Existence of solution for an elliptic problem involving $p(x)$-Laplacian in $\mathbb{R}^N$.

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Abstract

In this paper we study a class of nonlinear elliptic problems involving the $p(x)$-Laplacian operator. Under some additional assumptions on the nonlinearities, the corresponding functional verifies the Palais-Smale condition. So, we can use the Mountain Pass Theorem to prove the existence of nontrivial solution.

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1. Introduction

The aim of this paper is to prove some existence results for nonlinear elliptic problem

\[
\begin{cases}
-\Delta_{p(x)} u = \lambda V(x) |u|^{q(x)-2} u + f(x, u), \\ u \geq 0, u \neq 0, u \in W
\end{cases}
\]

(1.1)

\(\Delta_{p(x)}\) is so-called \(p(x)\)-Laplacian operator i.e. \(\Delta_{p(x)} u = \text{div}(|\nabla u|^{p(x)-2} \nabla u)\). In the case \(p(x) = p\), then \(\text{div}(|\nabla u|^{p-2} \nabla u)\) is well-known \(p\)-Laplacian and the problem is the usual \(p\)-Laplacian equation. \(f\) is real-valued function with domain \(\mathbb{R}^N \times \mathbb{R}\); \(u\) is unknown real valued function defined in \(\mathbb{R}^N\) and belonging to appropriate function spaces; \(\lambda\) is positive parameter; \(p\) and \(q\) are reals functions satisfying \(p(x), q(x) \in C_+(\mathbb{R}^n)\).

Problems involving the \(p(x)\)-Laplacian operator arise from many branches of mathematics as in the study of elastic mechanics (see [22]), electrorheological fluids (see [1], [7]), (see [17]) or image restoration (see [6]).

Let the eigenvalue problem involving variable exponent growth conditions intensively studied is the following

\[-\Delta_{p(x)} u = \lambda V(x) |u|^{q(x)-2} u, \text{ in } \Omega.\]

(1.2)

where \(\Omega\) is bounded domain in \(\mathbb{R}^N, n \geq 3\), with smooth boundary \(\partial \Omega\).

In [21] the author studied the problem (1.2) in bounded domain where \(V(x) = 1\), under the assumption \(1 < \min_{\Omega} q(x) < \min_{\Omega} p(x) < \max_{\Omega} q(x)\), the continuous spectrum is proved.

However [18] the author established in bounded domain, using the simple variational arguments based on the Ekeland’s principle, that there exists \(\lambda^{**} > 0\) such that for any \(\lambda \in (0, \lambda^{**})\) is an eigenvalue for the above problem.

This paper is organized as follows. In Section 1 we recall some previous results. In Section 2, we state some basic results for the variable exponent Lebesgue-Sobolev spaces, which are given in Fan and Zhao (see [11]), O. Kováčik, J. Rákosník (see [19]). In Section 3, we give sufficient conditions on \(V\) and \(f\) to obtain the existence of solution for the problem (1.1) above.

2. Preliminary results

We recall some background facts concerning the generalized Lebesgue-Sobolev spaces and introduce some notations used below.

Let

\[C_+(\Omega) = \{ p \in C(\Omega) : p(x) > 1, \text{ for every } x \in \Omega \}\]

\[p^+ = \max \{ p(x) \in \Omega \} \text{ et } p^- = \min \{ p(x) \in \Omega \} \text{ for every } p \in C_+(\Omega).\]

Denote by \(\mathcal{M}(\Omega)\) the set of measurable real-valued functions defined on \(\Omega\).
We introduce for \( p \in \mathbb{C}_+ (\Omega) \), the space

\[
L^{p(x)} (\Omega) = \left\{ u \in \mathcal{M} (\Omega) \text{ such that, } \int_{\Omega} |u(x)|^{p(x)} \, dx < +\infty \right\}
\]
equipped with the so-called Luxemburg norm

\[
|u|_{p(x),\Omega} = \inf \left\{ t > 0 : \int_{\Omega} \left| \frac{u(x)}{t} \right|^{p(x)} \, dx \leq 1 \right\}.
\]

In what follows \(|u|_{p(x)}\) will denote \(|u|_{p(x),\mathbb{R}^N}\). It is well-known that this norm confers a reflexive Banach structure.

Define the variable exponent Sobolev space \( W^{1,p(x)} (\mathbb{R}^N) \) as the closure of \( C_0^\infty (\mathbb{R}^N) \) under the norm

\[
\|u\|_{p(x)} = |\nabla u|_{p(x)}.
\]

Moreover, we recall some previous results.

**Proposition 2.1.** ([8]) If \( p \in \mathbb{C}_+ (\mathbb{R}^N) \), then \( L^{p(x)} (\mathbb{R}^N) \) and \( W^{1,p(x)} (\mathbb{R}^N) \) are separable and reflexive Banach spaces.

**Proposition 2.2.** ([8]) The topological dual space of \( L^{p(x)} (\mathbb{R}^N) \) is \( L^{p'(x)} (\mathbb{R}^N) \), where

\[
\frac{1}{p(x)} + \frac{1}{p'(x)} = 1.
\]

Moreover for any \( (u, v) \in L^{p(x)} (\mathbb{R}^N) \times L^{p'(x)} (\mathbb{R}^N) \), we have

\[
\left| \int_{\mathbb{R}^N} uv \, dx \right| \leq \left( \frac{1}{p} + \frac{1}{p'} \right) |u|_{p(x)} |v|_{p'(x)} \leq 2 |u|_{p(x)} |v|_{p'(x)}.
\]

Let us now define the modular corresponding to the norm \(|.|_{p(x)}\) by

\[
\rho (u) = \int_{\mathbb{R}^N} |u|^{p(x)} \, dx.
\]

**Proposition 2.3.** ([11],[19]) For all \( u \in L^{p(x)} (\mathbb{R}^N) \), we have

\[
\min \left\{ |u|_{p(x)}^{-}, |u|_{p(x)}^{+} \right\} \leq \rho (u) \leq \max \left\{ |u|_{p(x)}^{-}, |u|_{p(x)}^{+} \right\}.
\]

In addition, we have

(i) \(|u|_{p(x)} < 1 \text{ (resp. } = 1; > 1) \Leftrightarrow \rho (u) < 1 \text{ (resp. } = 1; > 1)\),

(ii) \(|u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{-} \leq \rho (u) \leq |u|_{p(x)}^{+}\).
(iii) \(|u|_{p(x)} > 1 \implies |u|_{p(x)}^{p^+} \leq \rho(u) \leq |u|_{p(x)}^{p^-},\)

(iv) \(\rho\left(\frac{u}{|u|_{p(x)}}\right) = 1.\)

**Proposition 2.4.** ([8]) Let \(p(x)\) and \(s(x)\) be measurable functions such that \(p(x) \in L^{\infty}(\mathbb{R}^N)\) and \(1 \leq p(x) s(x) \leq \infty\) almost everywhere in \(\mathbb{R}^N\). If \(u \in L^{p(x)}(\mathbb{R}^N)\), \(u \neq 0\), then

\[ |u|_{p(x)s(x)} \leq 1 \implies |u|_{p(x)s(x)}^{p^-} \leq \left| |u|_{p(x)} \right|_{s(x)} \leq |u|_{p(x)s(x)}^{p^+}, \]

\[ |u|_{p(x)s(x)} \geq 1 \implies |u|_{p(x)s(x)}^{p^+} \leq \left| |u|_{p(x)} \right|_{s(x)} \leq |u|_{p(x)s(x)}^{p^-}. \]

In particular, if \(p(x) = p\) is a constant, then

\[ \left| |u|^{p(x)} \right|_{s(x)} = |u|_{p s(x)}^{p(x)}. \]

**Proposition 2.5.** ([11]) If \(u, u_n \in L^{p(x)}(\mathbb{R}^N)\), \(n = 1, 2, \ldots\), then the following statements are mutually equivalent:

1) \(\lim_{n \to \infty} |u_n - u|_{p(x)} = 0,\)

2) \(\lim_{n \to \infty} \rho(u_n - u) = 0,\)

3) \(u_n \to u\) in measure in \(\mathbb{R}^N\) and \(\lim_{n \to \infty} \rho(u_n) = \rho(u).\)

Let \(p^*(x)\) be the critical Sobolev exponent of \(p(x)\) defined by

\[ p^*(x) = \begin{cases} \frac{N p(x)}{N - p(x)} & \text{for } p(x) < N \\ +\infty & \text{for } p(x) \geq N \end{cases}, \]

and let \(C^{0,1}(\mathbb{R}^N)\) be the Lipschitz-continuous functions space.

**Proposition 2.6.** ([11],[9]) If \(p(x) \in C^{0,1}(\mathbb{R}^N)\), then there exists a positive constant \(c\) such that

\[ |u|_{p^*(x)} \leq c_{p(x)} |\nabla u|_{p(x)}, \quad \text{for all } u \in W^{1,p(x)}(\mathbb{R}^N). \]

**Proposition 2.7.** ([9]) 1) If \(s \in L^{\infty}(\mathbb{R}^N)\) and \(p(x) \leq s(x) \ll p^*(x), \forall x \in \mathbb{R}^N\), then the embedding

\[ W^{1,p(x)}(\mathbb{R}^N) \hookrightarrow L^{s(x)}(\mathbb{R}^N) \]
is continuous but not compact.

2) If $p$ is continuous on $\Omega$ and $s$ is a measurable function on $\Omega$, with $p(x) \leq s(x) < p^*(x)$, $\forall x \in \Omega$, then the embedding

$$W^{1,p(x)}(\Omega) \hookrightarrow L^{s(x)}(\Omega)$$

is compact.

3. Main result and proof

**Definition 3.1.** $u \in W$ is a weak solution of (1.1) if for all $v \in W$,

$$\int_{\mathbb{R}^N} |\nabla u|^{p(x)-2} \nabla u \nabla v \, dx - \lambda \int_{\mathbb{R}^N} V(x) |u|^{q(x)-2} u v \, dx - \int_{\mathbb{R}^N} f(x,u) v \, dx = 0,$$

The present paper is studied under the following hypotheses. Put $F(x,u) = \int_0^u f(x,t) \, dt$.

(H1) We suppose that the functions $p, q$ are continuous and satisfy $p(x) < N$, along with $1 < p^- < p^+ < q^- < q^+ \leq p^*(x)$. In particular, $p$ verifies the weak Lipschitz condition, namely, $|p(x) - p(y)| \leq \frac{c}{\log|x-y|}$ holds for $|x-y| \leq \frac{1}{2}$ and $x, y \in \mathbb{R}^N$.

(H2) We assume $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function and satisfies $f \in C^1(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and

$$|f(x,u)| \leq a(x) |u|^{\frac{p(x)}{q(x)}} \quad \forall (x,u) \in \mathbb{R}^N \times \mathbb{R}.$$ 

Here $a \in L^{\frac{p(x)}{q(x)}}(\mathbb{R}^N)$, is nonnegative measurable function, along with $\frac{1}{\alpha(x)} + \frac{1}{p(x)} = 1$.

(H3) Suppose that $0 \leq \theta F(x,u) \leq uf(x,u)$, such that $p^- < \theta < q^-$, $x \in \mathbb{R}^N$.

(H4) The potential $V \in L^\infty(\mathbb{R}^N) \cap L^{r(x)}(\mathbb{R}^N)$ is nonnegative, and $\frac{1}{r(x)} + \frac{1}{q(x)} = 1$.

**Remark 3.2.** As in [3] the hypothesis (H3) implies that, for all $t > 1$, $F(x,tu) \geq t^\theta F(x,u)$. Moreover, in view of (H1), $W = W^{1,p(x)}$.

The main result for this paper is given by the following theorem.

**Theorem 3.3.** If the hypotheses (H1)–(H4) fulfilled, then the problem (1.1) has a non-trivial weak solution for all $\lambda > 0$. 

We need some lemmas to prove main result. The energy functional corresponding to problem (1.1) is defined by
\[
J_{\lambda}(u) = \int_{\mathbb{R}^n} \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx - \int_{\mathbb{R}^n} \lambda \frac{V(x)}{q(x)} |u|^{q(x)} \, dx - \int_{\mathbb{R}^n} F(x,u) \, dx
\]
and we put
\[
\varphi(u) = \int_{\mathbb{R}^n} \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx,
\]
\[
\psi(u) = \int_{\mathbb{R}^n} \frac{V(x)}{q(x)} |u|^{q(x)} \, dx,
\]
\[
K(u) = \int_{\mathbb{R}^n} F(x,u) \, dx.
\]

**Lemma 3.4.** The functional $J_{\lambda}$ is well defined and $C^1(W, \mathbb{R})$. Moreover,
\[
\langle J_{\lambda}'(u), v \rangle = \int_{\mathbb{R}^n} \left( |\nabla u|^{p(x)-2} \nabla u \nabla v - \lambda V(x) |u|^{q(x)-2} u v \right) dx - \int_{\mathbb{R}^n} f(x,u) v \, dx.
\]

By (H2) toghether with (H4), it is easy to see that $J_{\lambda}'$ belongs to the topological dual of $W$.

**Lemma 3.5.** There exists positives constants $R$ and $\rho$ such that
\[
J_{\lambda}(u) \geq \rho \text{ on } \|u\|_{p(x)} = R.
\]

**Proof.** By the Hölder inequality, we get
\[
\int_{\mathbb{R}^n} |F(x,u)| \, dx \leq \int_{\mathbb{R}^n} \frac{|a(x)|}{q(x)} |u|^{q(x)} \, dx
\]
\[
\leq \frac{2}{q^-} |a|_{r(x)} \|u\|^{q(x)}_{r(x)}
\]
\[
\leq \frac{2c_1}{q^-} |a|_{r(x)} \|u\|^{q_i}_{r(x)},
\]
where
\[
i = + \text{ if } \|u\|_{p(x)} > 1, \text{ and } i = - \text{ if } \|u\|_{p(x)} < 1
\]
and we are
\[
\int_{\mathbb{R}^n} \frac{V(x)}{q(x)} |u|^{q(x)} \, dx \leq \frac{2}{q^-} |V|_{r(x)} \|u\|^{q(x)}_{r(x)}
\]
\[
\leq \frac{2}{q^-} |V|_{r(x)} \|u\|^{q_i}_{r(x)}
\]
\[
\leq \frac{2c_2}{q^-} |V|_{r(x)} \|u\|^{q_i}_{p(x)},
\]
where
\[
i = + \text{ if } \|u\|_{p(x)} > 1, \text{ and } i = - \text{ if } \|u\|_{p(x)} < 1
\]
\[ J_{\lambda}(u) = \int_{\mathbb{R}^n} \left( \frac{1}{p(x)} |\nabla u|^{p(x)} - \frac{\lambda}{q(x)} |u|^{q(x)} \right) dx - \int_{\mathbb{R}^n} F(x,u) \, dx \]

where \( c_1, c_2 \) are positive constants. So, for all \( \lambda > 0 \), and \( u \in W \) with \( \|u\|_{p(x)} = R \) sufficiently small, there exists \( \rho > 0 \) such that

\[ J_{\lambda}(u) \geq \rho > 0 \]

Lemma 3.6. There exists \( e \in W \) with \( \|e\|_{p(x)} > R \) such that \( J_{\lambda}(e) < 0 \).

Proof. Choose \( u_0 \in W, \|u_0\|_{p(x)} > 1 \). For \( t \) large enough we obtain

\[ J_{\lambda}(tu_0) = \int_{\mathbb{R}^n} \left( \frac{1}{p(x)} |\nabla tu_0|^{p(x)} - \frac{\lambda}{q(x)} |tu_0|^{q(x)} \right) dx - \int_{\mathbb{R}^n} F(x,tu_0) \, dx \]

\[ \leq \frac{1}{p^+} \int_{\mathbb{R}^n} |\nabla tu_0|^{p(x)} dx - \frac{\lambda}{q^+} \int_{\mathbb{R}^n} V(x)|tu_0|^{q(x)} \, dx \]

This yields \( J_{\lambda}(tu_0) \to -\infty \), as \( t \to +\infty \) since

\[ 0 \leq \int_{\mathbb{R}^n} V(x)|u_0|^{q(x)} \, dx \leq 2c_2 |V|_{r(x)} \|u_0\|^{q(x)}_{p(x)} \cdot \]

Lemma 3.7. The functional \( J_{\lambda} \) satisfies the Palais-Smale condition (PS), for any \( c \in \mathbb{R} \).

Proof. Let \( (u_n) \) be a (PS)\(_c\) sequence for the functional \( J_{\lambda} \) in \( W \) i.e. \( J_{\lambda}(u_n) \) is bounded and \( J'_{\lambda}(u_n) \to 0 \). Then the sequence \( u_n \) is bounded in \( W \).

Indeed, since \( J_{\lambda}(u_n) \) is bounded, we have

\[ C_1 \geq J_{\lambda}(u_n) = \int_{\mathbb{R}^n} \left( \frac{1}{p(x)} |\nabla u_n|^{p(x)} - \frac{\lambda}{q(x)} |u_n|^{q(x)} \right) dx - \int_{\mathbb{R}^n} F(x,u_n) \, dx \]

\[ \geq \int_{\mathbb{R}^n} \left( \frac{1}{p(x)} |\nabla u_n|^{p(x)} - \frac{\lambda}{q(x)} |u_n|^{q(x)} \right) dx - \int_{\mathbb{R}^n} F(x,u_n) \, dx \]

\[ \geq \int_{\mathbb{R}^n} \left( \frac{1}{p(x)} |\nabla u_0|^{p(x)} dx - \frac{\lambda}{q(x)} |u_0|^{q(x)} \right) dx - \int_{\mathbb{R}^n} \frac{u_n}{\theta} f(x,u_n) \, dx \]
Furthermore

\[ \{ J'_\lambda (u_n) , u_n \} = \int_{\mathbb{R}^n} |\nabla u_n|^{p(x)} - \lambda V(x) |u_n|^{q(x)} \, dx - \int_{\mathbb{R}^n} f(x, u_n) u_n \, dx \]

Then

\[ C_1 \geq \frac{1}{p^+} \int_{\mathbb{R}^n} |\nabla u_n|^{p(x)} \, dx - \frac{1}{q^-} \int_{\mathbb{R}^n} \lambda V(x) |u_n|^{q(x)} \, dx + \frac{1}{\theta} \{ J'_\lambda (u_n) , u_n \} \]

\[ \geq \left( \frac{1}{p^+} - \frac{1}{\theta} \right) \int_{\mathbb{R}^n} |\nabla u_n|^{p(x)} \, dx + \left( \frac{1}{\theta} - \frac{1}{q^-} \right) \int_{\mathbb{R}^n} \lambda V(x) |u_n|^{q(x)} \, dx + \frac{1}{\theta} \{ J'_\lambda (u_n) , u_n \} \]

Arguing by contradiction, we assume that \((u_n)\) is unbounded in \(W\). In particular we can choose \(\|u_n\| \geq 1\) for \(n\) sufficiently large. Hence, there exists \(C_3 > 0\) such that

\[-C_3 \| u_n \|_{p(x)} \leq \{ J'_\lambda (u_n) , u_n \} \leq C_3 \| u_n \|_{p(x)} \]

since \(J'_\lambda (u_n) \to 0\). To this end,

\[ C_1 \geq \frac{1}{p^+} \| u_n \|_{p(x)}^{p^+} + \left( \frac{1}{\theta} - \frac{1}{q^-} \right) \int_{\mathbb{R}^n} \lambda V(x) |u_n|^{q(x)} \, dx - \frac{1}{\theta} C_3 \| u_n \|_{p(x)} \]

\[ \geq \left( \frac{1}{p^+} - \frac{1}{\theta} \right) \| u_n \|_{p(x)}^{p^+} - \frac{1}{\theta} C_3 \| u_n \|_{p(x)} \]

This implies a contradiction.

Hence the sequence \((u_n)\) is bounded in \(W\). Thus, there exists a subsequence, again denoted \((u_n)\), weakly convergent to \(u\) in \(W\). We prove that \((u_n)\) is strongly convergent to \(u\) in \(W\).

To this end, we consider the following equality

\[ \{ J'_\lambda (u_n) - J'_\lambda (u) , u_n - u \} = \{ \varphi' (u_n) - \varphi' (u) , u_n - u \} - \{ \varphi' (u_n) - \psi' (u) , u_n - u \} - \{ K' (u_n) - K' (u) , u_n - u \} \]

Obviously, the term in the left hand side tends to zero for \(n\) large enough. First, we show that \(\{ K' (u_n) - K' (u) , u_n - u \} \to 0\) as \(n \to \infty\).

Let \(B_R\) be the ball in \(\mathbb{R}^N\) of radius \(R\) centered at the origin and \(B'_R = \mathbb{R}^N - B_R\). We use well-know compactness argument in unbounded domains. Roughly speaking, we write

\[ \| \{ K' (u_n) - K' (u) , u_n - u \} \| = \left| \int_{\mathbb{R}^n} (f(x, u_n) - f(x, u)) (u_n - u) \, dx \right| \]

\[ \leq \int_{B_R} |f(x, u_n) - f(x, u)| |u_n - u| \, dx \]

\[ + \int_{B'_R} |f(x, u_n) - f(x, u)| |u_n - u| \, dx \]
Taking into account Proposition 2.7 together with the compact embedding $W^{1,p(x)}(B_R) \hookrightarrow L^{p(x)}(B_R)$, the first term in the right hand side of the above inequality vanishes as $n \to \infty$. Contrariwise, the second term vanishes as $R \to \infty$. In fact, we have
\[
\int_{B_R} |f(x,u_n) - f(x,u)| |u_n - u| \, dx \leq 2 |f(x,u_n) - f(x,u)|_{\alpha(x)} |u_n - u|_{p(x),B_R}.
\]

In virtue of (H2) the Nemyckii operator is bounded. Hence, we obtain
\[
\int_{B_R} |f(x,u_n) - f(x,u)| |u_n - u| \, dx \leq \frac{\varepsilon}{2}.
\]

On the other hand, we have
\[
\int_{B'_R} |f(x,u_n) - f(x,u)| |u_n - u| \, dx \leq 
\int_{B'_R} a(x) |u_n|^{p(x)} + a(x) |u_n|^{p(x)-1} |u| + a(x) |u|^{p(x)} + a(x) |u_n|^{p(x)-1} |u_n| \, dx \leq \frac{\varepsilon}{2},
\]
for $R$ sufficiently large. Indeed,
\[
\int_{B'_R} a(x) |u_n|^{p(x)} \, dx \leq 2 |a|_{\alpha(x)} \left| |u_n|^{p(x)} \right|_{p(x)} \leq \frac{\varepsilon}{8},
\]
for $R$ sufficiently large. Using the Young inequality, we get
\[
\int_{B'_R} a(x) |u_n|^{p(x)-1} |u| \, dx \leq \int_{B'_R} a(x) \left( |u_n|^{p(x)} + |u|^{p(x)} \right) \, dx \\
\leq 2 |a|_{\alpha(x)} \left( \left| |u_n|^{p(x)} \right|_{p(x)} + \left| |u|^{p(x)} \right|_{p(x)} \right) \leq \frac{\varepsilon}{8},
\]
for $R$ sufficiently large.

In the same way, according to $R$, we show that both the two last terms are less than $\frac{\varepsilon}{8}$.

Similarly, using the same arguments, the following holds
\[
\langle \psi'(u_n) - \psi'(u), u_n - u \rangle \\
\leq \lambda \int_{B_R} V(x) \left( |u_n|^{q(x)-2} u_n - |u|^{q(x)-2} u \right) |u_n - u| \, dx \\
+ \lambda \int_{B'_R} V(x) \left( |u_n|^{q(x)} + |u|^{q(x)-2} u_n u + |u|^{q(x)} + |u_n|^{q(x)-2} u_n u \right) \, dx \\
\leq c_1 \int_{B_R} V(x) \left( |u_n|^{q(x)-2} u_n - |u|^{q(x)-2} u \right) \left| u_n - u \right|_{q(x)} \\
+ c_2 \int_{B'_R} V(x) \left( \left| u_n |^{q(x)} \right|_{q(x)} + \left| u |^{q(x)} \right|_{q(x)} \right) \leq \varepsilon.
\]
for \( n, R \) large enough.

It appears from (1.3) that \( \langle \phi'(u_n) - \phi'(u), u_n - u \rangle \to 0 \) as \( n \to \infty \). Now, with the aid of an elementary inequality in \( \mathbb{R}^N \), we get if \( p(x) \geq 2 \)

\[
2^{2-p^+} \int_{\mathbb{R}^N} |\nabla u_n| - |\nabla u|^{p(x)} \, dx \leq \int_{\mathbb{R}^N} \left( |\nabla u_n|^{p(x)-2} \nabla u_n - |\nabla u|^{p(x)-2} \nabla u \right) (\nabla u_n - \nabla u) \, dx \to 0 \quad \text{as} \quad n \to \infty.
\]

Otherwise, use the following inequality in \( \mathbb{R}^N \)

\[
(p - 1) |\zeta - \eta|^2 (|\zeta| + |\eta|)^{p-2} \leq (|\zeta|^{p-2} \zeta - |\eta|^{p-2} \eta) (\zeta - \eta) \quad \text{if} \quad 1 < p < 2
\]

and consider the following sets

\[
U_p = \{ x \in \mathbb{R}^N, p (x) \geq 2 \}; \quad V_p = \{ x \in \mathbb{R}^N, 1 < p (x) < 2 \}
\]

**Proof [Proof of theorem 3.3].** Set

\[
\Gamma = \{ \gamma \in C ([0, 1], W) : \gamma (0) = 0, \gamma (1) = e \}
\]

\[
c := \inf \max_{\gamma \in \Gamma, t \in [0,1]} J_{\lambda} (\gamma (t)).
\]

According to lemma 3.5 and lemma 3.6, the energy functional \( J_{\lambda} \) satisfies the geometrical conditions of the Mountain pass theorem. Hence \( c \) is a critical value of \( J_{\lambda} \) associated with a critical point \( u \in W \), which is precisely one solution of (1.1). The proof is complete.

**References**


