Some Results On Generalized Kirk’s Process In Banach Spaces and Application

Abdelkader Dehici (work in collaboration with Nadjeh Redjel)

Laboratory of Informatics and Mathematics
University of Souk-Ahras, Algeria

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Work plan

1. Basic definitions and preliminaries
2. Fixed points formulas
3. The case of asymptotically regular mappings
4. Convergence of generalized Kirk’s processes
5. Application to a nonlinear system
In applied sciences, many problems are modeled by equations

\[ u - Tu = f \]  \hspace{1cm} (1)

where \( T \) is nonlinear and \( f \in X \) (convenable Banach space).

\( u_0 \) is a solution of (1) if and only if \( u_0 \) is a fixed point of \( T_f \)

\[ T_f u = Tu + f \]  \hspace{1cm} (2)

**Definition 1.1**

Let \( C \) be a nonempty subset of a normed space \( X \). \( T : C \rightarrow C \) is said to be nonexpansive if

\[ \|Tx - Ty\| \leq \|x - y\| \quad \text{for all } x, y \in C \]

In equations (1) and (2), it is easy to observe that \( T \) is nonexpansive if and only if \( T_f \) is nonexpansive.
Big Question:

Let $X$ be a Banach space and $C$ a closed bounded convex subset of $X$. Does every nonexpansive selfmapping $T$ on $C$ has a fixed point?

Some positive answers to the big question

1. If $\dim(X) < \infty$ then $T$ has a fixed point.  
   (Consequence of Brouwer’s Theorem (1912)).

2. If $\dim(X) = \infty$ and  
   1. $C$ compact then $T$ has a fixed point.  
      (Consequence of Schauder’s Theorem (1930)).

   2. $C$ weakly compact and has a normal structure then $T$ has a fixed point.  
      (W. Kirk, D. Göhde, F. E. Browder (1965-1966)).
A famous negative answer to the big question

- $X = L^1([0, 1])$
- $\|f\| = \int_0^1 |f(t)|\,dt$,
- $C = \{ f \in L^1([0, 1]), \int_0^1 f(t)\,dt = 1, 0 \leq f \leq 2 \}$
- $T : C \longrightarrow C$ defined by
  
  $T(f)(t) = \begin{cases} 
  \min\{2f(2t), 2\} & 0 \leq t \leq \frac{1}{2} \\
  \max\{2f(2t - 1) - 2, 0\} & \frac{1}{2} < t \leq 1. 
  \end{cases}$

Then $T$ is nonexpansive and fixed point free.

(D. Alpasch (1981)).
Definition 1.2

Let $C$ be a nonempty convex subset of a Banach space $X$.

1. Let $T: C \to C$ be a selfmapping. Define a sequence $(x_n)_n \subset C$ by

$$x_{n+1} = \lambda x_n + (1 - \lambda) T(x_n) \quad \lambda \in (0, 1)$$

$(x_n)_n$ is called Krasnoselskii process associated to $T$.

2. Let $T_1, T_2, \ldots, T_k$ be selfmappings on $C$. Define $(x_n)_n \subset C$ by

$$x_{n+1} = \lambda_0 x_n + \ldots + \lambda_k T_k(x_n),$$

where $\lambda_1 > 0$, and $\lambda_i \geq 0$, $i \neq 1$ with $\sum_{i=0}^{k} \lambda_i = 1$

$(x_n)_n$ is called generalized Kirk’s process associated to the mappings $T_1, \ldots, T_k$. 

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Remark 1.3

1. If $\lambda_2 = \ldots = \lambda_k = 0$ in the case of generalized Kirk’s process, then it is reduced to Krasnoselskii process associated to the mapping $T_1$.

2. If $T_i = T_i, \forall i \geq 1$ in the case of generalized Kirk’s process, then it reduced to the classical Kirk’s process associated to the mapping $T$.

3. In the following, we denote by $F(T)$ the set of fixed points of the mapping $T$. 
We start this section by the following lemma.

**Lemma 2.1**

Let $C$ be a nonempty convex subset of a Banach space $X$ and let $T_1, \ldots, T_k$ be selfmappings on $C$. For $(\lambda_i)_{i=0}^k \subset [0, 1]$ with $\sum_{i=0}^k \lambda_i = 1$, we denote by

$$S = \sum_{i=0}^k \lambda_i T_i,$$

with the notation $T_0 = \text{Id}_C$, then

$$\bigcap_{i=1}^k F(T_i) = F(S) \cap \left( \bigcap_{i=1}^k F(T_i S) \right).$$
Proof: Let $x_0 \in \bigcap_{i=1}^{k} F(T_i)$ then $x_0 \in F(T_i)$ for all integer $i = 1, \ldots, k$, which proves that $T_i(x_0) = x_0$ for all $i = 1, \ldots, k$ and consequently

$S(x_0) = \sum_{i=0}^{k} \lambda_i T_i(x_0) = x_0$, this gives that $x_0 \in F(S)$ and consequently

$x_0 \in F(S) \cap \bigcap_{i=1}^{k} F(T_i S)$.

Conversely, let $x_0 \in F(S) \cap \left( \bigcap_{i=1}^{k} F(T_i S) \right)$, then $S(x_0) = x_0$ and

$(T_i S)(x_0) = x_0$ for all integer $i = 1, \ldots, k$ by composition the equality $S(x_0) = x_0$ by $T_i$ ($i = 1, \ldots, k$), we get

$(T_i S)(x_0) = T_i x_0 = x_0$,

this implies that $x_0 \in F(T_i), \forall i = 1, \ldots, k$ and consequently

$x_0 \in \bigcap_{i=1}^{k} F(T_i)$, which achieves the proof.
Corollary 2.2

Let $C$ be a **nonempty subset** of a Banach space $X$ and let $T : C \rightarrow C$ be a selfmapping then for all $k \geq 1$, we have

$$F(T) = F(T^k) \cap F(T^{k+1}).$$

**Proof:** In the proof of Lemma 2.1, it suffices to take that $\lambda_i = 0$, $T_i = T^i$ for all integer $i \neq k$ and $\lambda_k = 1$ together with $T_k = T^k$.

**Remark 2.3**

It is easy to observe that **the assumption of the convexity** of the subset $C$ can be **dropped** in Corollary 2.2.
Theorem 2.4

Let $C$ be a **convex subset** of a Banach space $X$ and let $T_1, T_2, \ldots, T_k$ be a selfmappings satisfying that $\forall x \in C$, and $\forall i, j = 1, \ldots, k, (i < j)$ there exists an integer $n(x)$ with $1 \leq i \leq n(x) < j \leq k$ such that

$$\|T_i(x) - T_j(x)\| \leq \|x - T_{n(x)}(x)\| \quad (3)$$

Let $(\lambda_i)_{i=0}^k \subset [0, 1]$ with $\lambda_1 > 0$ and $\sum_{i=0}^k \lambda_i = 1$. We denote

$$S = \sum_{i=0}^k \lambda_i T_i \quad (\text{with the notation } T_0 = I_C).$$

Then

$$\bigcap_{i=1}^k F(T_i) = F(S).$$

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**Proof** It is easy to prove that \( \bigcap_{i=1}^{k} F(T_i) \subseteq F(S) \). For the converse, let

\[ x_0 \in F(S), \text{ thus} \]

\[ S(x_0) = \left( \sum_{i=0}^{k} \lambda_i T_i \right) (x_0) = x_0, \]

this gives that

\[ x_0 = \left( \sum_{i=1}^{k} \left( \frac{\lambda_i}{1 - \lambda_0} \right) T_i \right) (x_0) \quad (\lambda_0 \neq 1 \quad \text{since} \quad \lambda_1 > 0). \]

Let \( \delta = \sup\{\| T_i(x_0) - T_j(x_0) \|, i,j = 0, \ldots, k \} \). Assume that \( \delta > 0 \), the assumption (3) proves that there exists a smallest integer \( p(x_0) \in \{1, \ldots, k\} \) such that

\[ \delta = \| x_0 - T_{p(x_0)}(x_0) \|. \]
Since $\sum_{i=1}^{k} \frac{\lambda_i}{1 - \lambda_0} = 1$, it follows that
\[ x_0 = \gamma_0 T_1(x_0) + (1 - \gamma_0)z, \]
where $z \in \text{conv}\{T_2(x_0), \ldots, T_k(x_0)\} (\gamma_0 \in (0, 1])$. Thus
\[ \delta = \|x_0 - T_{p(x_0)}(x_0)\| = \|\gamma_0 T_1(x_0) + (1 - \gamma_0)z - T_{p(x_0)}(x_0)\| \]
\[ \leq \gamma_0 \|T_1(x_0) - T_{p(x_0)}(x_0)\| + (1 - \gamma_0)\|z - T_{p(x_0)}(x_0)\| \]
\[ \leq \gamma_0 \delta + (1 - \gamma_0)\delta = \delta. \]

(i) If $p(x_0) = 1$, this is a contradiction, since, we obtain that
\[ \|T_1(x_0) - T_1(x_0)\| = 0 = \delta. \]

(ii) If $p(x_0) > 1$, by the assumption (3), we obtain the existence of an integer $m(x_0) < p(x_0)$ such that
\[ \delta \leq \|T_1(x_0) - T_{p(x_0)}(x_0)\| \leq \|x_0 - T_{m(x_0)}(x_0)\| \]
which gives that $\|x_0 - T_{m(x_0)}(x_0)\| = \delta$ and contradicts the fact that $p(x_0)$ is the smallest integer such that $\delta = \|x - T_{p(x_0)}(x_0)\|$. Necessarily, we get $\delta = 0$ and $\|x_0 - T_i(x_0)\| = 0$ for all integer $i = 1, \ldots, k$, consequently $x_0 \in \bigcap_{i=1}^{k} F(T_i)$ which achieves the proof.
Corollary 2.5

Let $C$ be a convex subset of a Banach space $X$ and let $T : C → C$ be nonexpansive. Denote by

$$S = \sum_{i=0}^{k} \lambda_i T^i$$

with the notation $T^0 = I_C$ where $(\lambda_i)_{i=0}^{k} \subset [0, 1]$ together with $\lambda_1 > 0$.

and $\sum_{i=0}^{n} \lambda_i = 1$.

Then $F(S) = F(T)$.

Proof: The result follows from Theorem 2.4 by taking $T_i = T^i$ for all integer $i$. In this case, we have

$$\bigcap_{i=1}^{k} F(T^i) = F(T)$$

since $F(T) \subset F(T^i)$ for all integer $i \geq 1$ and $n(x) = j - i$ ($1 \leq i < j \leq k$) for all $x \in C$. 

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Definition 3.1

Let $C$ be a nonempty subset of a Banach space $X$ and let $T : C \rightarrow C$ is said to be asymptotically regular if, for all $x \in C$, we have

$$\lim_{n \rightarrow \infty} \| T^{n+1}(x) - T^n(x) \| = 0.$$ 

Remark 3.2

1. If $T$ is a Banach contraction then $T$ is asymptotically regular.

2. If $T$ is a nonexpansive, then $\delta_n = \| T^{n+1}(x) - T^n(x) \|$ is decreasing but does not converge necessarily to 0.

   Indeed, it suffices to take

   - $C = X = \mathbb{R}$ equipped with it’s usual norm.
   - $T : \mathbb{R} \rightarrow \mathbb{R}$ defined by $T(x) = 1 - x$. 

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Definition 3.3

A uniformly convex Banach space $X$ is a Banach space such that for every $0 < \epsilon \leq 2$ there is some $0 < \delta$ such that for any two vectors $x, y$ with $\|x\| = \|y\| = 1$, the condition $\|x - y\| \geq \epsilon$ implies $\frac{\|x + y\|}{2} \leq 1 - \delta$.

This concept was firstly introduced by James. A. Clarkson in (1936).

Remark 3.4

Intuitively, $X$ is a uniformly convex Banach space if it’s unit ball is sufficiently round.

Examples 3.5

1. Hilbert spaces and $L_p([0, 1])(1 < p < \infty)$ are uniformly convex
2. $L_1([0, 1])$ and $L_\infty([0, 1])$ are not uniformly convex.
Theorem 3.6

Let \( C \) be a convex subset of a uniformly convex Banach space \( X \) and let \( T_1, T_2, \ldots, T_k \) be nonexpansive selfmappings on \( C \) satisfying assumption (3). Denote by

\[
S = \sum_{i=0}^{k} \lambda_i T_i \quad \text{(with the notation } T_0 = \text{Id}_{C})
\]

where \((\lambda_i)_{i=0}^{k} \subset [0, 1]\) and \(\lambda_1 > 0\) with \(\sum_{i=0}^{k} \lambda_i = 1\). If \(\bigcap_{i=1}^{k} F(T_i) \neq \emptyset\).

Then \( S \) is asymptotically regular.
Proof: First of all, since $T_i$ is nonexpansive for all integer $i \in \{1, 2, ..., k\}$, then $S$ is nonexpansive. Moreover, Theorem 2.4 implies that $F(S) = \bigcap_{i=1}^{k} F(T_i) \neq \emptyset$. Assume that $y \in C$ is a fixed point of $S$ and let $x \in C$. Define a sequence $(x_n) \subset C$ by $x_n = S^n x$, $n \in \mathbb{N}$ with the notation $S^0 = \text{Id}_C$. It is easy to show that the sequence $\{\|x_n - y\|\}_n$ is decreasing, then $$\lim_{n \to \infty} \|x_n - y\| = \alpha \geq 0.$$ 

[(i)] If $\alpha = 0$, then $$\lim_{n \to +\infty} x_n = y,$$ since $S$ is continuous ($S$ is nonexpansive), it follows that $$\lim_{n \to +\infty} x_{n+1} = \lim_{n \to +\infty} S(x_n) = S(\lim_{n \to +\infty} x_n) = S(y) = y$$ and consequently $$\lim_{n \to +\infty} \|S^{n+1}(x) - S^n(x)\| = \|y - y\| = 0.$$  

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[(ii)] If $\alpha > 0$, thus

$$x_{n+1} - z_0 = S(x_n) - y = \sum_{i=0}^{k} \lambda_i T_i(x_n) - y = \lambda_0(x_n - y) - (1 - \lambda_0)z_n,$$

where

$$z_n = \frac{1}{1 - \lambda_0} \sum_{i=1}^{k} \lambda_i (T_i(x_n) - y).$$

Since $y \in \bigcap_{i=1}^{k} F(T_i)$, we get

$$\|T_i(x_n) - y\| = \|T_i(x_n) - T_i(y)\| \leq \|x_n - y\|. $$
The fact that \( \sum_{i=0}^{k} \lambda_i = 1 \) implies that \( \lim \|z_n\| \leq \alpha \). Moreover, since \( \lim_{n \to +\infty} \|x_n - y\| = \alpha \), gives that \( \lim_{n \to +\infty} \|x_{n+1} - y\| = \alpha \). From the uniform convexity of \( X \), we get that

\[
\lim_{n \to +\infty} \|x_n - y - z_n\| = 0,
\]

and consequently

\[
\lim_{n \to +\infty} x_{n+1} - x_n = \lim_{n \to +\infty} (1 - \lambda_0)(x_n - y - z_n) = 0,
\]

which achieves the proof.
Theorem 4.1

Let $X$ be a uniformly convex Banach space and let $T_1, T_2, \ldots, T_k$ be nonexpansive compact selfmappings on $X$ satisfying the assumption (3). Denote by $S$ the mapping

$$S = \sum_{i=0}^{k} \lambda_i T_i$$

with the notation $T_0 = Id_X$, where $(\lambda_i)_{i=0}^{k} \subset [0, 1], \lambda_1 > 0$ and \[ \sum_{i=0}^{k} \lambda_i = 1. \]

If $\bigcap_{i=1}^{k} F(T_i) \neq \emptyset$, then for each $x_0 \in X$ the Picard sequence $\{S^n(x_0)\}$ converges to a common fixed point of the mappings $T_1, T_2, \ldots, T_k$. 

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Proof: It follows from Theorem 3.6 that $S$ is asymptotically regular with

$$F(S) = \bigcap_{i=1}^{k} F(T_i) \neq \emptyset.$$ 

First of all, we prove that the mapping $I - S$ maps bounded closed subsets of $X$ into closed subsets of $X$. Indeed, let $C$ an arbitrary bounded closed subset of $X$ and assume that

$$\lim_{n \to +\infty} (y_n - Sy_n) = y, y_n \in C.$$ 

We will show that $y \in (I - S)(C)$. The fact that each $T_i, 1 \leq i \leq k$ is compact implies the existence of a subsequence $(y_{n_i(l)})_l$ such that $T_i(y_{n_i(l)})_l$ converges to $z_i \in X, 1 \leq i \leq k$ which proves the existence of a subsequence $(y_{f(l)})_l$ of $(y_l)_l$ (with $f(1)$ is the smallest integer multiple of $n^1(1), n^2(1), ..., n^k(1)$) such that $T_i(y_{f(l)})$ converges to $z_i \in X$. Thus

$$(I - S)(y_{f(l)}) = y_{f(l)} - \sum_{i=0}^{k} \lambda_i T_i(y_{f(l)})$$

$$=(1 - \lambda_0)y_{f(l)} - \sum_{i=1}^{k} \lambda_i T_i(y_{f(l)}).$$
Since $y_f(l) - S(y_f(l))$ converges to $y \ (l \to +\infty)$, we get

$$
\lim_{l \to +\infty} (1 - \lambda_0)y_f(l) = y + \sum_{i=1}^{k} \lambda_i z_i
$$

which implies

$$
\lim_{l \to +\infty} y_f(l) = \frac{y}{1 - \lambda_0} + \sum_{i=1}^{k} \left( \frac{\lambda_i}{1 - \lambda_0} \right) z_i \in C \ \text{(since } C \ \text{is closed)}
$$

then

$$
\lim_{l \to +\infty} y_f(l) = \tilde{y} \in C, \text{ which gives that}
$$

$$
\tilde{y} - S\tilde{y} = y,
$$

it proves that $y \in (I - S)(C)$ which is the desired result. Now the result follows from Theorem 6 in (F. E. Browder and W. V. Petryshin, *The solution by iteration of linear functional equations in Banach spaces*, Bull. Amer. Math. Soc., (72) (1966), 566-570).
Theorem 4.2

Let $X$ be a uniformly convex Banach space, $C$ a closed bounded convex subset of $X$, and let $T_1, T_2, ..., T_k$ be a nonexpansive mappings satisfying the assumption (3). Define

$$S = \sum_{i=0}^{k} \lambda_i T_i$$

with the notation $T_0 = Id_C$ where $(\lambda_i)_{i=0}^{k} \subset [0, 1], \lambda_1 > 0$ and

$$\sum_{i=0}^{k} \lambda_i = 1.$$  

Assume that $\bigcap_{i=1}^{k} F(T_i) = \{z_0\}$. Then for each $x_0 \in C$, the Picard sequence $\{S^n(x_0)\}$ converges weakly to $z_0$ in $C$.

Proof: Since $S$ is nonexpansive, then the mapping $I - S$ is demiclosed (F. E. Browder, *Nonexpansive nonlinear operators in a Banach space*, Proc. Nat. Acad. Sci. USA., (54) (1965), 1041-1044).
Now let $x_0 \in C$ and let $(x_n)_n$ the Picard sequence $x_n = S^n x_0 (n \in \mathbb{N})$, since $X$ is uniformly convex, then $X$ is reflexive (K. Goebel and W. A. Kirk, *Topics in metric fixed point theory*, Cambridge Studies in Advanced Mathematics. First edition (1990)), this implies the existence of a subsequence $\{x_{n_k}\}_k$ of $\{x_n\}_n$ such that $x_{n_k}$ converges weakly to $y_0$. Theorem 3.6 gives that $S$ is asymptotically regular, thus

$$\lim_{k \to +\infty} (I - S)(x_{n_k}) = \lim_{k \to +\infty} (S^{n_k}(x_0) - S^{n_k+1}(x_0)) = 0.$$ 

By definition of demiclosedness, it follows that

$$(I - S)(y_0) = 0,$$

which proves that $y_0$ is a fixed point of $S$. But $F(S) = \bigcap_{i=1}^{k} F(T_i)$ (see Theorem 2.4), hence $y_0 = z_0$ and $y_0$ is the unique fixed point of $S$. Consequently, every weakly convergent subsequence of $\{x_n\}$ converges weakly to $z_0$. By a standard argument using the reflexivity of $X$ and the fact that the sequence $\{x_n\}_n$ is bounded, we infer that $\{x_n\}_n$ converges weakly to $z_0$ which is the desired result.
Remark 4.3

Notice that Theorems 4.1 and 4.2 are extensions respectively of Corollary and Theorem 3 in (W. A. Kirk, On successive approximations for nonexpansive mappings, Glasgow. Math. J., Vol (2) (1), (1971), 6-9) by taking $T_i = T^i$ for all integer $i \geq 1$.

Lemma 4.4


If $\{x_n\}_n$ and $\{y_n\}_n$ are sequences in a uniformly convex space with

$$\|y_n\| \leq \|x_n\| \text{ and } x_{n+1} = (1 - \alpha_n)x_n + \alpha_n y_n \quad (0 \leq \alpha_n \leq 1)$$

where $\sum_{n=1}^{\infty} \min(\alpha_n, 1 - \alpha_n) = \infty$.

Then $0 \in \{x_n - y_n, n \in \mathbb{N}\}$ (where $\overline{C}$ denotes the closure of the set $C$).
Let \((\alpha_{ij})_{i=0}^{\infty} \ (j = 0, 1, \ldots, k)\) a set of positive reals such that

\[0 \leq \alpha_{ij}, 0 < \alpha \leq \alpha_{i1} \quad \text{with} \quad \sum_{j=0}^{k} \alpha_{ij} = 1 \quad \text{for each} \ i\]

\[\sum_{i=0}^{\infty} \min(\alpha_{i0}, 1 - \alpha_{i0}) = \infty.\]

Define the mappings \(S_i\) by

\[S_i = \alpha_{i0} I + \alpha_{i1} T_1 + \ldots + \alpha_{ik} T_k \quad (i = 0, 1, 2, \ldots,)

A non-stationary generalized Kirk’s process is given by the formula

\[x_{n+1} = S_n x_n \quad (n = 0, 1, 2, \ldots)\]
It is easy to observe that if

- $T_1, T_2, \ldots, T_k$ are nonexpansives mappings,

- $z_0 \in \bigcap_{i=1}^{k} F(T_i)$.

Then

$$\|x_{n+1} - z_0\| = \left\| \sum_{j=0}^{k} \alpha_{nj}(T_j x_n - T_j z_0) \right\| \leq \|x_n - z_0\|$$
Proposition 4.5

Let $C$ be a convex subset of uniformly convex Banach space and let $T_1, T_2, \ldots, T_k$ be nonexpansive selfmappings on $C$ with $\bigcap_{i=1}^{k} F(T_i) \neq \emptyset$ and let $(x_n)$ defined by equation (4), then $0 \in \{x_{n+1} - x_n, n \in \mathbb{N}\}$.

Proof: Let $x_0 \in \bigcap_{i=1}^{k} F(T_i)$. Define $y_n = x_n - x_0$ and

$$z_n = \frac{1}{1 - \alpha_n 0} \sum_{j=1}^{k} \alpha_{nj} (T_j x_n - T_j x_0).$$
It follows that

\[ y_{n+1} = x_{n+1} - x_0 = S_n x_n - x_0 = \alpha_{n0} x_n + \ldots + \alpha_{nk} T_k x_n - \left( \sum_{j=0}^{k} \alpha_{nj} \right) x_0 \]

\[ = \alpha_{n0} (x_n - x_0) + \sum_{j=1}^{k} \alpha_{nj} (T_j x_n - T_j x_0) \]

\[ = \alpha_{n0} y_n + (1 - \alpha_{n0}) z_n. \]

We have \( \|z_n\| \leq \|x_n - x_0\| = \|y_n\| \), because the mappings \( T_1, T_2, \ldots, T_k \) are nonexpansive. It follows by Lemma 4.4, that \( 0 \in \{ y_n - z_n, n \in \mathbb{N} \} \). On the other hand,
\[ \| y_n - z_n \| = \left\| x_n - x_0 - \frac{1}{1 - \alpha_n} \sum_{j=1}^{k} \alpha_{nj} T_j x_n + x_0 \right\| \]

\[ = \| x_n - \frac{1}{1 - \alpha_n} \sum_{j=0}^{k} \alpha_{nj} T_j x_n + \frac{\alpha_n}{1 - \alpha_n} x_n \| \]

\[ = \frac{1}{1 - \alpha_n} \| x_n - x_{n+1} \| \]

\[ \geq \| x_n - x_{n+1} \| \text{ since } \frac{1}{1 - \alpha_n} \geq 1 \]

this proves the existence of a subsequence \( \{ x_{n_k} \} \) such that

\[ \lim_{k \to +\infty} \| x_{n_k} - x_{n_k+1} \| = 0, \] which is the desired result.
Theorem 4.6

Assume in addition to the hypotheses of Proposition 4.5, that the mappings $T_1, T_2, \ldots, T_k$ satisfy the assumption (3) and each $T_i \ (1 \leq i \leq k)$ is compact. Then for each $x_1 \in C$, the sequence $\{x_n\}_n$ defined by the equation (4) converges to a common fixed point for the mappings $T_1, T_2, \ldots, T_k$.

Proof: By the previous Proposition, there exists a subsequence $\{x_{n_k}\}$ with $x_{n_{k+1}} - x_{n_k} \to 0$. The assumption given on the set $(\alpha_{ij})_{i=0}^\infty \ (j = 0, 1, \ldots, k)$ shows that, we can extract a subsequences $\alpha_{m_{kj}}$ of the sequence $\{\alpha_{n_kj}\}$ such that $\lim_{k \to +\infty} \alpha_{m_{kj}} = \alpha_j \in [0, 1]$ with $\alpha_1 > 0$.

Let

$$S = \alpha_0 I + \alpha_1 T_1 + \ldots + \alpha_k T_k.$$ 

We get

$$x_{m_k} - Sx_{m_k} = x_{m_k} - S_m x_{m_k} + S_m x_{m_k} - Sx_{m_k},$$
Where

\[ x_{m_k} - S_{m_k}x_{m_k} = x_{m_k} - x_{m_k+1} \to 0. \]

If \( x_0 \in \bigcap_{i=1}^{k} F(T_i) \), since the sequence \( \{\|x_n - x_0\|\} \) is decreasing and the mappings \( T_1, T_2, \ldots, T_k \) are nonexpansive, it follows that

\[ \|T_jx_{m_k} - x_0\| = \|T_jx_{m_k} - T_jx_0\| \leq \|x_{m_k} - x_0\| \leq \|x_1 - x_0\|. \]

Since

\[ \|T_jx_{m_k} - x_0\| \leq \|x_1 - x_0\|. \]

We obtain that

\[ \|T_jx_{m_k}\| \leq \|x_1 - x_0\| + \|x_0\| = \gamma \text{ for all } j = 0, 1, \ldots, k \]

Thus
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$$\| S_{m_k}x_{m_k} - Sx_{m_k} \| = \| \sum_{j=0}^{k} (\alpha_{m_kj} - \alpha_j) T_j x_{m_k} \|$$

$$\leq \gamma \sum_{j=0}^{k} |\alpha_{m_kj} - \alpha_j| \longrightarrow 0 \ (k \longrightarrow +\infty).$$

We infer that $x_{m_k} - Sx_{m_k} \longrightarrow 0 \ (k \longrightarrow +\infty)$. Since each $T_i \ (i \leq 1 \leq k)$ is compact, Theorem 4.1 shows that $I - S$ maps closed bounded subsets into closed subsets. On the other hand, from the decrease of the sequence $\{\|x_n - x_0\|\}_n$, we deduce that $\{\bar{x}_n, n \in \mathbb{N}\}$ is closed and bounded. Afterwards, Proposition 4.5 implies that $0 \in (I - S)(\{\bar{x}_n, n \in \mathbb{N}\})$. This proves the existence of $y_0 \in \{\bar{x}_n, n \in \mathbb{N}\}$ with $S(y_0) = y_0$ and here $y_0$ is a fixed point of $S$. Now, by Theorem 2.4, we get $y_0 \in \bigcap_{i=1}^{k} F(T_i)$. Apply for a second time the decrease of the sequence $\{\|x_n - y_0\|\}_n$, it follows that $x_n \longrightarrow y_0 \ (n \longrightarrow +\infty)$, which completes the proof.
Let be the nonlinear system

\[
\begin{align*}
  x - T_1 x &= f_1 \\
  \ldots &= \ldots \\
  \ldots &= \ldots \\
  \ldots &= \ldots \\
  x - T_k x &= f_k
\end{align*}
\]

in a convex subset \( C \) of a Banach space \( X \) where \( f_i \in C \) for all \( i = 1, \ldots, k \) and \( T_1, \ldots, T_k \) are selfmappings on \( C \).

Denote by \( B_i, i = 1, \ldots, k \) the mapping given by \( B_i x = T_i x + f_i \) with the notation \( B_0 = l d_X \). For all \( (\lambda_i)_{i=0}^k \subset [0, 1] \) with \( \lambda_1 > 0 \) and \( \sum_{i=0}^k \lambda_i = 1 \), if we denote by \( \gamma_i = \frac{\lambda_i}{1 - \lambda_0} \) \( (i = 1, \ldots, k) \), then we have
Lemma 5.1

Let $z_0 \in X$. Then $z_0$ is a solution of the system $(\star)$ if and only if $z_0$ is at the same time the solution of the nonlinear equation

$$x = \sum_{i=1}^{k} \gamma_i B_i x$$

and the system

$$x = B_i \left( \sum_{j=0}^{k} \lambda_j B_j \right) x, \quad i = 1, \ldots, k$$

$(\star \star)$
Lemma 5.2

Assume that the mappings $(B_i)_{i=1}^k$ given in $(\star)$ satisfy the assumption (3). Then $x$ is a solution of the system $(\star)$ if and only if $x$ is the solution of the nonlinear equation (5).

Let $X$ be a Banach space and $C$ a convex subset of $X$. For a finite family of nonexpansive selfmappings $\{T_i\}_{i=1}^k$ of $C$. For $\alpha \in ]0, 1[$, P. Kuhfittig (Common fixed points of nonexpansive mappings by iteration, Pacific. J. Math., Vol (97) (1), (1981), 137-139)) has defined the following iterative process

$$x_{n+1} = U_k(x_n), \quad n = 0, 1, \ldots,$$

where

$$\begin{cases}
U_0 = I \\
U_1 = (1 - \alpha)I + \alpha T_1 U_0 \\
\vdots \\
U_k = (1 - \alpha)I + \alpha T_k U_{k-1}
\end{cases}$$
Theorem 5.3

Let $C$ be a convex compact subset of a strictly convex Banach space $X$ and let $\{T_i\}_{i=1}^k$ be a family of nonexpansive selfmappings of $C$. If the nonlinear equation (5) has at least a solution and the mappings $\{B_i\}_{i=1}^k$ satisfy the assumption (3).

Then for an arbitrary $z_0 \in C$, the sequence $\{U^n_k z_0\}$ converges strongly to a solution of the system ($\ast$).

Theorem 5.4

If $X$ is a Hilbert space and $C$ is a closed convex subset of $X$. Assume that the mappings $\{T_i\}_{i=1}^k$ are nonexpansive selfmappings of $C$. If the nonlinear equation (5) has at least a solution and the mappings $\{B_i\}_{i=1}^k$ satisfy the assumption (3).

Then for any $z_0 \in C$, the sequence $\{U^n_k z_0\}$ converges weakly to a solution of the system ($\ast$).


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Thank you for your attention