_0}(w_0))
\]
\[
\leq f_0(w_0 - z) + n f_0(z) - n f_0(S_{l_0}(w_0))
\]
\[
= f_0(w_0 - z) + n f_0(S_{l_0}(z)) - n f_0(S_{l_0}(w_0))
\]
\[
= f_0[w_0 - z] + n f_0[S_{l_0}(z - w_0)]
\]
\[
= f_0[(I - nS_{l_0})(w_0 - z)]
\]
\[
\leq \|f_0\| \| I - nS_{l_0} \| \|w_0 - z\|
\]
\[
\leq \alpha_1 \text{ diam}(A_{l_0}).
\]
(4)

Thus following Equation (3), we obtain that
\[
n(1 - \alpha_0) \leq \alpha_2 \text{ diam}(A_{n+1}) + \text{ diam}(A_{n+1})
\]
\[
= (\alpha_2 + 1) \text{ diam}(A_{n+1})
\]
(5)

and from Equation (4), it follows that
\[
(\alpha_0 + 1) \leq (\alpha_1 + 1) \text{ diam}(A_{l_0}).
\]
(6)

Thus by summation of Equations (5) and (6), we get
\[
n[1 - (\alpha_1 + 1) \text{ diam}(A_{l_0}) + 1] \leq (\alpha_2 + 1) \text{ diam}(A_{n+1}).
\]

This gives that
\[
n[2 - (\alpha_1 + 1) \text{ diam}(A_{l_0})] \leq (\alpha_2 + 1) \text{ diam}(A_{n+1}),
\]
which implies that
\[
n[2 - (\alpha_1 + 1) \max(\text{diam}(A_i))_{i=1}^{n}] \leq (\alpha_2 + 1) \text{ diam}(A_{n+1}).
\]

Hence
\[
\frac{n[2 - (\alpha_1 + 1) \max(\text{diam}(A_i))_{i=1}^{n}]}{\alpha_2 + 1} \leq \text{ diam}(A_{n+1}).
\]
which is contradiction.

**Corollary 3.3.** Let $X$ be a reflexive Banach space and let $x_1, x_2, ..., x_{n+1} \in X$ ($x_i \neq x_j$) for all $i, j = 1, 2, ..., n+1, i \neq j$. Assume that $0 < r < \frac{n}{2(n+1)}$.

Then, if there exist bounded linear operators $(S_i)_{i=1}^n$ on $X$ satisfying assumptions 1) and 2) of Theorem 3.1 and if each $B(x_i, r)$ is invariant by a nonexpansive mapping $T$ for each $i = 1, 2, ..., n+1$ with $\bigcap_{i=1}^{n+1} B(x_i, r) \neq \emptyset$. Then $T$ has a fixed point in $n+1 \bigcap_{i=1} B(x_i, r)$.

**Proof.** In this case, for all $i \in \{1, 2, ..., n+1\}$, each $B(x_i, r)$ is weakly compact since $X$ is reflexive. Moreover, we have $\text{diam}(B(x_i, r)) = 2r$ for all $i \in \{1, 2, ..., n+1\}$.

Now, the result is an immediate consequence of Theorem 3.1.

Also, as an immediate consequence of Theorem 3.2, we have the following

**Corollary 3.4.** Let $X$ be a reflexive Banach space and let $x_1, x_2, ..., x_{n+1} \in X$ ($x_i \neq x_j$) for all $i, j = 1, 2, ..., n+1, i \neq j$. Assume that $r \in \left[0, \frac{1}{2} \left( \bigcap_{i=1}^n \frac{1}{\alpha_1 + 1} \frac{n}{(\alpha_2 + 1) + n(\alpha_1 + 1)} \right) \right]$ where $\alpha_1 \geq 1$ and $\alpha_2 > 0$.

Assume that there exist bounded linear operators $(S_i)_{i=1}^n$ on $X$ satisfying 1) and 2) of Theorem 3.2. If each $B(x_i, r)$ is invariant under a nonexpansive mapping $T$ for each $i = 1, 2, ..., n+1$ with $\bigcap_{i=1}^{n+1} B(x_i, r) \neq \emptyset$. Then $T$ has a fixed point in $\bigcap_{i=1}^{n+1} B(x_i, r)$.

In 2000, T. Suzuki [22] has introduced $C$-mappings or $C\lambda$-mappings as an extension of nonexpansive mappings.

**Definition 3.5.** Let $C$ be a nonempty set of a Banach space $X$ and let $T : C \rightarrow C$ be a nonexpansive selfmapping on $C$. If $\lambda \in [0, 1]$, then $T$ is said to be a $C\lambda$-mapping if

$$\forall x, y \in C, \lambda \|x - Tx\| \leq \|x - y\| \implies \|Tx - Ty\| \leq \|x - y\|$$

If $\lambda = \frac{1}{2}$, $T$ is said to be $C$-mapping.

**Remark 3.6.** If we denote $\tilde{S}_x$ the set defined by

$$\tilde{S}_x = \{y \in C : \lambda \|x - Tx\| \leq \|x - y\|\}$$

Then it is easy to observe that $Tx \in \tilde{S}_x$ and thus the set $\tilde{S}_x$ is nonempty.

**Remark 3.7.** It is easy to observe that every nonexpansive mapping is $C\lambda$-mapping for every $\lambda \in [0, 1]$ but the converse is not true as the following example shows.
Example 3.8. Let $T : [0, 3] \to [0, 3]$ defined by

$$Tx = \begin{cases} 
0 & \text{if } x \neq 3 \\
1 & \text{if } x = 3.
\end{cases}$$

Then $T$ is a $C$-mapping on $[0, 3]$ but $T$ is not nonexpansive mapping (for more details, see G. Falset [5]).

Remark 3.9. Let $K$ be a nonexpansive subset of a Banach space $X$ and $\lambda \in [0, 1]$. Assume that $T$ is a continuous $C_\lambda$-mapping. Then $T$ has an approximate fixed point sequence $(x_n)_n \subset K$. (See B. Pilarika- T. Benavides [1]).

Lemma 3.10. Let $K$ be a minimal weakly compact convex subset for a $C_\lambda$-mapping ($\lambda \in [0, 1]$) and let $(x_n)_n \subset C$ be an approximate fixed point sequence. Then, $\forall x \in C$, we have

$$\lim_{n \to +\infty} \|x_n - x\| = \text{diam}(C).$$

and if $\lambda = \frac{1}{2}$, then the continuity assumption can be dropped.

Remark 3.11. By combining Remark 3.9 and Lemma 3.10, we conclude that Theorems 3.1, 3.2 and Corollaries 3.3, 3.4 hold for the case of continuous $C_\lambda$-mappings ($\lambda \in [0, 1]$).

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Authors contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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